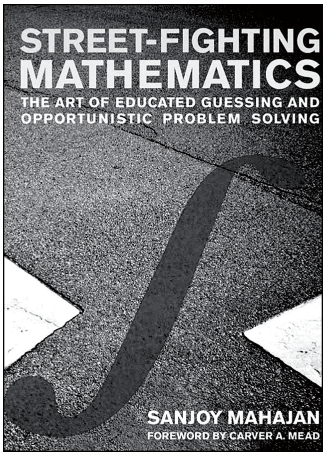
Book Review



Street-Fighting Mathematics

Reviewed by Brent Deschamp

Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving
Sanjoy Mahajan
The MIT Press, 2010
US$25.00, 134 pages
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I have read *Street-Fighting Mathematics* twice and most of the sections five or six times. I find myself working, and struggling with, problems from this book as much as I did with homework problems when I was a student. It is not that the material in this book is difficult; in fact the aim of the book is to provide simple tools for approximating solutions to complicated problems. The difficulty lies in the fact that the ideas this book presents are at times completely foreign to my way of thinking. Learning to see problems the way Mahajan sees them takes deep thought, time, and practice, but that is what makes *Street-Fighting Mathematics* an enjoyable read that provides an enlightening look at solving problems.

At only 134 pages in length, the book is small and covers only six major topics, but those topics have kept me busy for months. They are dimensional analysis, easy cases, lumping, pictorial proofs, taking out the big part, and reasoning by analogy. Along the way, Mahajan, a physicist by training, solves problems involving everyday calculations, geometry, calculus, differential equations, topology, and physics. He even finds a solution for the Navier-Stokes equations involving falling cones using nothing more than dimensional analysis and easy cases. The list of topics is short, but those topics are powerful. To illustrate some of these ideas, I present a few examples.

Consider the integral

$$\int \sqrt{1-\alpha x^2}\, dx.$$

The most common method for solving this integral would be to use a trigonometric substitution such as $x = (1/\sqrt{\alpha})\cos\theta$. Instead, Mahajan shows how dimensional analysis can be used to see how the parameter α influences the solution. In order to simplify notation, assign the dimension of length, L, to x. Since everything under the radical must have the same dimension and since 1 is dimensionless, it must be that α has dimension L^{-2}.

We now need to determine the dimension of the entire integral. The integrand is dimensionless, and the differential dx, which represents a small quantity of x, will have dimension L. Finally, the integral symbol represents both a limit and a sum, which are both dimensionless, and thus summing terms of dimension L yields a quantity with dimension L. Also, the integral will produce a function of α, $f(\alpha)$, and the only way for $f(\alpha)$ to have the dimension L is if $f(\alpha) \sim \alpha^{-1/2}$.

Using the trigonometric substitution mentioned earlier and a lot more work, we find the exact solution to the integral is

$$-\frac{1}{2\sqrt{\alpha}}\left[\cos^{-1}(x\sqrt{\alpha}) - (x\sqrt{\alpha})\sqrt{1-\alpha x^2}\right] + C.$$

Note that both terms inside the brackets are dimensionless and that the leading coefficient combines $\alpha^{-1/2}$ with a dimensionless constant. If learning how α influences the solution is all that is needed, then using dimensional analysis provides the same result with considerably less effort.

Another example involves "taking out the big part". Students often complain about having to remember the shortcuts for derivatives. Does the de-

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rivative of b^x involve $\ln b$ or its reciprocal? Which inverse trig functions have square roots in their derivatives? The issue is that the students remember the pieces of the answer but not how those pieces go together. Mahajan calls answers of this type "high-entropy" expressions. A solution that uses a "low-entropy" expression is one that has a few pieces that go together in a memorable way so there is little confusion when the method is used long after the exact process has been forgotten.

Consider the product 2.08×5.25. To estimate the product we could simply take out the big part and consider $2.08 \times 5.25 \approx 2 \times 5 = 10$, but what if we wanted a better approximation? We could use a correction factor to improve our estimate. Our first attempt is to try

$$(x + \Delta x)(y + \Delta y) = xy + x\Delta y + y\Delta x + \Delta x\Delta y,$$

but remembering which Δ term is multiplied with which whole term is not obvious until we return to the formula and complete the multiplication. This is a "high-entropy" expression.

Instead, consider the dimensionless correction factor

$$\frac{\Delta(xy)}{xy} = \frac{(x+\Delta x)(y+\Delta y)}{xy} = \frac{x+\Delta x}{x}\frac{y+\Delta y}{y} = \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta y}{y}\right).$$

The result has an apparent meaning. It is easy to interpret that $(\Delta x)/x$ and $(\Delta y)/y$ represent the fractional change in x and y, respectively. Moreover, we see that

$$\frac{\Delta(xy)}{xy} = \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta y}{y}\right) - 1 = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta x}{x}\frac{\Delta y}{y},$$

and if Δx and Δy are relatively small, then

$$\frac{\Delta(xy)}{xy} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}.$$

Thus we can improve our estimate by multiplying by a correction factor that consists of summing the fractional change of x and the fractional change in y, which is easy to remember. Therefore

$$2.08 \times 5.25 \approx 10 \times (1 + 0.04 + 0.05) = 10 \times 1.09 = 10.9.$$

The exact answer is 10.29, which means our approximation is within 0.18 percent of the exact value.

Two other topics in the book that warrant explanation are lumping and reasoning by analogy. The former involves such ideas as approximating an integral by a single, well-chosen rectangle. The latter is used to answer such questions as, "Into how many regions do five planes divide space?" The approach is to first determine how many regions are created by five lines in a plane and then use analogy to reason the three-dimensional case.

My biggest issue with *Street-Fighting Mathematics* is that in applying these ideas I do not know what I am doing. I was a structural engineer before I was a mathematician, and so approximating is nothing new to me, but as a mathematician I do not think of approximating unless I am using numerical methods, and even then the algorithm does the heavy lifting. So, I can estimate my time of arrival on a long drive based on a rough estimate of my average speed without giving it much thought, but it never occurs to me to approximate a formula for the solution to a differential equation using dimensionless factors.

Additionally, I do not think as Mahajan does. He spends a chapter on guessing formulas by considering easy cases. What is the volume of a frustum or truncated pyramid with a square base? Three easy cases present themselves: the length of the side of the top square is 0, which yields a pyramid; the length of the side of the bottom square is 0, which yields an inverted pyramid; the lengths of the sides of both squares are equal, which yields a cube. Fine, but now we need to determine the volume of a pyramid. Mahajan points out that six pyramids with height equal to half the length of the base, with their vertices located over the center of the square base, when joined at their vertices, form a cube. Thus the volume of a pyramid with a square base is one-sixth the volume of the cube. This is not so much an easy case as a clever trick.

Later we are asked to derive a formula for the period of a pendulum. The easy cases in this situation are when the pendulum is released from an angle of 0 or 90 degrees. When these cases do not provide enough information, Mahajan turns to an extreme case: when the pendulum is released from 180 degrees. Again, this is not obvious, and so it is more clever than easy.

My complaint is that this book can be too clever, and how to be clever is not always obvious. I cannot fault Mahajan for this. I remember sitting in my advisor's office and often thinking, "How does he get these ideas?" Years later, one of my master's advisees told me she often had the same thoughts sitting in my office. Being clever comes from deep thought and experience, and you cannot teach that in a book. But you can help people get started, and that is what *Street-Fighting Mathematics* does.

And so I have read parts of this book many times, and I have tried every problem, and I have learned a great deal. This book is based on a short course at MIT, and I have even visited the course website to find new problems and solutions as a way to gain further insight. But there are enough problems that I could not solve that I found myself wishing that the book had a solutions section. I suspect that without the correct trick some of these problems are genuinely difficult to solve.

Mahajan, though, is upfront about the difficulty of learning a new technique, and he often returns to Pólya's statement that a tool is a trick we use twice. Mahajan shows a trick once (or several times), but recognizing when that trick can be used in a new setting is often difficult to see. I do not want to sound like my students who complain that they cannot see how to do a problem when all they need to do is spend more time thinking, but after trying unsuccessfully for a week to solve a problem, I would at least like a hint. It would not have taken much effort to add a few pages to this 134-page book to give hints about or solutions to the more difficult problems. Like I said, I do not think as Mahajan does, and so it would be helpful if there were a way to peek inside his head every once in a while and gain a better understanding of how to pull a problem apart.

My other issue with this book lies in the Foreword by noted computer scientist Carver Mead. His opening line irks me: "Most of us took mathematics courses from mathematicians—Bad Idea!" He goes on to state that since mathematics courses are taught as their own subject, most courses "are seldom helpful and are often downright destructive." I agree that mathematics should not be taught in a vacuum, but I do not understand how any in-depth study of a subject can be "downright destructive".

I do not know the intent behind Mead's statement—it may have just been an attempt to promote the book that wound up using hyperbole; and if I did not encounter this sentiment regularly, I could simply ignore Mead's remarks. I mentioned I was a structural engineer before I was a mathematician, and I have encountered this prejudice for years. My engineering advisors always chastised me for "wasting my time on more useless math courses". My fellow engineers never understood why I spent time learning more math, and the engineering students in my calculus classes perpetually complain that none of what they are learning will ever be useful.

And yet Mahajan uses a wider range of math in this book than I ever would have expected. He doesn't take time to explain derivatives or integrals or series or operators. He simply uses them when they are the correct tool or shows how to estimate their value. The very courses people are decrying as useless teach the material that forms the backbone of this book.

For instance, several times Taylor series is used to simplify a problem. This tells me two things: First, Taylor series is useful. Second, since Mahajan is not explaining what a Taylor series is, he is assuming we already learned this concept somewhere else.

I get tired of engineers telling me I waste my time with mathematics, and I am equally tired of mathematicians looking at me like I grew a second head when I use an engineering-style thought process to solve a problem. Both modes of thinking have their place, but we teach a single approach to a single audience when both approaches would be useful.

A better statement about what this book represents is that it fills a void for those who do not think in approximations and back-of-the-envelope calculations. If you are someone who has been trained in theory and brute-force calculations, then this book would provide insight into a new way of thinking. If you do such things on a regular basis, then this book will provide you with more tools.

But the ideas presented in this book will never replace more exacting methods of mathematics. It is one thing to run a sequence of estimates and back-of-the-envelope calculations at the beginning of a project. It saves time and provides a framework for the future development of the project, and this book teaches such methods. But it would be another thing to base a final design on a sequence of quick approximations, and those "useless" courses that emphasize rigor and precision will be what provide the tools to calculate those final results.

A group of students recently approached me wanting to know how to design a catapult. I did not use any of the tools described in this book—I ran straight to free-body diagrams and calculus. I do not know if I did this because those are the tools with which I am most comfortable or because I felt my students, who were taking calculus, needed to practice using calculus or because street-fighting mathematics simply did not provide the right tools. This book has taught me a great deal. Am I comfortable with this material? Some of it, yes. So why did it not occur to me to try out some of these techniques? I don't know.

What I do know is that *Street-Fighting Mathematics* is an engaging, well-written, insightful book. I do know that the book will provide any reader with new tools for making quick estimates and will introduce new ways of viewing problem solving. And I do know that I will read this book again. And again. And again. And maybe one day I will take the leap and actually use these methods to solve a problem.