WHAT IS...

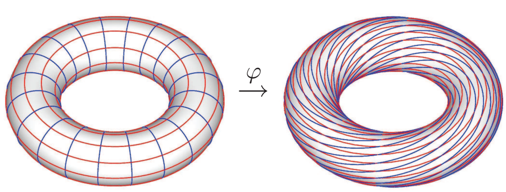
an Infra-nilmanifold Endomorphism?

Karel Dekimpe

Ever since the term "infra-nilmanifold endomorphism" arose in the 1960s, there has been confusion about its exact meaning, and different authors have used the term to refer to different concepts. Partly because of this confusion, two major results in dynamical systems, one on Anosov diffeomorphisms (1974) and one on expanding maps (1981), turn out to be incorrect.

With this historical background as motivation, let me introduce the reader gradually to the world of infra-nilmanifold endomorphisms, starting with one of the best known examples, Arnold's cat map φ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This map is given by

$$\varphi : \mathbb{T}^2 \to \mathbb{T}^2 : (x, y) + \mathbb{Z}^2 \mapsto (2x + y, x + y) + \mathbb{Z}^2.$$



In this picture one sees that φ stretches any small rectangle on the torus in one direction and shrinks it in the other. In dynamical systems one refers to such a map as an Anosov diffeomorphism. This diffeomorphism exhibits uniformly hyperbolic behavior: Around any point there are some directions in which the map is expanding and complementary directions in which it is contracting ([1] contains further details).

It is an open problem to determine the class of manifolds admitting an Anosov diffeomorphism, and it is also quite hard to construct examples of these mappings. The most obvious method to obtain new examples is to generalize Arnold's cat map to higher dimensions. The cat map lifts to a linear map l_φ of $\mathbb{R}^2$ that is given by the matrix $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, and φ being Anosov is equivalent to A having no eigenvalues of modulus 1. More generally, if $A \in \mathrm{GL}(n, \mathbb{Z})$, then the linear map $l_\varphi : \mathbb{R}^n \to \mathbb{R}^n$ determined by A induces a diffeomorphism $\varphi : \mathbb{T}^n \to \mathbb{T}^n : \mathbf{r} + \mathbb{Z}^n \mapsto l_\varphi(\mathbf{r}) + \mathbb{Z}^n$ on the n-torus. Such maps, called *toral automorphisms*, are prototype examples of infra-nilmanifold automorphisms, which we will describe below. When A has no eigenvalue of modulus 1, A is said to be hyperbolic, and the corresponding toral automorphism, which is called a hyperbolic toral automorphism, is an Anosov diffeomorphism. It is easy to construct a hyperbolic matrix $A \in \mathrm{GL}(n, \mathbb{Z})$, for any $n \geq 2$; hence any torus $\mathbb{T}^n$ of dimension $n \geq 2$ admits an Anosov diffeomorphism. Moreover, by a result of J. Franks, it is known that any Anosov diffeomorphism ψ on a torus $\mathbb{T}^n$ is topologically conjugate to such a hyperbolic toral automorphism. This means that there exists a hyperbolic toral automorphism φ of $\mathbb{T}^n$ and a homeomorphism h of $\mathbb{T}^n$ such that $\psi = h \circ \varphi \circ h^{-1}$.

The linear map l_φ can be seen as an automorphism of the abelian Lie group $\mathbb{R}^n$, leaving $\mathbb{Z}^n$ invariant. Now, $\mathbb{Z}^n$ is a discrete and cocompact (meaning that the quotient $\mathbb{R}^n/\mathbb{Z}^n$ is compact) subgroup of $\mathbb{R}^n$. More generally, let G be any Lie group, and assume that $l_\varphi \in \mathrm{Aut}(G)$ is an automorphism of G, such that there exists a discrete and cocompact subgroup Γ of G, with $l_\varphi(\Gamma) = \Gamma$. Then the space of right cosets $\Gamma\backslash G$ is a closed manifold, and l_φ induces a diffeomorphism $\varphi : \Gamma\backslash G \to \Gamma\backslash G : \Gamma g \mapsto \Gamma l_\varphi(g)$.

If we want this diffeomorphism to be Anosov, l_φ must be hyperbolic. It is known that this can happen only when G is nilpotent. So we restrict ourselves to that case, where the resulting manifold $\Gamma\backslash G$ is said to be *a nilmanifold*. Such a diffeomorphism φ, induced by an automorphism l_φ, is called a *nilmanifold automorphism* and is said to be a hyperbolic nilmanifold automorphism, when l_φ is hyperbolic. Just as in the case of tori, any hyperbolic infra-nilmanifold automorphism is Anosov. Again there is more: by

Karel Dekimpe is professor of mathematics at the K.U.Leuven Kulak, Belgium. His email address is `Karel.Dekimpe@kuleuven-kortrijk.be`.

a result of A. Manning, we know that any Anosov diffeomorphism on a given nilmanifold $\Gamma\backslash G$ is topologically conjugate to a hyperbolic nilmanifold automorphism on that nilmanifold.

At this point, one might think that the algebraic approach of finding Anosov diffeomorphisms ends with the class of nilmanifolds. However, there are manifolds that are not nilmanifolds but are finitely covered by a nilmanifold and that also admit Anosov diffeomorphisms obtained in an algebraic way. These manifolds are called *infra-nilmanifolds*. To define such a manifold, one constructs the affine group $\mathrm{Aff}(G)$ of a nilpotent Lie group G as being the semidirect product $G \rtimes \mathrm{Aut}(G)$. An element of $\mathrm{Aff}(G)$ is of the form (g, α), with $g \in G$ its "translational part" and $\alpha \in \mathrm{Aut}(G)$ its "linear" part. The element (g, α) acts on $x \in G$ by $(g, \alpha) \cdot x = g\alpha(x)$. Note that, for $G = \mathbb{R}^n$, we recover the usual affine group $\mathrm{Aff}(\mathbb{R}^n)$.

Let Γ be a torsion-free subgroup of $\mathrm{Aff}(G)$ such that $\Lambda = \Gamma \cap G$ is a discrete and cocompact subgroup of G and the index $[\Gamma : \Lambda]$ is finite. The orbit spaces $\Gamma\backslash G$ obtained in this way are the infra-nilmanifolds, and each of them is finitely covered by a nilmanifold $\Lambda\backslash G$. The most famous example of an infra-nilmanifold is, without any doubt, the Klein bottle which has $\mathbb{T}^2$ as a double cover. To construct the Klein bottle, one needs the group $\Lambda = \mathbb{Z}^2$ of translations, which together with a glide reflection, generates the needed group Γ.

Now assume that, for a given infra-nilmanifold $\Gamma\backslash G$, there exists a $l_\varphi \in \mathrm{Aut}(G)$, with $l_\varphi \Gamma l_\varphi^{-1} = \Gamma$ (where the conjugation is computed inside the affine group $\mathrm{Aff}(G)$); then again l_φ induces a diffeomorphism $\varphi : \Gamma\backslash G \to \Gamma\backslash G$, now called an *infra-nilmanifold automorphism*. Note that, for $\Lambda \subseteq G$, one can see that $l_\varphi \Lambda l_\varphi^{-1} = l_\varphi(\Lambda)$, showing that this new situation is a generalization of the previous one. When l_φ is hyperbolic, the corresponding infra-nilmanifold automorphism is Anosov. For a long time it was claimed that any Anosov diffeomorphism on an infra-nilmanifold is topologically conjugate to such a hyperbolic infra-nilmanifold automorphism. (And even more, it is conjectured that only infra-nilmanifolds admit an Anosov diffeomorphism.) However, as pointed out in [2], there do exist Anosov diffeomorphisms on certain infra-nilmanifolds that are not of this type. The idea is to look for an element $\delta = (g, l_\varphi) \in \mathrm{Aff}(G)$, with $g \neq 1$ such that $\delta\Gamma\delta^{-1} = \Gamma$. This δ also induces a diffeomorphism $\psi : \Gamma\backslash G \to \Gamma\backslash G : \ \Gamma x \mapsto \Gamma g l_\varphi(x)$, which is Anosov when l_φ is hyperbolic. Let us call ψ a (hyperbolic) *affine automorphism* of the infra-nilmanifold. Some authors, including me, have used the term infra-nilmanifold automorphism to refer to this more general type of diffeomorphisms. It is an open question whether all Anosov diffeomorphisms on an infra-nilmanifold can be obtained in this way.

Up to now, we have discussed only diffeomorphisms. However, if $\Gamma\backslash G$ is an infra-nilmanifold and $l_\varphi \in \mathrm{Aut}(G)$ is such that $l_\varphi \Gamma l_\varphi^{-1} \subseteq \Gamma$, then l_φ still induces a map φ on $\Gamma\backslash G$. This map will no longer be a diffeomorphism, but a self-covering. So such a map φ is a self-map of the infra-nilmanifold $\Gamma\backslash G$ that lifts to a Lie group automorphism of G, and, moreover, any self-map of $\Gamma\backslash G$ that lifts to an automorphism of G is of this form. M. W. Hirsch was the first person to use the term *infra-nilmanifold endomorphism* for these maps. However, when using the term infra-nilmanifold endomorphism, most researchers refer to a paper of J. Franks ([3]). It is important to realize that, although Franks himself attributes the terminology of an infra-nilmanifold endomorphism to Hirsch, the definition used by J. Franks is not consistent with the one used by Hirsch, and, in fact, Franks's definition is even mathematically incorrect, causing several researchers to misinterpret this notion. However, it is obvious from the rest of Franks's paper that he also intended to use the term infra-nilmanifold endomorphism for a map φ lifting to (and hence induced by) an automorphism as above.

These more general infra-nilmanifold endomorphisms are also important in dynamical systems: e.g., when l_φ has only eigenvalues of modulus > 1, φ is an expanding map. As a corollary to his famous paper on groups of polynomial growth (1981), M. Gromov proved that any expanding map is topologically conjugate to an infra-nilmanifold endomorphism. Unfortunately, this corollary depended not only on Gromov's own main result but also on false results by other researchers, and recently a counterexample to this statement was constructed ([2]). However, again we can consider the slightly more general class of maps, obtained by taking $\delta \in \mathrm{Aff}(G)$, with $\delta\Gamma\delta \subseteq \Gamma$ and constructing the map ψ as above. We say that ψ is an *affine endomorphism* of the infra-nilmanifold. It turns out that any expanding map is topologically conjugate to such an expanding affine endomorphism.

We have seen that the class of infra-nilmanifold endomorphisms provides interesting examples of maps, but one should really study the more general class of affine endomorphisms of an infra-nilmanifold. More evidence for this claim follows from the fact that any self-map of an infra-nilmanifold is homotopic to a map induced by an affine map δ of the corresponding Lie group. So perhaps after more than forty years of lack of clarity about the notion of infra-nilmanifold endomorphisms, it is time to consider instead what we have referred to here as affine endomorphisms.

Further Reading

[1] Boris Hasselblatt, *Hyperbolic Dynamical Systems*, Handbook of dynamical systems, Vol. 1A, 239–319, North-Holland, Amsterdam, 2002.

[2] Karel Dekimpe, *What an infra-nilmanifold endomorphism really should be...*, preprint (18 pages); see `http://arxiv.org/abs/1008.4500`.

[3] John Franks, *Anosov Diffeomorphisms*, 1970 Global Analysis, Proc. Sympos. Pure Math., Vol. XIV, Berkeley, CA, 1968, pp. 61–93, Amer. Math. Soc., Providence, RI.