

Directional Regularity vs. Joint Regularity

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We start by referring to an experience many of us have had, namely, being diagnosed in a hospital by a computer tomograph. What is the idea behind the computer tomograph investigation? The machine is measuring linewise (directionally) the density of the material, and out of this information a global (joint) picture of the material is created. This kind of principle, “from the simple case to the complicated one”, is often used, also for studying various problems in mathematics, e.g., properties of functions in one and many variables. Recall various notions such as continuity, differentiability, harmonicity, real analyticity, holomorphy, etc., which may be defined for functions of one variable and also for functions of many variables. So the problem arises of whether the simpler one-dimensional property along “many” test directions implies the global property. Or, in our language, is joint regularity a consequence of the directional one? Exactly this kind of question will be studied in this article.

To be more precise, let $f : D \times G \rightarrow \mathbb{C}$ be a function. We say that f is *directionally (separately) regular* (e.g., directionally (separately) continuous) if for all pairs $(a, b) \in D \times G$ the functions $G \ni w \mapsto f(a, w)$ and $D \ni z \mapsto f(z, b)$ are regular (e.g., continuous) with respect to the other variable. Notice that directionally continuous

functions are sometimes called *linearly continuous*. In the same spirit one says that a function $f : D_1 \times \cdots \times D_n \rightarrow \mathbb{C}$ is *n-directionally regular* if for all points $(a_1, \dots, a_n) \in D_1 \times \cdots \times D_n$ and every j , $1 \leq j \leq n$, the function $D_j \ni z_j \mapsto f(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_n)$ is regular. Instead of speaking of directional regularity we sometimes will also use the term *separate regularity*. We point out that, when we speak of directional regularity, even in \mathbb{R}^n , then we always discuss regularity only in directions parallel to the coordinate axes.

In the case in which directional regularity implies joint regularity, one may even sharpen the question in the following way: what are subsets $A \subset D$, $B \subset G$ such that whenever $f(a, \cdot)$, $a \in A$, and $f(\cdot, b)$, $b \in B$, are regular, then f is jointly regular; i.e., we ask how thick the test sets A, B have to be so that joint regularity follows.

On the other hand, if the answer to the problem above is negative, then one should either study the shape of singularities with respect to the joint regularity, exhibit additional conditions under which the joint regularity follows, or find a weaker joint regularity that all such functions share.

So far we have discussed the general idea of the problems we are going to describe. Of course, we expect that the answer will heavily depend on the class of functions discussed, i.e., on the notion of “regular”. Therefore, phenomena appearing in different classes of regularity will be discussed in “almost independent” subsections. The reader is free to jump to those of particular interest.

Directional Continuity

In 1821, in his book *Cours d'analyse*, Augustin-Louis Cauchy claimed that if a function f of two real variables is directionally continuous, then it is jointly continuous. As every student nowadays knows, this statement is not correct. Nevertheless,

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it took time until people realized this mistake. Only in 1870, in the book *Abriss einer Theorie der complexen Functionen und der Thetafunctionen einer Veränderlichen* by J. Thomae, could one find a counterexample, which, according to Thomae, is due to E. Heine. The example is the following function defined on \mathbb{R}^2 :

$$f(x, y) := \begin{cases} \sin(4 \arctan \frac{x}{y}), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}.$$

If we approach the origin along the line $y = tx$ ($t \neq 0$), then we get for $x \neq 0$:

$$f(x, tx) = \sin(4 \arctan(1/t)).$$

From here it is obvious that f is not continuous at $(0, 0)$ as a function of two variables, although it is directionally continuous.

The example that is nowadays usually taught in classes is the following one due to G. Peano:

$$(*) \quad f(x, y) := \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } x = y = 0 \end{cases}.$$

Observe that f is even directionally real analytic but not continuous at the origin. In other words, the directional regularity of f is as good as possible, but nevertheless its weakest joint regularity fails to hold.

Using this example, it is an easy exercise to construct a directionally continuous function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ that is discontinuous exactly at all points with rational coordinates; in particular, the set $S_C(f)$ of all points at which f is not continuous is dense. Moreover, there exists a directionally continuous function $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ such that $S_C(f)$ is a full measure set (see G. Tolstov (1949)), i.e., the set of singularities, although small in the sense of topology, is large in the sense of measure theory.

The first general formulation of a weaker joint regularity (to be of Baire class 1) that all directionally continuous functions share was found by R. Baire (1899) in his thesis *Sur les fonctions de variables réelles*. Using modern notation we say that a function $f : A \rightarrow \mathbb{R}$ ($A \subset \mathbb{R}^n$) is said to be of *first Baire class* (or *Baire class 1*) if there is a sequence of continuous functions $f_j : A \rightarrow \mathbb{R}$ with $f_j \rightarrow f$ pointwise. Then the result by Baire reads as follows:

Theorem (Baire 1899). *Any directionally continuous function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is of first Baire class and $S_C(f)$ is of first Baire category. In particular, each such function is Borel measurable.*

Functions for which S_C is of first Baire category are also known as *pointwise discontinuous functions*.

To comment on the last implication in the former result we quote from W. Rudin's paper "Lebesgue's first theorem" (1981): "Several years ago I used to pose this question (i.e., is a separately

continuous function on $\mathbb{R} \times \mathbb{R}$ Borel measurable?) to randomly selected analysts. The typical answer was something like this: 'Hmm—well—probably not—why should it be?' The only group that did a little better were the probabilists. And there was just one person who said: 'Let's see, yes, it is—and it is of Baire class 1—and...'. He knew."

So we propose this question for any math exam.

Moreover, Baire proved the following stronger result.

Theorem (Baire 1899). *If $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is directionally continuous, then there exist sets of first Baire category $A, B \subset \mathbb{R}$ such that $S_C(f) \subset A \times B$.*

This result may give a first idea of how small $S(f)$ should be.

These results have been generalized by Baire in a weak sense to the case of three-directional continuity; in fact, he showed that *any three-directionally continuous function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is jointly continuous on dense subsets of all two-dimensional planes parallel to the coordinate axes.*

In arbitrary dimensions it was proved by H. Lebesgue (1905) in "Sur les fonctions représentable analytiquement" that any n -directionally continuous function $f : \mathbb{R} \times \cdots \times \mathbb{R} \rightarrow \mathbb{R}$ is a *Baire function of class $n - 1$* (i.e., f is a pointwise limit of functions of Baire class $n - 2$; continuous functions are functions of Baire class 0). An example in his paper shows that this result is even sharp. Then, in 1919, H. Hahn generalized the three-dimensional result of Baire to n -directionally continuous functions. He proved that, for any n -directionally continuous function and any $(n - 1)$ -dimensional plane H parallel to the coordinate axes, there exists a dense set H' of points of H such that $H' \cap S_C(f) = \emptyset$. Much later, namely only in 1943, R. Kershner succeeded in giving a general version of the Baire theorem for n -directionally continuous functions. In fact, his result is the following one:

Theorem (Kershner 1943). *Let f be n -directionally continuous on the unit cube $I \times \cdots \times I$.*

(a) *Then $S_C(f)$ is an \mathcal{F}_σ -set, i.e., a set which is a countable union of closed sets, and all projections of $S_C(f)$ on all coordinate hyperplanes, i.e., $x_j = 0$, are sets of first Baire category.*

(b) *For any subset S of the unit cube in \mathbb{R}^n which satisfies the above assumptions, there is an n -directionally continuous function f on the cube with $S_C(f) = S$.*

Let us add a somewhat strange result by K. Bögel (1926) for the two-dimensional case. Let f depend on the variables x and y . Assume that f is continuous in the direction of x and differentiable in the direction of y , in particular, continuous. Then f is jointly continuous except on a nowhere dense set. Observe that a set of the first category may be strictly larger than a nowhere dense set, so this set

of points of discontinuity is really very small. Similar results are also found in the paper by Kershner. The last remark is due to R. L. Kruse and J. J. Deely (1969): if f is n -directionally continuous on \mathbb{R}^n and monotone with respect to the variable x_j for all fixed $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, $j = 1, \dots, n-1$, then f is jointly continuous everywhere.

But this is not the end of the story. Later, questions from above were discussed in more general settings. The component spaces (initially \mathbb{R}) were substituted by more general topological spaces. This kind of discussion is nowadays connected with the so-called *Namioka spaces*. We will not go into details here.

In any case, we are wondering why we have never learned about these results in a standard analysis course. We therefore suggest adding this part of interesting mathematics to such courses or, at least, discussing it in seminars following the introductory course.

We would like to point out that we learned most of the above results from the paper "The genesis of separate versus joint continuity" by Z. Piotrowski (1996).

Partial Differentiability

Recall that a partially differentiable function f in \mathbb{R}^2 need not even be continuous (see (*)). There are also partially differentiable functions that are continuous but that are not totally differentiable. For example take $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given as follows

$$(**) \quad f(x, y) := \begin{cases} \frac{x^2 y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } x = y = 0 \end{cases}.$$

The reason for this phenomenon lies in the fact that the graph of a jointly differentiable function has to be almost equal (at least locally) with its tangent hyperplane. Of course, this fact is much stronger than to approximate the directional graph by simple lines. In the case in which the partial derivatives are assumed to be continuous at some point, joint differentiability at that point follows (as we are taught in standard analysis courses).

It was already known to Baire that a partially differentiable function in \mathbb{R}^2 is differentiable at all points of a dense subset. This result has been sharpened by E. B. van Vleck (1907) in the sense that he only assumed the existence of the partial derivative with respect to the first variable together with the continuity with respect to the second variable.

Moreover, a result due to K. Bögel (1926) shows that any partially differentiable function f in two variables such that $\frac{\partial f}{\partial x}$ is continuous with respect to y and $\frac{\partial f}{\partial y}$ is continuous with respect to x is jointly differentiable except on a set of first category.

This may explain why all examples we present during our analysis course look similar. In each of

them, the set of points at which f is not totally differentiable is in some sense very small.

Nevertheless, for any $\varepsilon \in (0, 1)$ there exists a function $f_\varepsilon: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ (due to G. Tolstov (1949)) that has all partial derivatives at all points of the square $[0, 1] \times [0, 1]$ but for which the measure of $S_C(f_\varepsilon)$ is larger than ε . In particular, the set in which f_ε is not jointly differentiable is a set of positive measure.

Now let $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a partially differentiable function which, in addition, has locally bounded partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Then f is locally Lipschitz and therefore, using a result of H. Rademacher (1919), almost everywhere jointly differentiable. Nevertheless, (**) shows that there may exist points at which such a function is not differentiable.

Finally, let us also mention a result of J. Boman (1967), who discusses the situation in which the regularity information is known for a lot of test curves, not only along lines parallel to the axes.

Theorem (Boman 1967). (a) Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and assume that $f \circ u \in C^p(\mathbb{R}^n)$ for all $u \in C^\infty(\mathbb{R}, \mathbb{R}^n)$, where $p \geq 1$. Then $f \in C^{p-1,1}(\mathbb{R}^n)$, i.e., $f \in C^{p-1}(\mathbb{R}^n)$ and all partial derivatives of f of order $p-1$ are locally Lipschitz.

(b) There is an f as in (a) such that $f \notin C^p(\mathbb{R}^n)$.

We note that we have mentioned only a special case of Boman's result.

This discussion shows that, in comparison with directional continuity, the situation does not change dramatically. To obtain better results one has to discuss big varieties of test curves. Nevertheless, one gets only a weaker joint regularity.

Directionally Lipschitz

It is a simple observation that every uniformly partially Lipschitz function is jointly Lipschitz. We will discuss certain analogues (involving also derivatives). For $0 < \alpha \leq 1$ let $\Lambda_\alpha(\mathbb{R}^k)$ denote the space of all functions $f: \mathbb{R}^k \rightarrow \mathbb{R}$ that satisfy the Hölder condition with exponent α . If $\alpha > 1$, then

$$\Lambda_\alpha(\mathbb{R}^k) := \left\{ f \in C^1(\mathbb{R}^k) : \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k} \in \Lambda_{\alpha-1}(\mathbb{R}^k) \right\}$$

is the α -order Lipschitz space. S. Bernstein (1912) proved a first version of the following general result.

Theorem. Let $\alpha > 0$ and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be partially of class $\Lambda_\alpha(\mathbb{R})$ with

$$(\dagger) \quad \sup \{ \|f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n)\|_{\Lambda_\alpha(\mathbb{R})} : x \in \mathbb{R}^n, j = 1, \dots, n \} < +\infty.$$

Then $f \in \Lambda_\alpha(\mathbb{R}^n)$.

Of course, the main point of the theorem is the case $\alpha > 1$. Note that (*) shows that the conclusion may be false without (†).

Two different proofs are presented by S. G. Krantz (1983).

As an immediate consequence we get: if a function $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ admits $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ and both “pure” derivatives satisfy the uniform Lipschitz condition, then the function is of class C^2 . The question arose of what might happen if the pure second derivatives are only assumed to be continuous. An example (based on discussion of special logarithmic potentials) was given by B. S. Mitjagin (1959). Below we present a simpler explicit one due to V. I. Judovič (which may be found in the book of O. V. Besov, V. P. Il’in, S. M. Nikol’skiĭ (1978)).

Let $\mathbb{B}_2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$,

$$f(x, y) := \begin{cases} xy \log(-\log(x^2 + y^2)), & \text{if } (x, y) \in \mathbb{B}_2 \setminus \{(0, 0)\} \\ 0, & \text{if } x = y = 0 \end{cases}.$$

Then $f \in C^1(\mathbb{B}_2) \cap C^\infty(\mathbb{B}_2 \setminus \{(0, 0)\})$, f is partially C^2 , and the partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ are continuous on \mathbb{B}_2 , but $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ does not exist, and

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\partial^2 f}{\partial x \partial y}(x, y) = +\infty.$$

The above theorem may be extended to the case of Sobolev spaces. A first attempt was made also by S. Bernstein (see *Collected Works*, vol. I, 1952, pp. 96–98 (in Russian)). Let $W^s(\mathbb{R}^k)$ be the space of all functions $f: \mathbb{R}^k \rightarrow \mathbb{R}$ whose derivatives of order $\leq s$ are in $L^2(\mathbb{R}^k)$.

Theorem. Let $s > 0$ and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be partially of class $W^s(\mathbb{R})$ with

$$\sup\{\|f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n)\|_{W^s(\mathbb{R})} : x \in \mathbb{R}^n, j = 1, \dots, n\} < +\infty.$$

Then $f \in W^s(\mathbb{R}^n)$.

Applying this result, it turns out that the mixed second-order derivatives of Judovič’s example, although they have a singularity at the origin, are locally square integrable.

Notice that the assumption of uniform boundedness forces the function under discussion to behave in a similar way along nearby test directions. This explains the joint regularity in the above results.

Looking at the above three subsections, one might get the impression that, working from the point of view of real analysis, directional regularity (without additional assumptions) never implies joint regularity. But this is not the whole truth, as we will see now.

Directional Holomorphy

For an open set $\Omega \subset \mathbb{C}^k$, let $\mathcal{O}(\Omega)$ be the space of all functions holomorphic on Ω .

Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be domains. We say that a function $f: D \times G \rightarrow \mathbb{C}$ is *directionally (separately) holomorphic* (and we write $f \in \mathcal{O}_s(D \times G)$), if $f(a, \cdot) \in \mathcal{O}(G)$ for each $a \in D$ and $f(\cdot, b) \in \mathcal{O}(D)$ for each $b \in G$.

Obviously, $\mathcal{O}(D \times G) \subset \mathcal{O}_s(D \times G)$. The problem is whether equality holds, i.e., whether every directionally holomorphic function is jointly holomorphic. For $f \in \mathcal{O}_s(D \times G)$, let $S_\mathcal{O}(f)$ denote the set of all points $(a, b) \in D \times G$ such that f is not jointly holomorphic in any neighborhood of (a, b) .

At the end of the nineteenth century, using Cauchy integral representation, it was well known that every continuous directionally holomorphic function is jointly holomorphic, i.e., $\mathcal{O}_s(D \times G) \cap C(D \times G) = \mathcal{O}(D \times G)$. Next, using classical methods from that time, W. F. Osgood (1899, 1900) proved that if $f \in \mathcal{O}_s(D \times G)$ is locally bounded, then f is continuous and, consequently, $\mathcal{O}_s(D \times G) \cap L^\infty_{\text{loc}}(D \times G) = \mathcal{O}(D \times G)$. He also proved that for every $f \in \mathcal{O}_s(D \times G)$, the set $S_\mathcal{O}(f)$ is nowhere dense. Moreover, he made an observation that in order to prove that $\mathcal{O}_s(D \times G) = \mathcal{O}(D \times G)$ (for arbitrary p, q, D, G) we only need to check (for $p = 1$) the following lemma.

Lemma (Hartogs lemma (1906)). Let $\mathbb{C}^p \times \mathbb{C}^q \supset \mathbb{B}(r) \times \mathbb{B}(s) \xrightarrow{f} \mathbb{C}$ ($\mathbb{B}(a, r)$ stands for the Euclidean ball centered at a with radius r , $\mathbb{B}(r) := \mathbb{B}(0, r)$) be such that $f(a, \cdot) \in \mathcal{O}(\mathbb{B}(s))$ for every $a \in \mathbb{B}(r)$ and $f \in \mathcal{O}(\mathbb{B}(r) \times \mathbb{B}(\delta))$ for some $0 < \delta < s$. Then $f \in \mathcal{O}(\mathbb{B}(r) \times \mathbb{B}(s))$.

In his proof Hartogs used for the first time methods from potential theory in complex analysis. But finally it turned out (see K. Koseki (1966)) that this lemma can also be verified with a pure complex analysis argument. Hartogs also observed that the lemma is not true without the assumption that $f \in \mathcal{O}(\mathbb{B}(r) \times \mathbb{B}(\delta))$ for some $0 < \delta < s$. Thus we have the following fundamental result.

Theorem (Hartogs theorem (1906)). $\mathcal{O}_s(D \times G) = \mathcal{O}(D \times G)$ (for arbitrary p, q, D, G).

A more general question is to allow nonlinear fibers—a first step in this direction was done by G. M. Chirka (2006 = 1906 + 100).

The Hartogs lemma suggests the following problem, called the *Hukuhara problem*. We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a nonempty set $B \subset G$, and a function $f: D \times G \rightarrow \mathbb{C}$ that is *directionally holomorphic* in the following sense: $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in D$, $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$. We ask whether $f \in \mathcal{O}(D \times G)$.

In the situation above we write $f \in \mathcal{O}_s(X)$ with $X := (D \times G) \cup (D \times B)$. Notice that from the point of view of set theory, the set X is nothing other than

the Cartesian product $D \times G$, which is, of course, independent of B . Writing $\mathbf{X} = (D \times G) \cup (D \times B)$ we point out the role played by the test set B .

Observe that the answer must be negative if B is too “thin”. For example, if $B := g^{-1}(0)$, where $g \in \mathcal{O}(G)$, $g \not\equiv 0$, then for arbitrary function $\varphi : D \rightarrow \mathbb{C}$, the function $f(z, w) := \varphi(z)g(w)$, $(z, w) \in D \times G$, belongs to $\mathcal{O}_s(\mathbf{X})$.

M. Hukuhara (1942) proved an analogue of Osgood’s result (with less “horizontal” test directions) showing that if B is an identity set at a point $b_0 \in G$ (i.e., for any open connected neighborhood U of b_0 and $f \in \mathcal{O}(U)$, if $f = 0$ on $B \cap U$, then $f \equiv 0$), then every locally bounded function $f \in \mathcal{O}_s(\mathbf{X})$ is holomorphic on $D \times G$, i.e., $\mathcal{O}_s(\mathbf{X}) \cap L_{\text{loc}}^\infty(D \times G) = \mathcal{O}(D \times G)$.

It was T. Terada who finally answered the question raised by Hukuhara, applying results of pluripotential theory—a new tool at that time.

Theorem (Terada 1967, 1972). *If B is not pluripolar (i.e., B is not “thin” from the point of view of the pluripotential theory), then $\mathcal{O}_s(\mathbf{X}) = \mathcal{O}(D \times G)$. Moreover, if D is bounded and B is a pluripolar set of type \mathcal{F}_σ , then $\mathcal{O}_s(\mathbf{X}) \subsetneq \mathcal{O}(D \times G)$.*

Just a few words to get an intuitive meaning of the notion “pluripolar”. Recall that a convex function on \mathbb{R} may be thought of as a sublinear function, i.e., whenever this function is majorized on the boundary of any subinterval by a linear function, then the same remains true inside of the interval. Note that linear functions are nothing other than the solutions of the simple differential equation $u'' = 0$. Subharmonic functions in the complex plane may be understood in the analogous sense substituting $u'' = 0$ by $\Delta u = 0$, Δ is the Laplace operator. And finally, plurisubharmonic functions are essentially functions that are subharmonic in all complex directions. By $\mathcal{PSH}(\Omega)$ we denote all plurisubharmonic functions on an open set $\Omega \subset \mathbb{C}^n$. A set $B \subset \mathbb{C}^n$ is called *pluripolar* if there is a $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $B \subset u^{-1}(-\infty)$.

Summarizing: The reader should be aware of the essential difference between real and complex differentiation—holomorphic functions are much less flexible than differentiable ones. This is so because of the identity principle, even along small (nonpluripolar) sets.

Directionally Polynomial Functions

To understand better the general problem of directional regularity, we first consider the case of polynomials. For $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), let $\mathcal{P}(\mathbb{K}^k)$ be the space of all complex polynomials of k real (resp. complex) variables.

Let $\emptyset \neq A \subset \mathbb{C}^p$, $\emptyset \neq B \subset \mathbb{C}^q$, $\mathbf{X} := (A \times \mathbb{C}^q) \cup (\mathbb{C}^p \times B)$. We say that a function $f : \mathbf{X} \rightarrow \mathbb{C}$ is *directionally polynomial on \mathbf{X}* , if $f(a, \cdot) \in \mathcal{P}(\mathbb{C}^q)$ for each $a \in A$ and $f(\cdot, b) \in \mathcal{P}(\mathbb{C}^p)$ for each $b \in B$. Let

$\mathcal{P}_s(\mathbf{X})$ be the space of all functions directionally polynomial on \mathbf{X} . Clearly, $\mathcal{P}(\mathbb{C}^p \times \mathbb{C}^q)|_{\mathbf{X}} \subset \mathcal{P}_s(\mathbf{X})$. The problem is to characterize those pairs of test sets (A, B) for which $\mathcal{P}_s(\mathbf{X}) = \mathcal{P}(\mathbb{C}^p \times \mathbb{C}^q)|_{\mathbf{X}}$, i.e., every function that is directionally polynomial on \mathbf{X} extends to a polynomial of $n := p + q$ complex variables.

A set $C \subset \mathbb{C}^k$ is a *determining set* (for polynomials) if for every $f \in \mathcal{P}(\mathbb{C}^k)$ with $f = 0$ on C vanishes identically. We say that a set $C \subset \mathbb{C}^k$ is a *strongly determining set* (for polynomials) if for every representation $C = \bigcup_{s=1}^\infty C_s$ with $C_s \subset C_{s+1}$, there exists an s_0 such that C_{s_0} is a determining set. Observe that if $C \subset \mathbb{C}^k$ is nonpluripolar, then C is strongly determining. A set $C \subset \mathbb{C}$ is strongly determining iff C is uncountable. The set $C := \{1/k : k \in \mathbb{N}\} \subset \mathbb{C}$ is determining, but not strongly determining.

With this notion at hand we have the following description of directionally polynomially functions.

Theorem (Siciak 1995). *The following conditions are equivalent:*

(i) *A and B are determining, and at least one of them is strongly determining;*

(ii) *for every $f \in \mathcal{P}_s(\mathbf{X})$ there exists exactly one $\hat{f} \in \mathcal{P}(\mathbb{C}^p \times \mathbb{C}^q)$ such that $\hat{f} = f$ on \mathbf{X} .*

We mention that there is a similar result for real-valued functions of two real variables due to Z. Sasvári (1992). As a consequence of the Siciak result (or the one of Sasvári in the case of \mathbb{R}^2) we get the following result.

Corollary. *Let $f : \mathbb{K}^p \times \mathbb{K}^q \rightarrow \mathbb{C}$ be directionally polynomial, i.e., $f(a, \cdot) \in \mathcal{P}(\mathbb{K}^q)$ for each $a \in \mathbb{K}^p$ and $f(\cdot, b) \in \mathcal{P}(\mathbb{K}^p)$ for each $b \in \mathbb{K}^q$. Then $f \in \mathcal{P}(\mathbb{K}^p \times \mathbb{K}^q)$.*

Finally, we quote a result due to R. S. Palais discussing a similar question over an arbitrary field K .

Theorem (Palais 1978). *If K is a field, then a necessary and sufficient condition for every directionally polynomial function $f : K \times K \rightarrow K$ to be a polynomial function is that K is either finite or uncountable.*

To summarize: to decide whether a function in many variables is a polynomial (in these variables), it suffices to prove its directionally polynomial behavior. Hence in this simple class of functions the directional regularity and the joint regularity coincide. Even more is true. If a function f defined on some cross-subset $\mathbf{X} \subset \mathbb{C}^n$ is directionally polynomial in many directions, then f is the restriction of a uniquely defined joint polynomial.

Directional Holomorphy—Crosses

Now we discuss the situation in which we only have few “horizontal” and “vertical” test directions. We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ and two nonempty sets $A \subset D$, $B \subset G$. Define the cross $\mathbf{X} := (A \times G) \cup (D \times B)$. A function $f : \mathbf{X} \rightarrow \mathbb{C}$ is said to be *directionally (separately) holomorphic on \mathbf{X}* ($f \in \mathcal{O}_s(\mathbf{X})$), if $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$ and $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$. We ask whether there exists an open neighborhood $\widehat{\mathbf{X}} \subset D \times G$ of \mathbf{X} such that every function $f \in \mathcal{O}_s(\mathbf{X})$ extends holomorphically to $\widehat{\mathbf{X}}$.

Observe that the Hukuhara problem was just the case in which $A = D$ and $\widehat{\mathbf{X}} = D \times G$. Notice once again that different crosses may have the same geometric image. In view of the Hartogs theorem, if A and B are open, then $\mathcal{O}_s(\mathbf{X}) = \mathcal{O}((A \times G) \cup (D \times B))$.

To get an intuition of the situation we are discussing, have a look at Figure 1.

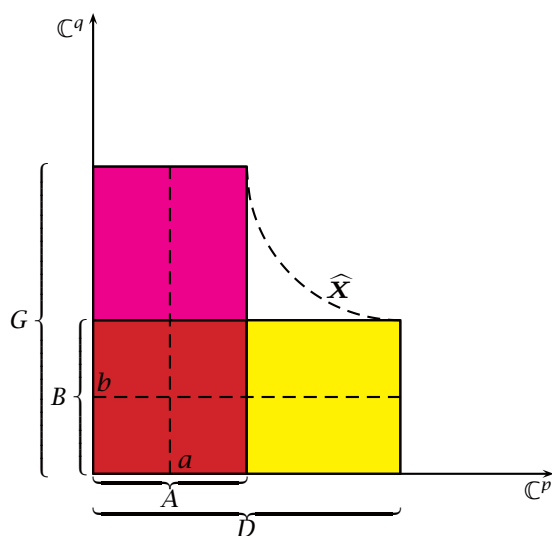


Figure 1.

Investigations of this kind of question began with S. Bernstein (1912) and have been continued in papers by J. Siciak (1969), N. I. Akhiezer and L. I. Ronkin (1973), V. P. Zahariuta (1976), J. Siciak (1981), B. Shiffman (1989), Nguyen Thanh Van and A. Zeriahi (1991, 1995), Nguyen Thanh Van (1997), and O. Alehyane and A. Zeriahi (2001). The problem has been completely solved. The breakthrough method, the so-called double basis method, was due to V. P. Zahariuta (1976). Recently, a new method of proof was found by Việt-Ahn Nguyễn that enables one to formulate the theorem even for arbitrary complex manifolds.

Theorem (Cross theorem (1912–2001)). Assume that A and B are locally pluriregular, i.e., A (respectively, B) is “thick” from the point of view of

the pluripotential theory at every point $a \in A$ (respectively, $b \in B$). Then for every $f \in \mathcal{O}_s(\mathbf{X})$, there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}})$ such that $\widehat{f} = f$ on \mathbf{X} and $\sup_{\widehat{\mathbf{X}}} |\widehat{f}| = \sup_{\mathbf{X}} |f|$, where

$$\widehat{\mathbf{X}} := \{(z, w) \in D \times G : h_{A,D}^*(z) + h_{B,G}^*(w) < 1\}$$

and $h_{C,\Omega}^*$ is the upper regularization of the relative extremal function

$$h_{C,\Omega} := \sup\{u \in \mathcal{PSH}(\Omega) : u \leq 1, u|_C \leq 0\}.$$

The result may reflect how holomorphic structures are rigid—a holomorphic function is in fact completely determined by its values along very small sets of test directions. Notice that in many cases the envelope $\widehat{\mathbf{X}}$ may be effectively calculated. To get a feeling of the power of this result, we add a simple example:

Let $A = B = \partial\mathbb{D}$ (here \mathbb{D} stands for the open unit disc in the complex plane) and $D = G = \mathbb{C}$. If $f \in \mathcal{O}_s((\partial\mathbb{D} \times \mathbb{C}) \cup (\mathbb{C} \times \partial\mathbb{D}))$, then f is automatically the restriction of an entire function $\tilde{f} \in \mathcal{O}(\mathbb{C}^2)$.

We point out that, in contrast to real analysis, holomorphic functions are already globally determined by their values along small sets such as \mathbf{X} .

Directional Real Analyticity

For an open set $\Omega \subset \mathbb{R}^k$, let $C^\omega(\Omega)$ be the space of all complex-valued real analytic functions on Ω .

Let $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ be domains. A function $f : D \times G \rightarrow \mathbb{C}$ is said to be *directionally real analytic* ($f \in C_s^\omega(D \times G)$) if $f(a, \cdot) \in C^\omega(G)$ for each $a \in D$ and $f(\cdot, b) \in C^\omega(D)$ for each $b \in G$.

Obviously $C^\omega(D \times G) \subset C_s^\omega(D \times G)$. Observe that the function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x, y) := \begin{cases} xye^{-\frac{1}{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & x = y = 0 \end{cases}$$

is of the class $C_s^\omega(\mathbb{R} \times \mathbb{R}) \cap C^\infty(\mathbb{R} \times \mathbb{R})$ but is not analytic near $(0, 0)$.

For $f \in C_s^\omega(D \times G)$, let $S_{C^\omega}(f)$ be the set of all $(a, b) \in D \times G$ such that f is not jointly real analytic in any neighborhood of (a, b) .

The central question is to characterize $C^\omega(D \times G) \setminus C_s^\omega(D \times G)$ and $S_{C^\omega}(f)$ for $f \in C_s^\omega(D \times G)$. The cross theorem may be applied to prove the following sufficient criterion for a directionally real analytic function to be jointly real analytic.

Theorem (Browder 1961, Lelong 1961). The space $C^\omega(D \times G)$ consists of all $f \in C_s^\omega(D \times G)$ such that for every $(a, b) \in D \times G$ there exists an open neighborhood $U \subset \mathbb{C}^p$ (resp. $V \subset \mathbb{C}^q$) of a (resp. b) with $U \cap \mathbb{R}^p \subset D$ (resp. $V \cap \mathbb{R}^q \subset G$) such that for every $x \in U \cap \mathbb{R}^p$ (resp. $y \in V \cap \mathbb{R}^q$) the function $f(x, \cdot)$ (resp. $f(\cdot, y)$) extends holomorphically to V (resp. U).

Moreover, results due to J.-S. Raymond, J. Siciak, and Z. Błocki describe precisely the singular sets of directionally real analytic functions.

Theorem (S. Raymond 1989, 1990, Siciak 1990, Błocki 1992). *If $f \in C_s^\omega(D \times G)$, then the projections $\text{pr}_{\mathbb{R}^p}(S_{C^\omega}(f))$, $\text{pr}_{\mathbb{R}^q}(S_{C^\omega}(f))$ are pluripolar as subsets of \mathbb{C}^p and \mathbb{C}^q , respectively.*

Conversely, for every relatively closed set $S \subset D \times G$ for which the projections $\text{pr}_{\mathbb{R}^p}(S)$ and $\text{pr}_{\mathbb{R}^q}(S)$ are pluripolar, there exists an $f \in C_s^\omega(D \times G)$ such that $S = S_{C^\omega}(f)$.

Thus singular sets of directional real analytic functions are completely characterized.

Although real analytic functions seem to be very similar to holomorphic ones, they are restrictions of holomorphic functions; the above discussion shows that their basic properties are different.

Directional Harmonicity

For an open set $\Omega \subset \mathbb{R}^k$, let $\mathcal{H}(\Omega)$ be the space of all harmonic functions on Ω .

Let $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ be domains. A function $h : D \times G \rightarrow \mathbb{R}$ is said to be *directionally harmonic* ($h \in \mathcal{H}_s(D \times G)$), if $h(a, \cdot) \in \mathcal{H}(G)$ for each $a \in D$ and $h(\cdot, b) \in \mathcal{H}(D)$ for each $b \in G$.

It is clear that $\mathcal{H}_s(D \times G) \cap C^2(D \times G) \subset \mathcal{H}(D \times G)$. In the context of the previous theorem, it is surprising that we have the following result (which also may be obtained as a consequence of the cross theorem).

Theorem (Browder 1961, Lelong 1961). $\mathcal{H}_s(D \times G) \subsetneq \mathcal{H}(D \times G)$.

The Extension Theorem with Singularities

So far our directionally holomorphic functions $f : X \rightarrow \mathbb{C}$ had no singularities on X . The fundamental paper by E. M. Chirka and A. Sadullaev (1988) and applications to mathematical tomography (see O. Öktem (1998, 1999)) show that the following is important.

Let $A \subset D \subset \mathbb{C}^p$, $B \subset G \subset \mathbb{C}^q$ be as before, let $M \subset X := (A \times G) \cup (D \times B)$ be fiberwise closed, and let

$$M_{(a,\cdot)} := \{w \in G : (a, w) \notin M\}, \quad a \in A,$$

$$M_{(\cdot,b)} := \{z \in D : (z, b) \notin M\}, \quad b \in B.$$

A function $f : X \setminus M \rightarrow \mathbb{C}$ is said to be *directionally (separately) holomorphic* ($f \in \mathcal{O}_s(X \setminus M)$) if $f(a, \cdot)$ is holomorphic in $G \setminus M_{(a,\cdot)}$ for every $a \in A$, $f(\cdot, b)$ is holomorphic in $D \setminus M_{(\cdot,b)}$ for every $b \in B$. We ask whether there exists a relatively closed set $\widehat{M} \subset \widehat{X}$ such that every function $f \in \mathcal{O}_s(X \setminus M)$ extends holomorphically to $\widehat{X} \setminus \widehat{M}$ (see Figure 2).

Observe that the case in which $M = \emptyset$ reduces to the one we discussed before. A positive solution has been found in a series of papers by O. Öktem (1998, 1999), J. Siciak (2001), and M. Jarnicki and P. Pflug (2001–2008). It turns out that if M

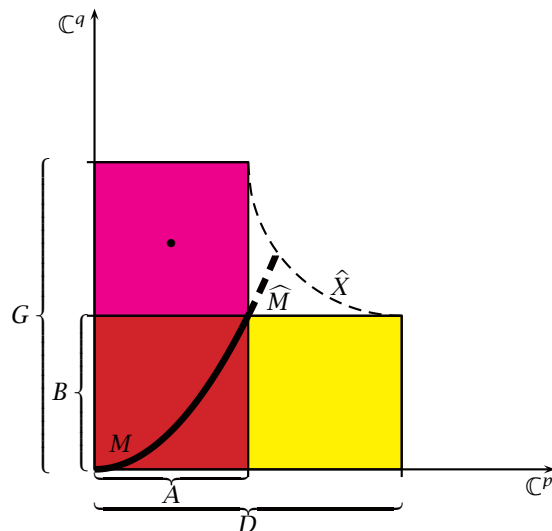


Figure 2.

is fiberwise pluripolar (resp. analytic), then \widehat{M} is pluripolar (resp. analytic). So we meet again the well-known phenomenon in complex analysis that the type of the singularity does not change under holomorphic extensions. For example, take $D = G = \mathbb{C}$, $A = B = \mathbb{R}$, $M := \{(x, x) : x \in \mathbb{R}\}$. Then $\widehat{X} = \mathbb{C}^2$ and $\widehat{M} = \{(z, z) : z \in \mathbb{C}\}$, which is the set of singularities of the function $f(z_1, z_2) := 1/(z_1 - z_2)$. In particular, $\widehat{X} \setminus \widehat{M}$ is the maximal extension domain for $\mathcal{O}_s(X \setminus M)$.

We conclude with a simple example due to T. Barth (1975). Let $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be given by

$$f(z_1, z_2) := \begin{cases} \frac{(z_1 + z_2)^2}{z_1 - z_2}, & \text{if } z_1 \neq z_2 \\ \infty, & \text{if } z_1 = z_2 \neq 0 \\ 0, & \text{if } z_1 = z_2 = 0. \end{cases}$$

Observe that now f is a directionally holomorphic map, but, nevertheless, it is not continuous at the origin. This example shows that our story is definitely not finished for holomorphic mappings.

Final Remarks

The results presented here were obtained during the last 100 years. They clearly show that real analysis is very flexible and, therefore, that the possibility to derive something from directional information is very rare. Simultaneously, this leads to a lot of interesting questions to be solved and understood. On the other hand, complex analysis is, in fact, governed by the identity theorem, i.e., it is strongly rigid. That is the reason why Hartogs's result is true. Here the main point is to extend objects as far as possible. Of course, both disciplines have their beauty and challenging problems. Although the rigidity of holomorphic functions is fascinating for us as mathematicians working in several complex variables, there are, of course, other points of view.

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