Gerhard Hochschild (1915–2010)

Walter Ferrer Santos and Martin Moskowitz, Coordinating Editors

In the mid-1940s and early 1950s, homological algebra was ripe for formalization, or to put it in Hochschild's own words—see his now classic review of the book *Homological Algebra* by Cartan and Eilenberg ([1])— "The appearance of this book must mean that the experimental phase of homological algebra is now surpassed."

From its topological origins in the work of Riemann (1857), Betti (1871), and Poincaré (1895) on the "homology numbers" and after 1925 with the work of E. Noether showing that the homology of a space was better viewed as a group and not as Betti numbers and torsion coefficients, the subject of homology of a space became more and more algebraic. A further step was taken with the appearance of the concept of a chain complex in 1929 as formulated by Mayer and in a different guise by de Rham in his 1931 thesis. In 1935 Hurewicz showed that the homology of an aspherical space depended only on its first homotopy group, hence defining implicitly the (co)homology of a group. In parallel, and purely from the viewpoint of algebra, the low-dimensional cohomology of a group had been considered earlier. For example, it appeared implicitly in Hilbert's Theorem 90 in 1897—known also to Kummer before—as well as in the study of so-called factor sets used to classify group extensions. It also occurred in other contexts—in the early work of Hölder (1893), Schur (1904), and Dickson (1906); and later Brauer (1928), Baer (1934), and Hall (1938). In 1941 H. Hopf gave an explicit formula for the second homology group in terms of a description of the first homotopy group using generators and relations which, according to Mac Lane, provided the justification for the study of group cohomology. The actual definition of homology and cohomology of a group first appears in the famous papers by Eilenberg and Mac Lane: first in the 1943 announcement [2] and later in 1947 in [3] in full detail. More or less simultaneously and independently, Freudenthal considered the same concepts in the Netherlands (c. 1944) and, in a different guise, Hopf was doing the same in Switzerland. In his 1944 paper, instead of using explicit complexes as in [3], Hopf uses free resolutions to define homology groups with integral coefficients, the concept of free resolution having been used in algebra since its introduction by Hilbert in his famous 1890 paper on invariant theory. For the above historical comments the authors followed closely the information appearing in Chapter 28, History of Homological Algebra by C. A. Weibel in [13].

While still a graduate student at Princeton, Hochschild submitted a paper for publication dealing with the study of the behavior of Lie algebras and associative algebras with respect to derivations. In the Introduction to [4]—which was published in 1942 shortly after he was drafted into the army at the end of 1941—he states: "These 'generalized derivations'…were found to be significant for the structure of an algebra. In fact we shall obtain a characterization of semisimple Lie algebras and semisimple associative algebras in terms of these generalized derivations."

Gerhard's dissertation committee, cochaired by Chevalley and Lefschetz, reports: "The thesis deals with certain important problems in Lie algebras

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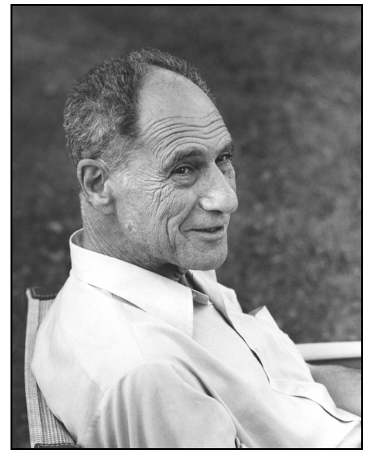
All photographs, unless otherwise noted, are courtesy of the Hochschild family.

and related questions in associative algebras. It contains in particular a highly interesting characterization of semisimple Lie algebras in terms of the operation of formal derivation. The thesis is highly worthy of publication as it contains many new results in addition to those indicated above. Furthermore, Hochschild set the problem himself and also did the research in an essentially independent way."[1] In modern terms Gerhard proved in his thesis that an associative algebra has the property that all its derivations are inner if and only if it is separable. The method he uses is basically homological, and with the tool of a separability idempotent, constructs from the given derivation the element that is used to characterize it as a conjugation. In these terms he was dealing with what is now called the first Hochschild cohomology group and proving its triviality. It still took Gerhard three years to publish a proof of the natural extension of his thesis results to a general cohomology theorem, and for that it was necessary first to construct an adequate cohomology theory. With that purpose in mind he generalized the complexes used in group cohomology to define what is now called Hochschild cohomology. Putting it in his own words: "The cohomology of an associative algebra is concerned with the m-linear mappings of an algebra into a two-sided module…A linear mapping…analogous to the coboundary operator of combinatorial topology and leading to the notion of 'cohomology group' has been defined by Eilenberg and Mac Lane (unpublished)…The present paper is concerned primarily with the connections between the structure of an algebra and vanishing of its cohomology groups"—see the Introduction to [5]. All these aspects of Hochschild cohomology, as well as its applications and many other topics, are treated in detail in the contribution by M. Gerstenhaber, "Hochschild's Work on Cohomology", that appears later in this article. In that contribution the author also describes the particular circumstances in which these first papers were written. Particularly interesting are his considerations concerning the back-and-forth interactions between the results of [5] and of [3]—e.g., concerning the concept of dimension shifting.

Paul Bateman contributed a note, "Gerhard Hochschild at Urbana (1948–58)", about the participation of Hochschild in the early development of the Department of Mathematics at the University of Illinois at Urbana-Champaign in the late 1940s and 1950s. That was an extremely fruitful period of Gerhard's professional as well as personal life. He married Ruth Heinsheimer in 1950—then a mathematics graduate student at UIUC—and his daughter and son were born in 1955 and 1957, respectively. Some relevant aspects of his mathematical work during this period are described in the following works.

In the contribution by John T. Tate, "Memories of Hochschild, with a Letter from Serre", the author writes about his personal and mathematical interaction with Gerhard in the genesis of the cohomological perspective of class field theory. He also adjoins a letter by Serre describing the process that led to the construction of the so-called Hochschild-Serre spectral sequence.



Gerhard Hochschild at his daughter Ann's wedding, 1978.

There are many mathematical objects carrying Gerhard's name. In another contribution to this memorial, Andy Magid, in "The Hochschild-Mostow group", reviews this concept (named in that fashion by Lubotzky in 1979), which appeared initially under the name of the "group of proper automorphisms of $R(G)$", $R(G)$ being the algebra of representative functions of the group G. This construction appeared in a series of papers by these authors published between 1957 and 1969. G. Hochschild and G. D. Mostow together wrote seventeen papers in different subjects. This very rich mathematical collaboration started while both were members of the Institute for Advanced Study in the academic year 1956–1957. Dan Mostow, in "Gerhard Hochschild as Collaborator", writes a very personal and illustrative note on their personal and mathematical interaction. One of its landmarks is the important paper on the invariant theory of unipotent groups [10]. Another is their theorem on faithful representations of real and complex connected Lie groups which is a global extension of Ado's theorem on Lie algebras (see the contribution to this memorial by M. Moskowitz).

The subject of representative functions is also dealt with in the interview that Pierre Cartier gave to one of the authors, which appears later. In particular, Cartier mentions the influence that his view of Tannaka duality and the concept of Hopf algebras had in this line of research. This is also explicitly spelled out in the introduction to [6], in which Hochschild writes: "We are concerned with the analogues for Lie algebras of the problems centered around the Tannaka duality theorem…The analogue of the Tannaka theorem for semisimple Lie algebras was established by Harish-Chandra in 1950. More recently P. Cartier has sketched (without proofs) a general duality

[1]*We thank the librarians of the Seeley G. Mudd Manuscript Library, Princeton, NJ, and Hochschild's family for providing access to Hochschild's records at Princeton University.*

theory for algebraic groups and Lie algebras which absorbs Harish-Chandra's result."

Cartier—as well as Tate in his article—also comments on the not-too-well-known interaction of Gerhard with the Bourbaki group, which began in the early 1950s when he attended three of the group congresses.[2]

The crucial institutional role he played in the development of mathematics at U. C. Berkeley from the 1960s on is described by Calvin Moore in "Gerhard Hochschild at Berkeley" in this article; in particular he describes in detail the institutional effort made by the department under the direction of J. Kelley (known by Gerhard since they served together at the Aberdeen Proving Ground) to recruit senior mathematicians in order to "aggiornare" the department to new mathematical developments.

Another result that has Hochschild's name attached to it is described by Bertram Kostant. In his contribution, "Hochschild Memorial", he relates from a very personal perspective the very close interaction they had as colleagues at that time at Berkeley. In particular he mentions the paper "Differential forms on regular affine algebras" that they wrote together with Alex Rosenberg (see [8]). This result, which now is known as the "HKR theorem", has played a very important conceptual role in the development of noncommutative geometry.

Concerning the ascribing of names to mathematical concepts and theorems, Gerhard used to say: "The personal names attached to mathematical objects are in general wrong." In this regard he insisted throughout his life that Hochschild cohomology should be called algebraic cohomology and that maybe it would be fairer to use that name for the rational cohomology of algebraic groups, as introduced in [7].

Concerning his interaction with colleagues at the Berkeley mathematics department, the contribution by G. Bergman, "Some Inadequate Recollections of Gerhard", gives a very personal description of Gerhard's personality.

Two of his students and then colleagues give us their personal viewpoint of Gerhard's role as an advisor and later mentor and friend throughout their mathematical careers. A constant subtheme in all the contributions, and in particular in these two, separated by more than twenty years, is a deep appreciation of Gerhard's personality that went together with, but often transcended, his mathematical influence. These two notes are by N. Nahlus, his last student (who finished in 1986), "Gerhard Hochschild as My Advisor and Friend", and M. Moskowitz, who finished in 1964, "Some Reminiscences".

Finally, with the help of Hochschild's family files, the science writer and Gerhard's son-in-law, James Schwartz, in "Gerhard Hochschild's Early Years: A Biographical Sketch", reconstructs the sometimes painful but very eventful early years of Gerhard's life until he sailed from Cape Town to New York in mid-1938 to begin Ph.D. studies at Princeton.

We conclude this introduction by addressing certain details not mentioned in the various contributions.

For one of the leading algebraists of his generation, his choice of courses at Princeton is rather peculiar.

1938/39: Calculus of variations, I.A.S. (Mayer); Elementary theory of functions of a real variable (Bohnenblust); Continuous groups (Eisenhart); Advanced theory of functions of a real variable (Bochner); The theory of relativity (Robertson); Continuous groups (Eisenhart).

1939/40: Applications of the theory of functions of a complex variable (Strodt); Riemannian geometry (Eisenhart); Topological groups (Chevalley); Algebraic geometry (Chevalley); Applications of analysis to geometry (Bochner); Riemannian geometry (Eisenhart).

1940/41: Applications of analysis to geometry (Bochner); Probability and ergodic theory, I.A.S. (Halmos and Ambrose); Differential equations (Chevalley); Research and work on dissertation under the direction of Chevalley—two semesters; Advanced theory of functions of a real variable (Bochner); Ergodic theory, I.A.S. (von Neumann).

An interesting story of his days at Princeton is the following: during the first lectures of Chevalley's course on differential equations the room was packed with people curious to know what he had to say on this subject. But at the end only three people remained: Hochschild, von Neumann, and Weyl.[3]

His writing style has been described by some of his colleagues as crystal clear and sometimes as "relentless". In the review of Gerhard's book *The Structure of Lie Groups*, which is nowadays

[2] *In accordance with the Bourbaki files provided to the editors by J.-P. Serre from Viviane le Dret, Hochschild participated at the following instances: the congress at Pelvoux-le-Poet (June 25 to July 8, 1951), foreign visitors: Hochschild and Borel, "cobayes": Cartier and Mirkil; the congress at Pelvoux-le-Poet (June 25 to July 8, 1952), foreign visitors: Borel, De Rham, and Hochschild; at the congress at Murols (August 17 to 31, 1954), foreign visitors: Hochschild and Tate, Honorable foreign visitors: Iyanaga and Yoshida; efficiency expert Mac Lane, "cobaye": Lang.*

[3] *A chronicler of Princeton's mathematical department in that period describes Chevalley as playing an "endless game of Go". Gerhard developed then a lasting passion for the game, acquiring approximately the level of a seventh kyu.*

considered a classic "fast and deep" introduction to the subject, Hirzebruch states: "It's amazing the extent to which the author achieved his goal to enable a self contained reading to someone who only knows the basics of multi-linear algebra, group theory, set theoretic topology and calculus," mentioning in particular his eighteen-page treatment of Tannaka duality that starts from scratch.

His originality was not limited only to style. One of the main ingredients of the singularity of his mathematics is his methodological consistency, even though that consistency did not always generate the approval of his colleagues. This is illustrated by some of the comments concerning his approach to algebraic groups in [9] and [11], in which the author "relentlessly" and almost exclusively used the Hopf algebra perspective in the development of the theory. Some of his colleagues thought that this viewpoint detracted from the geometric content of the results.

The adjective "relentless" could perhaps also be applied to his teaching style. In the introduction to his not-too-well-known monograph "A second introduction to analytic geometry", dedicated to his son Peter when he was a high school student, and in a direction surely orthogonal to the current trends in mathematical teaching, he says: "What follows is an examination of the basic geometrical features of Euclidian three-space from the view of rigorous mathematics…[and] our program here involves algebra and analysis as much as it involves geometry."

In his theory and practice of teaching, one also finds a tendency to include aspects of what is called "applied mathematics", and that could be considered somewhat surprising in someone deeply involved in the development of the more abstract areas of pure mathematics. In a letter to the *Notices of the AMS* published in July 2009, writing about calculus teaching, he says: "The educational potential of computers can be illustrated […] by elementary examples from classical mechanics construct[ing], by simple numerical integration, orbits like that of the earth around the sun […and…] explor[ing] paths generated when the acceleration depends in various ways on position, velocity, time, and path length from the origin." He bases these convictions on the courses he took in 1934 as an undergraduate in South Africa, where "in 'applied mathematics' […] you learned to formulate simple settings from classical mechanics in terms of differential equations [while] the introductory courses in formal calculus were as discouraging then as they are now."

Throughout his life he had a strong passion for photography. That hobby, which he began when he was a boy in Berlin, enabled him to obtain part-time work as a photographer's assistant in South Africa, a job that turned out to be extremely helpful under the difficult financial circumstances he found himself in while a student. Later in life, and especially after retiring, he dedicated many of his days to photography. In this article we show some pictures illustrative of his work.

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Short Biography of Gerhard Hochschild

Gerhard Paul Hochschild was born on April 29, 1915, in Berlin, of a middle-class Jewish family. He completed gymnasium there, and, as a consequence of the Nazi takeover in Germany, he emigrated together with his brother to Cape Town in South Africa in 1933, where he took a B.S. degree in science in 1936 and a M.S. in mathematics in 1937. Before beginning Ph.D. studies at Princeton University, he worked as a junior lecturer at the University of Cape Town during the 1937–38 academic year. He completed his thesis, entitled *Semisimple algebras and generalized derivations* and directed by Claude Chevalley, at Princeton in 1941.

After defending his thesis, he was appointed as a part-time instructor and research assistant at Princeton University for the academic year 1941–42 starting in September, but in November he was drafted into the U.S. Army and was mostly stationed at the Aberdeen Proving Ground. In June 1942, he became a naturalized citizen, and three years later, after the war was over, he left the Army to take a part-time position for a semester—November 1945 to June 1946—as an instructor at Princeton University. Two of his papers on what later was to be called Hochschild cohomology list his address as Aberdeen.

Heiner Hochschild and his two sons. Gerhard stands at his left, c. 1925.

Gerhard was a Benjamin Peirce Instructor at Harvard University during the academic years 1946–48 and in September of that year took a position at the University of Illinois at Urbana-Champaign. He spent the academic year 1951–52 visiting Yale University, 1955–56 visiting the University of California at Berkeley, and 1956–57 as a member of the Institute for Advanced Study at Princeton. Gerhard met Ruth Heinsheimer while in Urbana and they married in July 1950. Their daughter Ann was born in 1955 and their son Peter in 1957. Gerhard remained at UIUC until September 1958, when he moved to Berkeley as a professor of mathematics. He retired from Berkeley in 1982 and continued teaching until 1985.

He died peacefully at home on July 8, 2010, in El Cerrito, where he raised his family and lived during his years at Berkeley.

James Schwartz

Gerhard Hochschild's Early Years: A Biographical Sketch

Gerhard, the younger son of Heiner and Lilli, was born in Berlin in 1915. His father, who was a patent attorney with a degree in engineering, nurtured a love of science and engineering in both of his sons. At the age of nine, Gerhard's childhood took an abrupt turn when his mother was diagnosed with a lung ailment and was encouraged by the family doctor to seek a cure in a Swiss sanatorium. Reluctant to be parted from her young son, Lilli, with the support of the same doctor, became convinced that Gerhard shared her affliction and would benefit from a similar cure. Consequently, both mother and son were sent to Switzerland, she to Davos and he to a nearby "Kinderheim" in Arosa.

To the young boy, life in the sanatorium was a form of imprisonment, but Gerhard's spirit was not broken. In one incident reported by a staff member of the Kinderheim, Gerhard sold books and a treasured stamp collection given to him by his parents in order to raise money to buy himself a dog.[4] However, circumstances soon again conspired against the young boy. Far from the comfort of home and without the support of his father and brother, Gerhard was forced to witness at first hand his mother's gradual descent into mental illness, as Lilli began to suffer from delusions and gradually drifted further and further into her own world. After two years in Switzerland, Lilli was transferred to a mental asylum in Germany, where she was later murdered by the Nazis, and Gerhard returned alone to his father and brother in Berlin.

The tragic loss of his mother imprinted him with an acute sense of the fragility of life and at the same time seemed to instill in him a fierce determination to shape his own destiny. Back in Berlin, he entered gymnasium, where he found the German curriculum for the most part tedious and oppressive and developed a lifelong skepticism about formal education. Specializing in doing the minimum amount of work necessary to pass his courses, Gerhard nonetheless maintained his passion for learning and also pursued other interests, including photography and hiking. Though he found most of his high school teachers petty and tyrannical, there was one exception: Herr Doctor Flatow, a mathematics teacher. Five years later, when he was a university student in Cape Town, he wrote Flatow to express his gratitude: "I still remember with pleasure our hours of mathematics in school, and am so grateful to you for interesting me in mathematics."

Meanwhile, Germany itself was tottering on the verge of economic ruin as the National Socialists vied for power. Some seventy-five years after the fact, Gerhard still vividly remembered a particularly threatening incident. He had picked up his light-haired, fair-skinned friend Eva from school, and the two of them were walking home when a young Nazi roared up to them on a motorcycle. In a menacing voice, the Nazi boy said, "A Christian girl like you should not be with a dirty Jew." Gerhard, who had noticed that the Nazi was carrying a gun, avoided eye contact, and he and Eva managed to walk on without further engagement.

With Hitler's assumption of power in the spring of 1933, Gerhard's father made immediate plans to send his sons to safety in South Africa. Because it was impossible to send money out of the country, the boys were forced to earn their own way, and shortly after his arrival in Cape Town

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[4]*This incident and other events described in this sketch were gleaned from correspondence that Gerhard saved.*

in May of 1933, Gerhard found employment as a photographer's assistant. Over the following years his photographs appeared in various Cape Town newspapers.

Meanwhile, however, he was trying to find a way to enroll at the University of Cape Town. His father, who had now himself emigrated to Paris and had no access to his money in Germany, was still unable to help. However, Gerhard received money for his university expenses from the Hochschild Family Foundation, which had been established with remarkable prescience in 1924 by a cousin of his grandfather's, Berthold Hochschild, to aid members of the extended family in times of need. By January of 1934, Gerhard had been accepted at the University, with just enough money to pay the tuition. Supplementing his job in the photography studio with private tutoring in math, he was able to earn enough to cover his modest living expenses. Around this time, he began to frequent a well-known circle of political activists, artists, and academics that met weekly at the home of Dr. Abdul Hamid Gool and his wife Zainunnissa (known as Cissie), a highly educated political organizer who founded the National Liberation League in 1935 and would go on to serve on the Cape Town City Council for thirteen years. Gerhard must have felt at home among these free thinkers, people of all races and political persuasions, meeting in defiance of the rigid social conventions of Cape Town.

At the University, Gerhard studied physics and applied mathematics, receiving his bachelor of science degree in 1936. The following year he completed a master of science degree in pure mathematics. To Flatow, his high school mathematics teacher, he described his interests as follows: "I plan to specialize in tensor calculations and in the basis of modern relativity theory and 'geometrized mechanics' and then pure mathematics, differential geometry and certain parts of the theory of differential equations. I think the ideal middle ground for me is pure and applied mathematics in which I am equally interested."

Later in his life, Gerhard spoke of the great debt he owed Stanley Skewes,[5] a lecturer (and later professor) in the mathematics department at Cape Town, who taught him in his undergraduate years and subsequently became his primary advisor and supporter. Skewes recommended him for a position as a junior lecturer in pure math, which enabled him to continue his studies from 1937 to 1938, and encouraged him to apply to the mathematics department at Princeton for further graduate work. When Gerhard sailed for America in the summer of 1938, Skewes saw him off. Gerhard remembered Skewes standing at the pier, calling out his final words of advice: Never doubt your own abilities.

[5] *Stanley Skewes was best known for his discovery in 1933 of what became known as the Skewes number.*

Murray Gerstenhaber

Hochschild's Work on Cohomology

Hochschild cohomology has become indispensable in pure algebra for its applications, among others, to representation theory and algebraic deformation theory, but it is emerging also as a new and valuable tool in physics, particularly quantum theory. It is all the more remarkable that the first two defining papers ([10], [11]) were written in 1945 and 1946, not at any prestigious academic institution nor, in fact, at any academic institution at all but at the Aberdeen Proving Ground, Maryland, where Hochschild did his military service during World War II. Despite this semi-isolation, Hochschild was aware of simultaneous yet unpublished work by Eilenberg and Mac Lane on group cohomology, which Hochschild acknowledged in [10], and they in turn acknowledged and adopted a fundamental result of [10] when their paper [3] finally appeared in 1947. In [10] Hochschild defined the complex that bears his name: If A is a k algebra and M an A bimodule (a concept first made explicit in [10]), then let $C^n(A,M)$ be the module of k multilinear maps $A \times \cdots \times A$ (n times) $\to M$ (or linear maps $A^{\otimes n} \to M$) with coboundary map $\delta : C^n \to C^{n+1}$ given, for $F \in C^n$, by

$$\begin{aligned}\delta F(a_1,\ldots,a_{n+1}) = a_1F(a_2,\ldots a_{n+1})\\ + \sum_{i=1}^{n}(-1)^n F(\ldots,a_ia_{i+1},\ldots)\\ + (-1)^{n+1}F(a_1,\ldots,a_n)a_{n+1}.\end{aligned}$$

(Here $C^0(A,M)$ is just M.) Then $\delta\delta = 0$, and one can define cocycles, coboundaries, and cohomology groups in analogy with simplicial cohomology theory, except that here, as Hochschild proves in [10], if one knows $H^1(A,M)$ for all M then all higher cohomology groups are, in principle, already determined. (This fundamental "dimension shift" theorem, best understood in terms of resolutions that Hochschild had already mastered by [11], is cited in [3].) Accordingly, Hochschild described his theory as a "truncated" one (ending at H^1), whereas in fact he had discovered the fundamental fact that one must consider *all* modules over the object whose cohomology one is studying, not just the trivial module, as in the case of simplicial or group cohomology, or as originally in Lie cohomology. (What may be the first explicit consideration of Lie cohomology with coefficients in an arbitrary module seems to be in [17].) In fact, simplicial cohomology is actually a special case of

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Hochschild cohomology in the sense that to any simplicial complex Σ and coefficient ring k there is naturally associated a k algebra $A = A(\Sigma, k)$ such that the simplicial cohomology $H^*(\Sigma, k)$ is canonically isomorphic to the Hochschild cohomology $H^*(A, A)$; cf. [6].[6]

Both of these are rings (the latter known to Hochschild in the more general context of pairings between modules), and the isomorphism is actually one of rings. What was not yet known to Hochschild was the "Gerstenhaber algebra" structure (cf. [4]) carried by $H^*(A, A)$, which has become useful in physics. In particular, $H^*(A, A)$ is commutative (in the graded sense), so the commutativity of $H^*(\Sigma, k)$ may be viewed as a consequence of this more general theorem. (Every cohomology theory must, however, be viewed on its own terms; the Steenrod operations, for example, are not generally present in Hochschild cohomology. In fact, the graded Lie structure of $H^*(A, A)$ is an obstruction to their existence; in $A(\Sigma, k)$ the Lie multiplication vanishes identically.)

Hochschild's original interest in, and development of, his cohomology seems to have begun with the observation that Wedderburn's third structure theorem is a consequence of the following: If $H^2(A, M) = 0$, then any "singular extension" $0 \to M \to B \to A \to 0$, i.e., one in which M is an ideal of B where the product of any two elements is zero (and which is therefore naturally a bimodule over A), must split; it follows by induction on the index of nilpotence of the radical that, if $H^2(A, M) = 0$ for all M, then any Artinian algebra B with "semisimple part" $A = B/J (= \text{radical of } B)$ has the property that the sequence $0 \to J \to B \to A \to 0$ also splits. Hochschild showed that, for finite-dimensional algebras A over a field (the only kind then generally considered), the condition that $H^1(A, M)$ vanish for all M (and hence that all higher cohomology groups of A also vanish) was equivalent to the classical definition of separability of A. This cohomological criterion continues to hold for the Auslander-Goldman definition of separability [1], namely that A be projective in the category of A bimodules, where now one may allow any (commutative, unital) coefficient ring k. (Either criterion immediately implies, and is equivalent to, the existence of a separability idempotent, the construction of which is frequently the easiest way to prove separability.) The intimations of this are already present in Hochschild's first paper [9], which appeared in 1942, the year he got his Ph.D. from Princeton.

Hochschild saw applications for his cohomology in many places. André Weil had introduced cohomology into class field theory, and Hochschild (after an earlier paper on class field theory in 1949) collaborated with Nakayama in 1951 to rederive some of Weil's results and go even further [15]. Simultaneously, Hochschild was interested in Lie groups and algebras and in algebraic groups. Two papers produced in collaboration with Serre in 1953 have become classics ([16], [17]). The first, extending work of Roger Lyndon, introduces the Hochschild-Serre spectral sequence to study the relation between the cohomology of a group, that of a normal subgroup, and of the quotient; the second applies the same techniques to Lie algebras. By 1956 Hochschild had published his fundamental paper [12] introducing the relative Tor and Ext, the importance of which is indicated by the length of the review by Henri Cartan. (In fact, Hochschild cohomology is a relative theory—one can compute the Hochschild cohomology of a k algebra using resolutions, but algebra and module "morphisms" must be understood as ones that split when considered simply as k module morphisms.)

It is not clear when it was first recognized that Hochschild cohomology could be computed relative to any algebra separable over the coefficient ring. This is not difficult to prove (once it is known) using projective resolutions and, in turn, sometimes makes Hochschild's original complex into an effective computational tool; it is the essential step in proving that simplicial cohomology is a special case of Hochschild cohomology. Expressed in terms of Hochschild's original complex for a k algebra A and bimodule M, if S is a k subalgebra of A, then the S relative cochains $F \in C^n(A, M)$ are those where, for all $a_1, \dots, a_n \in A$, all $s \in S$, and all $i = 1, \dots, n-1$, one has

$$\begin{aligned} F(\dots, a_i s, a_{i+1}, \dots) &= F(\dots, a_i, s a_{i+1}, \dots), \\ F(s a_1, \dots, a_n) &= s F(a_1, \dots, a_n), \\ F(a_1, \dots, a_n s) &= F(a_1, \dots, a_n) s, \end{aligned}$$

and where F vanishes whenever any argument is in S. These form a subcomplex of the usual Hochschild complex, and if S is separable over k, then the inclusion induces an isomorphism of cohomology.

In the midst of his work on Lie groups (later partly in collaboration with G. D. Mostow), Hochschild joined with Bertram Kostant and Alex Rosenberg, publishing [14]. This contains the celebrated HKR theorem, which in its simplest form (and stripped of the unnecessary hypothesis that the coefficient ring k be a perfect field) asserts that, if A is a finite separable algebraic extension of a polynomial ring $k[x_1, \dots, x_n]$, then $H^*(A, A)$ is just the exterior algebra over k generated by the derivations of A.

Hochschild's last paper explicitly devoted to cohomology theory, [13], appeared in 1974; his

[6] *S. D. Schack, my former student, friend, and collaborator in some of the works mentioned here, died on February 9, 2010.*

last published work, a text with a somewhat philosophical bent intended for beginning graduate students, appeared in 1983.

There is much that Hochschild did not know about his own cohomology, and he probably greatly underestimated (as did most) the impact that it would have, but fortunately he lived long enough to see many important developments. One was the recognition that $H^2(A, A)$ was the module of infinitesimal deformations of A with $H^3(A, A)$ containing the obstructions (see [5]), and that by extending Hochschild cohomology to presheaves of algebras the same idea could be applied to view at least the formal aspects of the Froelicher-Nijenhuis-Kodaira-Spencer deformation theory of analytic manifolds as special cases of Hochschild cohomology (see [7]). Hochschild cohomology was introduced into quantum theory in the groundbreaking paper of Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [2]. They showed that the Poisson bracket on the algebra of functions on phase space should be viewed as an infinitesimal deformation of that algebra and that quantization could be viewed as (or, alternatively, result from) deformation of the algebra using the given Poisson bracket as infinitesimal. Phase space being symplectic, this led to the question of whether every symplectic manifold could be similarly quantized, subsequently proven by Dewilde and Lecomte and ultimately to the question of whether every Poisson manifold could be quantized. The affirmative answer to this much more difficult question was part of Kontsevich's Fields Medal-winning work. This work used in an essential way L_∞ algebras (strong homotopy Lie algebras) and indirectly A_∞ algebras (associative algebras up to homotopy), basic concepts due to Stasheff. These, too, have their cohomology and deformation theories, patterned after Hochschild. Deformation quantization, introduced in [2], has sometimes been referred to as capturing the bronze medal in quantization, after the Hilbert space operator approach (which in a way formalizes and is equivalent to both the Heisenberg and Schrödinger approaches) and Feynman's approach by path integrals. It avoids some of the troublesome infinities, and recent work, in particular by Dito and by Stasheff in homological physics, suggests that we have so far seen only the very tip of the iceberg—Hochschild cohomology will ultimately prove as important a tool in physics as it has already been in pure algebra. The universe, it seems, is not flat, not commutative, and in important aspects not even associative, but it does respect Hochschild cohomology.

Hochschild steadfastly rejected the term "Hochschild cohomology", insisting always on "algebraic cohomology" and correcting you when you slipped. He was generous, once offering to give up his office to Oscar Goldman, who was starting on a year's leave at Berkeley at a time when no visitors' offices were available. Time will surely reveal many more applications of Hochschild's fundamental ideas, and were he present he would disclaim credit; nevertheless, I think he would be pleased.

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Paul Bateman

Gerhard Hochschild at Urbana (1948–58)

Gerhard Paul Hochschild, an algebraist of the highest caliber, died July 8, 2010, in El Cerrito, California. His fields of interest included cohomology theory for algebras, algebraic groups, and Lie algebras. He was on the staff of the University of Illinois at Urbana-Champaign from 1948 to 1958. In 1958 he accepted an offer from the University of California at Berkeley, a position he held until his retirement. In 1979 he was elected to the National Academy of Sciences. In 1980 he was awarded the Leroy P. Steele Prize of the American Mathematical Society.

Gerhard Hochschild was born April 29, 1915, in Berlin, Germany. After the Nazi takeover in the early 1930s he suffered considerable harassment from Hitler Youth. Fortunately, his extended family was widely distributed geographically. Thus in due course he became a student at the University of Cape Town in South Africa, from which he received a B.S. in 1937 and an M.S. in mathematics in 1938. He then entered graduate school at Princeton University. In 1941 he produced an outstanding Ph.D. thesis under the supervision of Claude Chevalley. This was entitled *Semisimple algebras and generalized derivations* and was subsequently published in the *American Journal of Mathematics*, vol. 64 (1942), pp. 677–694.

Gerhard with wife Ruth on his right—partying at Urbana, c. 1952.

In 1941 Hochschild was drafted into the U.S. Army. He spent three years in the Army, most of it at the Aberdeen Proving Grounds in Maryland. In the army he found it desirable to use his middle name, Paul, rather than the more Teutonic name Gerhard. After his military service, Hochschild held a Benjamin Peirce Instructorship at Harvard University from 1946 to 1948. In 1948 Gerhard was appointed to an assistant professorship at Illinois. He was promoted to associate professor in 1950 and to full professor in 1952. During his ten years on the Illinois staff, Hochschild had three leaves of absence. He was a visiting professor at Yale for the academic year 1951–52, a visiting professor at Berkeley for the academic year 1955–56, and a member of the Institute for Advanced Study at Princeton for the academic year 1956–57. The Institute membership was supported by a Guggenheim Fellowship and by a sabbatical leave at half pay from the University of Illinois.

During his service at the University of Illinois, Gerhard taught a graduate course every semester. As someone who audited one of these courses, I can report that his lectures were very thoroughly prepared and well presented. Personally, I found Hochschild to be a very helpful consultant on matters related to algebra and algebraic number theory. While at Urbana Gerhard supervised the following three Ph.D. dissertations: George Francis Leger (1951), *On cohomology theory for Lie algebras*; Kung-Sing Shih (1953), *Cohomology of associative algebras and spectral sequences*; Ronald Alvin Macauley (1955), *Analytic group kernels and Lie algebra kernels.*

Reinhold Baer's return to Germany in 1956 and Gerhard Hochschild's departure for Berkeley in 1958 were both serious blows to the algebraic side of the Illinois mathematics department; however, the loss of Hochschild was a more serious blow in that he was at the top of his game mathematically in 1958, whereas Baer was near the end of his career in 1956. The two men differed in another respect: Baer's departure from Nazi Germany in the 1930s was almost painless, but Hochschild's definitely was not. In fact, although the two men were close friends in Urbana, Gerhard made it clear to Reinhold that he would never travel to Germany to visit him.

A story that is very illustrative of Gerhard's personality is the following: while Baer was on sabbatical, he asked Hochschild to take care of one of his students, Arno Cronheim. At the end of the process Arno wanted to give some mention of Hochschild in his thesis, but Gerhard insisted that he not do so and prevailed. The close and attentive relationship that Gerhard had with his students is illustrated by the following citations appearing in the acknowledgments in two of his students' theses.[7]

Shih says: "The author wishes to express his heartiest thanks to Prof. G. Hochschild for the warm encouragement, for the privilege of reading the manuscript of the paper [2][8] while it was not yet in print, and for many helpful suggestions generously given him throughout the preparation of this paper."

Macauley says: "The author takes this opportunity to express his most sincere gratitude

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[7] *We thank the staff at the Reference, Research, and Government Information Services, UIUC Library, for promptly providing this very useful information about this period of Hochschild's activities.*

[8] *The mentioned paper is G. P. Hochschild and J.-P. Serre, Cohomology of group extensions,* Trans. Amer. Math. Soc., *Vol. 74, 1953, pp. 110–134.*

to Professor G. P. Hochschild, who offered and gave more aid and encouragement in the preparation of this thesis than could reasonably be expected of any advisor and friend. Indeed, Professor Hochschild's patience has withstood an arduous test."

The quality and diversity of Hochschild's mathematical work during the period of his stay at Urbana is quite remarkable. Some of the topics studied in the period are: extensions and representations of Lie groups, cohomology theories, relative homological algebra, number theory, spectral sequences, theory of restricted Lie algebras, simple algebras, and more. His landmark work on the cohomological methods in class field theory appeared in three papers during that period; also, the two papers on the so-called Hochschild-Serre spectral sequences date from that time. The first two of the series of papers on representative functions, written jointly with G. D. Mostow, appeared in the years 1957 and 1958.

It was in Urbana that Gerhard met his wife, Ruth Heinsheimer, who, like Gerhard, was born in Germany. She and her mother escaped Germany in early 1939, first settling in Paris and then fleeing to a small village in the Pyrenees before sailing for New York from Lisbon in February of 1941. Ruth graduated from Bryn Mawr College in 1947 and then enrolled in the graduate program in mathematics at the University of Illinois at Urbana-Champaign, where she obtained an M.A. in mathematics in 1948. When Gerhard arrived as an assistant professor, she was working under the direction of R. Baer. They got acquainted then, and she and Gerhard were married in July 1950.[9] When they left for Berkeley in 1958, Ruth had also finished an M.A. degree in French literature. Their daughter Ann was born in 1955 in Urbana and their son Peter in Princeton in 1957.

John T. Tate

Memories of Hochschild, with a Letter from Serre

Gerhard Hochschild was my first friend in the mathematical world after my student days. Though his world outlook was pessimistic, it was usually expressed with humor, and in personal relations he was very generous, positive, and outgoing. I have many happy memories. Gerhard and his wife Ruth were fun to be with.

Our paths crossed often in the early 1950s. Our mathematical discussions at that time were mainly about the then quite new theory of cohomology of groups and its application to class field theory. Hochschild was the first to advocate this application in his 1950 *Annals* paper on local class field theory. The even greater importance of cohomology in the global theory became clear from André Weil's construction of the global Weil group as a group extension of a Galois group $G_{K/k}$ by the idele class group C_K made with a "canonical class" $\alpha_{K/k}$ in $H^2(G_{K/k}, C_K)$, but Weil used cohomology only because he had to and then as little as possible. He was not interested in using it to simplify proofs, nor in determining the exact structure of the groups $H^n(G_{K/k}, C_K)$. It was Hochschild and Nakayama, in their joint 1952 *Annals* paper, who determined the structure for $n = 1, 2$ in reworking global class field theory in terms of cohomology. Their work and Weil's inspired the Artin-Tate seminar of 1951–52 in Princeton. Later I showed that the results of Hochschild and Nakayama for $n = 1, 2$ imply purely algebraically that the cup product with the canonical class gives isomorphisms $H^{n-2}(G_{K/k}, \mathbb{Z}) \xrightarrow{\sim} H^n(G_{K/k}, C_K)$ for all n, a disappointment in that it shows that the higher-dimensional cohomology groups give no new arithmetic information.

The Hochschild-Serre spectral sequence was another preoccupation of Gerhard at that time. When I asked Serre[10] for the story behind his joint papers with Hochschild, he sent the following letter:

Paris, 3/11/10

Cher Tate,

J'ai surtout connu Hochschild en 51/53, à l'époque des "Hochschild-Serre". Voici comment ça s'est passé:

En septembre 1950, un peu avant de clarifier ce qui allait être ma thèse, j'ai vu que les suites spectrales de Leray s'appliquaient aussi à la cohomologie des extensions de groupes. C'était d'ailleurs presque évident vu ce qu'avaient déjà fait Cartan-Leray: ils avaient montré que, si $X \to Y$ est un revêtement galoisien de groupe Γ, il y a une suite spectrale qui part de $H^\bullet(\Gamma, H^\bullet(X))$ et aboutit à $H^\bullet(Y)$. [Cette suite spectrale est souvent appelée maintenant "de Hochschild-Serre"— je n'y peux rien.] Si $1 \to H \to G \to \Gamma \to 1$ est une extension de groupes, on applique ça à l'action de Γ sur E_G/H, où E_G est un fibré universel pour G. On trouve directement ce

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[9] *They both enjoyed very much those early years of their relationship at Urbana, and, frequently in his later years, Gerhard mentioned nostalgically the loss of that close group of friends that included not only mathematicians but also some people in literature [eds.].*

[10] *Jean-Pierre Serre is honorary professor at the Collège de France, Paris, where he held the chair "Algèbre et Gèométrie" from 1956 to 1994. His email address is* serre@noos.fr.

qu'on veut. Je m'exprimais en termes plus algébriques dans la Note, mais cela revenait au même. Je citais Lyndon en ajoutant "on remarquera l'analogie de cette démonstration avec celle employée par R. C. Lyndon", affirmation très gratuite car nos démonstrations étaient complètement différentes (et Lyndon ne connaissait pas les suites spectrales).

Au début de l'année suivante (février-mars-avril?) je suis parti faire du ski à Serre-Chevalier avec Josiane. C'est là que j'ai reçu ma première lettre de mathématicien étranger, celle de Hochschild. Il me disait qu'il avait vu ma Note et qu'il avait fabriqué une démonstration différente, par des calculs explicites de cochaines. Je crois me souvenir (je n'ai malheureusement pas gardé sa lettre) qu'il me proposait aussi de publier ensemble nos deux démonstrations. Je me souviens du très grand plaisir que m'avait causé sa lettre; j'étais malade (angine), et la lettre m'avait guéri! J'ai accepté cette collaboration. Il est probable que j'ai rédigé (en français) la première partie du texte: "General Methods", c'est-à-dire la méthode de ma Note. La version définitive du texte s'est faite lors de mon séjour à Princeton en janvier-février 1952: Hochschild m'a invité chez lui (il était à Yale), et j'ai passé deux ou trois jours avec lui, à lire et à réviser notre texte. Je lui ai fait adopter la méthode Bourbaki que tu connais bien: lecture ligne à ligne à haute voix. Il était un peu surpris, mais il a accepté. Ça nous a coûté un grand nombre d'heures, mais on y est arrivé.

Quelques commentaires mathématiques sur ce texte:

- 1 - La méthode explicite par filtration de cochaînes (celle que Hochschild avait fabriquée) est en fait très voisine de celle de Lyndon. Nous ne le savions pas, car Lyndon n'avait publié qu'une partie de sa thèse. De plus, Lyndon était essentiellement un topologue, et pour lui les seuls coefficients intéressants étaient les coefficients constants. Hochschild et moi nous intéressions à bien d'autres choses, comme tu sais.

- 2 - Il y a une petite erreur idiote (les erreurs sont rarement intelligentes …) dans notre exposé: nous parlons de $C^i(G/H, C^j(H,M))$, *où* C = *cochaînes, comme si ça avait un sens. Or ça n'en a pas, car* G/H *n'opère pas sur les* $C^j(H,M)$; *ça n'a aucune importance car on peut interpréter* $C^i(G/H, C^j(H,M))$ *simplement comme des applications*

$$G/H \times \cdots \times G/H \to C^j(H,M),$$

et l'opération de cobord est relative à H *uniquement. Cette erreur a été signalée en 1981 (seulement!) par F. R. Beyl; tu trouveras la référence à la p. 587 de mes C. P. vol. I.*

- 3 - Dans sa construction, Hochschild a choisi de mettre H *en première position, et* G/H *en deuxième. Or, en Topologie (que ce soit Leray, ou ma thèse), on écrit les variables dans l'ordre* $H(\text{base}, H(\text{fibre}))$ *en mettant d'abord (à gauche) les variables qui représentent la base, et à droite celles qui représentent la fibre. Cela peut paraître un détail, mais c'est un détail qui a des conséquences quand on fait des calculs explicites. Suppose par exemple que* G *soit abélien libre de rang 2, que* H *soit de rang 1, et que l'on ait choisi des orientations de* H *et de* G/H, *i.e., des isomorphismes de ces groupes avec* $\mathbb{Z}$. *"La" suite spectrale donne un isomorphisme de* $H^2(G)$ *(à coefficients dans* $\mathbb{Q}$, *disons) avec* $H^1(G/H, H^1(H))$ *qui est visiblement* $\mathbb{Q}$. *Bien sûr, une orientation du sous-truc et une orientation du quotient donnent une orientation du tout. Mais quelle orientation? La théorie de Leray (ou de ma thèse) en donne une, et la théorie de Hochschild donne l'opposée. C'est désagréable. Et que donne le point de vue Grothendieckien: suite spectrale des foncteurs composés? C'est pire: il ne donne rien, car Grothendieck, dans Tôhoku, se borne à dire qu' "il existe" une suite spectrale sans préciser laquelle (elle dépend de la méthode employée). Ce n'est pas par hasard qu'il n'a jamais exposé la théorie des cup-produits dans les suites spectrales: il aurait eu besoin d'être plus précis.*

Voilà. Excuse-moi de m'être un peu écarté de Hochschild! Quoi d'autre? Que j'ai fait en 1952 la connaissance de sa femme, Ruth, et que je l'avais trouvée charmante. Que notre article sur les algèbres de Lie ne nous a posé aucun problème; seul détail: Chevalley était referee pour les Annals *et voulait nous faire remplacer nos cochaînes alternées par de l'algèbre extérieure. Je l'ai envoyé paître (how do you say that in polite English, and in not-so-polite English?), en donnant l'argument que notre théorie s'appliquait à des algèbres de Lie de dimension infinie.*

J'ai revu Hochschild en 1954, quand il est venu avec toi assister à un congrès Bourbaki. Je l'ai sûrement revu plusieurs fois ensuite, mais je ne crois pas que nous ayons discuté sérieusement de maths: nous avions choisi des directions différentes.

Bien à toi
J.-P. Serre

In the last paragraph of his letter Serre writes that, in 1954, Hochschild attended a Bourbaki meeting with me. I remember that well. Here's how it happened. On my first trip to Europe, that summer before the ICM in Amsterdam, the Hochschilds invited me to visit them at their then favorite vacation spot, the Riederfurka, a mountain resort on the Riederalp, just north of Brig and south of the Aletsch glacier. It's a tranquil place with breathtaking views. (Google "Riederfurka" to find hundreds of pictures.) After a few days there I took, on Gerhard's recommendation, the train to Zermatt and a cog railway up to a lookout area for a magnificent view of the mountains on the Italian border, from the Matterhorn to Monte Rosa, the highest mountain in Switzerland, which Gerhard had climbed with a group earlier that summer. I then met him and Ruth, to tag along with them to a Bourbaki meeting in Murols. Our landlady there referred to Ruth as "the woman with the two men". I was a fan of Bourbaki. There I was, seeing him at work and meeting most of the younger members for the first time. These memorable new experiences in Europe were made possible by the Hochschilds. It's hard to express how much their friendship meant to me.

Gerhard had a great sense of humor. He informed me of Ruth's first pregnancy by a postcard with the formula $H(G,R) \neq 0$. He enjoyed hobbies. I remember one time in Urbana he was growing plants in his kitchen under a special light. A more permanent interest was photography. From time to time he took trips by car in the desert or in the mountains to capture their beauty on film. On such a trip he usually sent me the funniest or most absurd postcard he could find, with a brief comment. I tried to return this gesture, but he won our quirky postcard contest hands down.

After Gerhard moved to Berkeley I saw him less often, except during a visit there for the spring term and summer of 1963 that he arranged. Though our mathematical interests were not as close then as earlier, he was a great host as always. I remember many pleasant hours spent at his house in El Cerrito with its magnificent view from the living room across the bay to San Francisco and the Golden Gate bridge. I am sad that we were together so rarely after that time, for he was a true friend who made a real difference in my life, especially in my postdoctoral days.

Andy Magid

The Hochschild-Mostow Group

Gerhard Hochschild is well known to the many mathematicians who employ his eponymous homology theory or spectral sequence (the latter is also named for Serre). Another algebraic notion that bears his name, the Hochschild-Mostow group, is probably less familiar; it is my object in this section of this memorial article to share this beautiful and important construction with a wider audience.

The Hochschild-Mostow group is a functor from groups to complex pro-affine algebraic groups. More precisely, let G be an analytic group, an algebraic group, or a finitely generated group. Consider all the complex representations $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$, where ρ is also required to be analytic or algebraic if G is. The Hochschild-Mostow group $A(G)$ of G is a complex pro-affine algebraic group for which there exists a homomorphism $P\colon G \to A(G)$ such that for any representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ as above there is associated a unique algebraic representation $\hat{\rho}\colon A(G) \to \mathrm{GL}_n(\mathbb{C})$ such that $\hat{\rho} = \rho \circ P$. Of course algebraic representations of $A(G)$, when preceded by P, yield representations of G; these turn out to be analytic or algebraic if G is. Thus the representation theory of G is the same as the representation theory of $A(G)$.

A pro-affine algebraic group, by definition, is an inverse limit, with surjective transition maps, of affine algebraic groups. Its coordinate ring is the corresponding direct limit of the coordinate rings of the groups in the inverse system; these are all commutative Hopf algebras, as is their direct limit. Any commutative Hopf algebra over a field is a direct limit of its finitely generated Hopf subalgebras. Over a characteristic zero algebraically closed field, the affine algebraic groups associated to these finitely generated Hopf subalgebras form an inverse system with surjective transition maps. So, over C, pro-affine algebraic groups and commutative Hopf algebras are simply dual. In the papers introducing and studying $A(G)$, the authors treat the Hopf algebra as the basic object.

Those authors, of course, are G. D. Mostow and Gerhard P. Hochschild. Their collaborative work on this topic, which was published in the *American Journal of Mathematics* between 1957 and 1969, stresses the concept of a representative function on the group G. By this is meant a function $f : G \to \mathbb{C}$ whose left (equivalently right, or both) translates by the elements of G span over $\mathbb{C}$ a finite-dimensional vector space. If G is analytic or algebraic, f is required to be so as well. An example of such a function is a matrix coordinate function of a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, and in fact every representative function arises from some such matrix representation.

The set of all representative functions on G is denoted $R(G)$. The authors show it to be a Hopf algebra and hence associated to a pro-affine algebraic group as above. This latter is, of course, $A(G)$. In fact, the authors introduce $A(G)$ more directly: G acts on $R(G)$ as right translations, and then $A(G)$ can be seen to be the group of all $\mathbb{C}$-algebra automorphisms of $R(G)$ which commute with all these translations (the group of proper

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automorphisms of $R(G)$, in the authors' original terminology).

The theory, therefore, can be expressed strictly in terms of the algebra $R(G)$ of representative functions on G and the group of proper automorphisms of this algebra. Notice that in this formulation neither Hopf algebra nor pro-affine group methods are required, a fact today's readers of the papers can appreciate as did those of the 1950s and 1960s.

One of the significant accomplishments of the original papers was the description of $A(G)$ in the case in which G is an analytic group. It turns out in this case that $A(G)$ is a semidirect product $H \rtimes T$ where T is a pro-torus and H is an analytic group that has the structure of an algebraic variety such that left translations of H by elements of H are morphisms (the right translations need not be morphisms). It should be intuitively clear how these notions—pro-torus and left algebraic group—are intended. In fact, they are much simpler to explain in terms of the algebra $R(G)$, and this is what Hochschild and Mostow do: they show how to write $R(G)$ as a tensor product $R \otimes Q$ where R is an finitely generated subalgebra of $R(G)$ stable under right translations and Q is the group algebra of an infinite-dimensional $\mathbb{Q}$ vector space.

The description of $A(G)$ in the case where G is affine algebraic is much simpler: $P\colon G \to A(G)$ is an isomorphism. The case where G is a compact topological group and the representative functions are continuous and real, although it doesn't fit into the context surveyed here, also yields an isomorphism, which Hochschild demonstrates, for example, in *The Structure of Lie Groups*, where he points out that this is a way to understand the Tannaka duality theorem.

More generally, the Hochschild-Mostow group is linked with the Grothendieck-Saavedra theory of Tannakian categories: the category C of finite-dimensional complex G modules is a tensored abelian category, and $\mathrm{Hom}_G(-, R(G))$ can be seen to be a fiber functor on C. Its tensor automorphisms, then, can be seen to be the group of proper automorphisms of $R(G)$, namely $A(G)$.

The case where G is finitely generated is harder than both the analytic and algebraic cases, although its investigation has proven to be very fruitful. This has been done by a number of authors using a number of names, usually some variant of "pro-algebraic completion" for $A(G)$. The terminology "Hochschild-Mostow group" was introduced by Alexander Lubotzky in his thesis (Bar-Ilan University, 1979, in Hebrew), where he used it to point out the connection between the Tannaka duality property and the congruence subgroup property for discrete groups. Lubotzky and his students and collaborators (including myself) have continued the study of $A(G)$ for G finitely generated to the present day.

And to end on a more personal note: in the 1970s I had stumbled across left algebraic groups in the characteristic p context in trying to understand universal étale covers of affine algebraic groups in positive characteristic. A helpful reviewer of a research proposal pointed me toward the complex left algebraic groups of Hochschild and Mostow's work. In studying that work in the geometric language I favored I was able to make some additional progress for complex analytic groups. One of Hochschild's former students took me with him to see Gerhard during an AMS meeting in San Francisco, and eventually this resulted in my spending a sabbatical at Berkeley for the spring quarter of 1980. I was in an office right down the hall from Gerhard. We met in the late afternoon nearly every day to discuss mathematics and other topics. Although Gerhard was already then, thirty years ago, the age I am now, his energy and enthusiasm were remarkable, as well as his generosity with his time for junior colleagues. I believe it was then that he told me (although it could have been a couple of years later when he came to a conference at the University of Oklahoma) that during World War II, when he had been stationed at Ft. Sill in Oklahoma, he, along with the other noncitizen soldiers in his unit, had been taken to the local county courthouse for naturalization. Ever since, I have proudly claimed that Gerhard was a citizen of Oklahoma. Of course, he was a citizen of the world, and all of us who came to know him personally and mathematically are proud of that connection.

G. D. Mostow

Gerhard Hochschild as Collaborator

Gerhard Hochschild arrived in Princeton in September 1938 with a master's degree from Cape Town University, where he had majored in both math and physics. He had decided to drop physics; Gerhard said "I can raise and lower indices of tensors as well as the best physicists, but I find mathematics more satisfying." He chose Claude Chevalley for his thesis advisor. After three and a half years in Princeton and finishing his thesis, Gerhard had to go into the U.S. military service. He returned to Princeton for a semester after the war and in 1956 for a one-year visit to the Institute for Advanced Study.

I first became acquainted with Gerhard Hochschild at the Institute for Advanced Study. In those days at the Institute, everyone came to

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A typical Hochschild landscape photo.

tea, and members discussed their work with one another enthusiastically.

One cannot write about Gerhard comprehensively without mentioning his charisma. Some of his charisma resulted from his colorful criticism of the hypocrisies that abound in all large organizations. In the army, even though he was a recent immigrant to the United States, he impressed his fellow soldiers with the virtuosity of his profanity. I learned this from the famous geometric measure theorist Herbert Federer, who served in Gerhard's unit at Aberdeen Proving Grounds.

In general, he exhibited a tolerant viewpoint in his relations with people. As a teenager growing up in Berlin, he had suffered from the ignominies imposed on Jews—for example, he was once accosted by Nazi thugs and pushed around, with the police standing idly by. Although he was adamant about never returning to Germany, he did not allow his feelings to affect his personal relationships. For example, later he developed a lasting friendship with Professor Friedrich Hirzebruch and his wife that began when they visited the University of California at Berkeley in the early 1960s.

Gerhard had certain fixed standards that nothing could change. For example, in the many visits that I made to his photography lab, he showed me pictures of landscapes principally consisting of interestingly shaped stones colored gray. Never once did he show me a picture that included a person or that included color. He took many such pictures that I thought were repetitious, but to him they represented an effort to capture perfection. Anything short of that was a compromise that would disappoint him.

His striving for perfect rigor also is reflected in his textbook on Lie Groups for graduate students, which is "uncompromising in requiring austerity of thought", in the words of Gerhard's thesis advisor Claude Chevalley.

My mathematical collaboration with Hochschild began in 1957, when we were both members of the school of mathematics of the Institute. Our mathematical backgrounds were quite different. Gerhard had published papers on the theory of bimodules, cohomology groups of associative algebras, and the application of cohomology to number theory. My previous publications were on geometric aspects of Lie Groups and had virtually no overlap with his at that time. Also, our temperaments were very different, hardly predictive of a joint collaboration that produced seventeen papers.

Our collaboration resulted from the fact that Gerhard was interested in relating Tannaka duality to his earlier work on bimodules. At the time we began to work together, he was fascinated with the Tannaka duality theorem, which was carefully presented in the 1946 Claude Chevalley book *Theory of Lie Groups*. The only drawback was that the indirect definition of multiplication in the "dual" could have been more satisfactory.

Tannaka duality was initially intended to extend Pontryagin duality to a wider class of Lie groups. In order to give a satisfactory statement of Tannaka duality that is formulated in terms of standard mathematical objects, one needs to clarify some elementary definitions.

Let X be a set, G be a group, k be a field, and $F = F(X,k)$ be an algebra of functions on X with values in k. A *left action* of G on X is a map $\mu\colon G \times X \to X$ satisfying

$$\mu(a_2, \mu(a_1, x)) = \mu(a_2 a_1, x) \quad \text{for all } (a_1, a_2, x) \text{ in } G \times G \times X.$$

A *right action* of G on X is a map $\nu\colon X \times G \to X$ satisfying

$$\nu(\nu(x, a_1), a_2) = \nu(x, a_1 a_2) \quad \text{for all } (x, a_1, a_2) \text{ in } X \times G \times G.$$

For any left action μ on X, $f \in F$ and $a \in G$, let $(f.a)$ denote the function on X defined by

$$(f.a)(x) = f(\mu(a, x)).$$

Assume $f.a$ is in F for all f in F, a in G. Then the map $f \to f.a$ is easily seen to be a right action on F.

Similarly, for any right action ν on X, $f \in F$, and $a \in G$, $(a.f)$ denotes the function defined by

$$(a.f)(x) = f(\nu(x, a)).$$

Assume $a.f$ is in F for all f in F, a in G. The map f to $a.f$ is easily seen to be a left action on F.

The maps $f \to f.a$ and $f \to a.f$ are called right and left translates of f, respectively.

The k-valued function f on G is called a *representative function* if the linear span of the set $\{f.a, a \in G\}$ of right translate functions is a finite-dimensional vector space over k.

It is an easy theorem that a k-valued function f is a representative function if and only if the left translates of f span a finite-dimensional vector space over k. A similar result is true for the span of the right translates and, furthermore, for the span of both translates.

Let $\mathrm{Repr}(G)$ denote the set of all representative functions on G; this is a ring. An automorphism of the ring $\mathrm{Repr}(G)$ is called a proper automorphism if it commutes with all right translations by G and fixes all the constant functions. For example, any left translation on $\mathrm{Repr}(G)$ is a proper automorphism.

Let $A(G)$ denote the set of all proper automorphisms. $A(G)$ is a group with multiplication given by composition.

Tannaka duality is equivalent to the assertion: if $k = \mathbf{R}$ and G is a compact Lie group, then $A(G)$ coincides with the group of all left translations of G.

This formulation makes it possible to derive many relationships. For example, if $k = \mathbf{C}$ then $A(G)$ is the "universal" complexification of G. Moreover, the ring $\mathrm{Repr}(G)$ is a Hopf algebra. Many interesting relations between G, $\mathrm{Repr}(G)$, and $A(G)$ can be observed. For example, these relations were useful to Alex Lubotzky in his study of discrete subgroups.

During our long collaboration, we worked out the various insights provided by the study of the pro–algebraic group $A(G)$. The functor $G \to A(G)$ was named the Hochschild-Mostow functor by Alex Lubotzky.

It gives me much pleasure to recall our years of collaboration and warm friendship.

Walter Ferrer Santos

Interview with Pierre Cartier

The present interview occurred while the author and Pierre Cartier were participating in the "Segundo encuentro de historia conceptual de la matemática" that took place November 22–27, 2010, Córdoba, Argentina.

Walter Ferrer: Professor Cartier, when did you meet Gerhard Hochschild for the first time?

Pierre Cartier: To the best of my recollection, it was in June 1951, at the end of my first year as a student at the École Normale Supérieure, Paris. I was invited by Henri Cartan and Samuel Eilenberg—both were my advisors for the year—to participate in one of the closed-door meetings of the Bourbaki group. This meeting took place in Pelvoux, a small resort town in the Alps, and there I met for the first time people like C. Chevalley, J. Delsarte, J. Dieudonné, and A. Weil, the founders of Bourbaki. The senior participants asked me to go and pick up Gerhard at the train station, where he had arrived on the night train. [11] In the meeting, I remember discussions about Lie groups, according to a draft written by Laurent Schwartz, and also on commutative algebra. Hochschild was very much interested in both.

WF: Can you tell me about his later participation in meetings of the Bourbaki group?

PC: According to the minutes of Bourbaki, he participated again in the meetings in June 1952 and August 1954. I was told that he participated in the 1954 congress accompanied by his wife Ruth and John Tate. On the light side, I was told by the participants that the innkeeper used to refer to the visitors as the "lady with the two Americans". I did not go to either of these meetings, but I heard, from Chevalley, about Hochschild's visit and his deep interest in the topics covered. In that period the group was very much interested in the subjects of commutative algebra (including the new homological methods of Serre and M. Auslander) and Lie groups. It was about that time that J.-P. Serre—by then a very active member of the group—published his two papers with Hochschild on what are now called Hochschild-Serre spectral sequences.

WF: Did you meet him afterward while he was at Urbana or Berkeley?

PC: I remember very well two of my visits with him. In the fall 1957, I was a member of the Institute for Advanced Study at Princeton—shortly after he left it. I had been invited by Dieudonné—who by that time had a position at Northwestern University—to visit the Chicago area, including Urbana. In Urbana I was very much impressed by Joseph Doob, the well-known probabilist, who served there until his retirement in 1978, and also by Gerhard Hochschild, who was on the faculty at that time. In those days there was a great interest in algebraic groups, and I had just participated in the famous Chevalley seminar, and we talked extensively about the subject. On this occasion, he handed me the first of his series of papers on representative functions.[12] That was just after I had published a short note on the Tannaka duality for algebraic groups in 1956,[13] and we talked at length about these topics from that perspective.

My next meeting with him was in Berkeley [in] 1984. He had just recently retired, but we still had many common mathematical interests.

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[11] *Hochschild mentioned once to the author about his participation in the meeting, and the surprise that he felt when he saw the "very young boy with a Boy Scout look that went to pick me up at the station" later participating fully in the discussions of the group.*

[12] *This paper was the start of a long collaboration with G. D. Mostow.*

[13] *Cartier is referring to his paper "Dualité de Tannaka des groupes et des algèbres de Lie",* C. R. Acad. Sci. Paris, *vol. 242 (1956), pp. 322–325.*

I remember his pride when he showed me the miniature train—a very sophisticated device—that he had built in the basement of his house in El Cerrito. To me the train was a very symbolic illustration of his precise and accurate mind.

I remember that he attended the seminar on cyclic cohomology that I gave at MSRI on that occasion and that at the end of the day he took me to the airport as I was leaving for Asia. Since the plane was late, we could enjoy a wonderful sunset together.

WF: What was the influence of Hochschild's mathematics in your own personal work?

PC: At the time of our first contacts, I was deeply influenced by the methods of homological algebra, especially after some lectures by Henri Cartan and the publication of the now classic *Homological Algebra* by Cartan and Eilenberg. In the fall of 1955, having Mac Lane attending my lectures, I invented the cohomology theory of coalgebras. It was a dual version of Hochschild's work on what is now called Hochschild cohomology, a cohomology theory of algebras with coefficients on a bimodule. Then I was avidly reading all his papers on homological algebra, and I recall A. Weil mentioning to me very enthusiastically Gerhard's work with Nakayama on the homological methods in class field theory. Also, my thesis—defended in September 1958—was devoted to problems of Lie algebras and algebraic groups in characteristic p, and in the process of finishing it, his papers in this field were very useful. I would also like to mention that at the end of the 1950s I had been drafted into the French army and hence was unable to travel abroad for a period of almost three years. J.-P. Serre—who was regularly visiting the USA—was very often the link between Gerhard and myself.

WF: In many conversations Gerhard told me that you were the person who introduced him to the notion of a Hopf algebra. Can you tell me about this?

PC: My work on Hopf algebras was previous to my thesis and was developed in relation to the cohomology theory that I mentioned earlier, even if it did not appear explicitly in the thesis. I think that my major discovery in this direction was that, in order to apply the full power of Hopf theory to the realm of algebraic groups and Lie algebras, it was necessary to relax all kinds of restrictions that were customary in this theory, in particular the commutativity and the condition of being graded.[14] I suppose that, when we discussed his paper on representative functions in Urbana, I mentioned to him the possibility of using Hopf algebras in this setting.

WF: Is there any additional comment about Hochschild you would like to share with us?

[14] *I recall that Henri Cartan advised me in that respect not to "overbourbakaise" (P.C.)*

PC: I would like to comment that, even though we were not in direct personal contact too frequently, we corresponded regularly, and throughout his career he sent me his reprints which I read avidly. On a more personal note, concerning his youth in South Africa, I remember very well that my mother in France had helped many German and Austrian Jews to escape the Nazi regime. For many years I was able to collect stamps from many countries like Canada, Israel, and South Africa, taken from the extensive correspondence she had with the emigres she had helped. Sometimes I wonder if some of the stamps came from Gerhard or from his family.

Books by Gerhard Hochschild

The Structure of Lie Groups, Holden-Day Series in Mathematics, San Francisco-London-Amsterdam: Holden-Day, Inc. (1965).

A Second Introduction to Analytic Geometry, San Francisco-Cambridge-London-Amsterdam: Holden-Day, Inc. (1968).

Introduction to Affine Algebraic Groups, San Francisco-Cambridge-London-Amsterdam: Holden-Day, Inc. (1971).

Basic Theory of Algebraic Groups and Lie Algebras, Graduate Texts in Mathematics, 75, New York-Heidelberg-Berlin: Springer-Verlag (1981).

Perspectives of Elementary Mathematics, New York-Heidelberg-Berlin: Springer-Verlag (1983).

Calvin Moore

Gerhard Hochschild at Berkeley

Gerhard Hochschild's connection with Berkeley dates to 1955, when he accepted a position as visiting professor of mathematics at the University of California, Berkeley, for the academic year 1955–56. When he arrived, he found a mathematics department that was very different from the Berkeley mathematics department of today or indeed from what it became in the early 1960s. It was a small department with a total of nineteen faculty members in all professorial ranks. The department had some very real strengths, including a distinguished group of statisticians and probabilists, but this group had split off into a newly formed, separate department of statistics in 1955.

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There were also strengths in a number of areas—analysis, including PDEs and functional analysis; computational number theory; and logic. But algebra, as well as geometry and topology, fields that were rapidly developing at the time, were seriously underrepresented among the faculty in 1955. The department was poised to grow substantially in the next few years, and the hope was, through this growth, to remedy these programmatic weaknesses. Inviting Hochschild, a prominent senior algebraist of considerable stature, to visit for the 1955–56 year was a first step toward this program. Hochschild did enjoy his year's visit.

However, from past experience, the department knew that it would encounter difficulty in attracting to Berkeley permanently senior faculty in these areas of weakness because of concern about isolation. Further, it would be difficult to attract outstanding junior faculty in these areas absent senior faculty. Nevertheless, the department succeeded in hiring three outstanding assistant professors in these areas in 1956—Emery Thomas, Bertram Kostant, and James Eells. The department, under John Kelley's leadership, decided on a strategy of hiring clusters of senior faculty in the underrepresented areas. In the fall of 1957 the department approached Hochschild and Maxwell Rosenlicht and offered them both full professorships at Berkeley to begin in 1958. Each was informed of the offer to the other, and the strategy was proven to be a success when both accepted, thus providing a core of senior algebraists at Berkeley.

A shack in the Bay Area, photograph by Hochschild.

The following year, this cluster strategy was applied in geometry and topology, with offers made to Edwin Spanier and Shiing-Shen Chern. Both accepted, with Spanier arriving in 1959 and Chern deferring arrival for a year because of prior plans. Knowing that Hochschild and Rosenlicht had already moved to Berkeley perhaps increased the likelihood that Chern and Spanier would accept these offers. Also, in 1960, Stephen Smale, Morris Hirsch, and Glen Bredon accepted appointments as assistant professors. By 1960 the number of faculty in professorial ranks had grown to forty-four from nineteen five years earlier, with a much improved balance of fields and a vigorous and vibrant intellectual atmosphere. Hochschild's appointment and his decision to come to Berkeley were among several key factors in this change.

Beginning a bit before he first visited Berkeley, Gerhard's research interests began to shift from the homology of associative algebras and applications of homological algebra to class field theory, to the study of Lie algebras and Lie groups, especially algebraic Lie algebras and Lie groups, their linear representations, and their cohomology, and Lie group and Lie algebra extensions. When he arrived permanently in Berkeley in 1958, Gerhard found in Bert Kostant someone with similar interests in Lie algebras and Lie groups, and this common interest led to a collaboration, with two published papers on differential forms and cohomology of Lie algebras. Kostant, however, was lured away by MIT in 1961. In that same year the author arrived as a brand-new assistant professor with interests in extensions and cohomology of topological groups. These common interests led to many discussions of these topics but not to any joint publications.

Over the next twenty-five years Gerhard produced a steady stream of important and fascinating papers on Lie algebras, Lie groups, their representations, and cohomology. Many of these were done in collaboration with Dan Mostow of Yale University and represented a career-long scientific collaboration and friendship. He also became interested in Hopf algebras and wrote several papers on them and their connections with Lie groups. Gerhard supervised the doctoral dissertations of 22 students at UC Berkeley, and overall in his career he had 26 doctoral students with 122 descendants, according to the Mathematics Genealogy Project. His achievements were recognized in his election to the National Academy of Sciences and to the American Academy of Arts and Sciences. In 1980 the American Mathematical Society awarded Gerhard its Steele Prize for work of fundamental or lasting importance, citing in particular five papers published from 1945 to 1952 on homological algebra and its applications. Gerhard replied that although he was deeply honored by the Society's consideration of his work for a prize, he was, for personal reasons, unable to accept the Steele award. A friend has commented that the reason was simply that Gerhard did not believe in prizes.

As a senior algebraist in the department, he was called on frequently for advice and counsel on departmental matters. His opinions and advice were wise and incisive and offered with his characteristic ironic wit. He also served as advisor and mentor for many junior faculty members in algebra, and his guidance was deeply appreciated by the many colleagues who sought it. He had, however, a lifelong aversion to and dislike of what he termed academic bureaucracy. One result of this was that he consciously, consistently, and with self-deprecating good humor avoided all attempts to convince him to assume any administrative positions such as chair or vice chair in the department.

Gerhard was a longtime smoker who, when he did give it up, continued for some time to carry around with him an unlit cigarette between his fingers and sometimes between his lips, perhaps as a memento of his former smoking days. He changed the cigarette occasionally when it became too decrepit. But he never lit it!

The university had a policy in place up until July 1, 1982, that required mandatory retirement of tenured faculty members on July 1 following their sixty-seventh birthday. Effective July 1, 1982, this policy had been changed to require mandatory retirement on July 1 following the seventieth birthday. As Gerhard was born on April 29, 1915, he was by two months in the last cohort that was subject to the earlier mandatory retirement age. Gerhard saw this policy as academic bureaucracy at its worst. The end result was that he retired on July 1, 1982, and entered the university's phased retirement program for three years, teaching part time until retiring fully on July 1, 1985.

During his years in Berkeley, Gerhard became increasingly interested in landscape photography, pursuing this hobby with a deep and abiding commitment. Entirely self-taught, he relied on reading books on photography. He was particularly attracted to the desert landscape of the U.S. southwest. He would periodically go off on expeditions by himself, with his camera gear, which included a Hasselblad 4×4 and later a view camera, driving thousands of miles looking for just the right scene and just the right lighting, sometimes staying away for a month. His favorite site was in southeastern Utah, although Alaska was also a destination of his expeditions, and he also photographed the San Francisco Bay. The famous California photographer Ansel Adams was his model and his hero, and Gerhard's work does remind one of some of the work of Ansel Adams. Gerhard was encouraged by friends to have a show featuring his work, but he declined all such efforts. His photography occupied him for decades, but toward the end of his life, his health did not permit him to go off on these long expeditions.

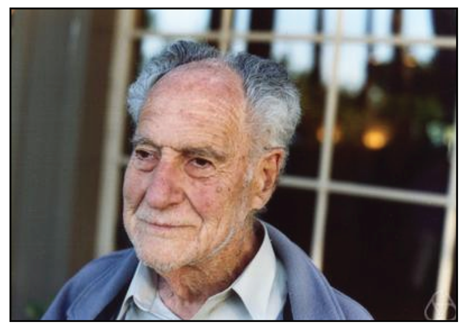
Gerhard died peacefully at home after a long and satisfying life on July 8, 2010, at the age of ninety-five with his daughter Ann at his bedside. He is survived by his daughter Ann, his son Peter, and two grandchildren. His beloved wife Ruth predeceased him in 2005.

Bertram Kostant

Hochschild Memorial

In 1956 I accepted an offer from UC Berkeley for an assistant professorship. About five years later I made an agonizing decision to leave Berkeley and accept an offer for a full professorship at MIT. One of the many reasons this decision was so painful to me was that it would severely diminish my close relationship with Gerhard Hochschild.

Hochschild arrived in Berkeley (I believe) in the late 1950s, and right from the beginning we recognized in each other kindred spirits. Whenever I came to the department I looked forward to spending time with him, either in his smoke-filled office or over lunch. Any and every topic, mathematical or otherwise, was up for discussion. Even when we strongly disagreed on some topic, he exuded so much charm and likeability that I found it impossible to become upset with him.



Photograph courtesy of George Bergman.

Gerhard in 2003.

Mathematically we collaborated on two papers about which I will shortly comment. Aside from these papers, we both became interested in the newly developing field of algebraic groups. We focused our attention on the 1956–58 Séminaire C. Chevalley on the Classification des groupes de Lie algébriques. However, in order to read the output of this Séminaire, we both needed to learn some basic algebraic geometry. To do this we alternated in lecturing to each other, using as a basic text Chevalley's book *Fondements de la Géométrie Algébrique*. Happily, this was a successful effort.

Our first joint paper, coauthored with Alex Rosenberg, was entitled *Differential forms on regular affine algebras*. It appeared in *TAMS* **102** (1962), No. 3, 383–408. The main result of the paper (the HKR theorem) is still highly cited and played an important role in the development of cyclic cohomology. The second paper proved that, for complex reductive homogeneous spaces, the de Rham cohomology can be computed using only holomorphic differential forms. It was cited by A. Grothendieck in a work establishing a very interesting, far more general theorem.

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Hochschild disliked ceremony and academic glorification of any kind. I understand it took a great deal of arm-twisting for him to accept membership in the National Academy of Sciences. Even so, he refused to cooperate and fill out the necessary forms that went with membership. Hochschild cohomology and homology are central objects in modern algebra. Nevertheless, he enjoyed making light of his own fundamental discoveries and all the fuss made about them. In other matters as well, he seemed to take pleasure in swimming upstream against the flow of academic behavior. I must say that this kind of rebelliousness resonated with me, but he was much more courageous than I was. At any rate my friendship with him was one of the happiest experiences of my life.

George M. Bergman

Some Inadequate Recollections of Gerhard

I wish I had known Gerhard better, mathematically and personally.

When, while a senior at Berkeley, I got my first mainstream mathematical result (an answer to an old ring-theoretic question), I went to Professor Hochschild, and he helped me put it into good form for publication. When I came back to Berkeley as a faculty member four and a half years later, Gerhard was a friendly presence, and over the years, he made many little helpful comments on drafts of my papers.

Courtesy of Martin Moskowitz.

A computer-generated picture presented by Gerhard to Martin Moskowitz.

He liked to talk pessimistically: "If we were really happy, of course, we wouldn't be doing mathematics." (I think this was typical of the impact of Freud on many people of my father's generation.)

As Berkeley's mathematics department photographer, I inform each of my colleagues, when five years have passed since the last photograph I've taken of them, that it is time to do another. In recent decades, Gerhard would respond with "You are recording the course of senile decay, Bergman?" But he never resisted letting me take the picture.

He had an interest in photography himself, and one time, when I brought him outside to re-photograph him, he held up a large envelope in which he had gotten some material from Kodak and wanted me to include it in the picture. I didn't know why, but I complied. Unfortunately, my photo didn't include the whole envelope; when I showed him the picture, he chided me for having left out the part showing the product name: *Ultra Filter*.

Dennis Sullivan came here to speak in 2008, long after Gerhard had stopped coming by. He was disappointed not to see him at his talk. He commented, "I wonder whether he knows that people eat Hochschild cohomology for breakfast these days?"

Looking through a birthday collection for Anthony Joseph in the library not long ago, I noticed a contribution written in Latin. One section was about the "Complexus de Rham-Koszul-Hochschildianus". I emailed Gerhard and let him know that he had been Latinized.

I sometimes thought that if, as a retiree, I was eventually required to share my office, I would ask whether the person I shared it with could be Hochschild. Though he was no longer coming in frequently, the times when he did would be a pleasure. But that was not to be.

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Martin Moskowitz

Some Reminiscences

I met Gerhard in 1962, when I took qualifying exams at UC Berkeley and he was chair of my Algebra Committee for that exam. When the results of the exam were in, I asked him if he would agree to be my thesis advisor. He did, and as an example of his generosity, he put off a sabbatical to do it.

Several of his students would meet Gerhard once a week, after which he would take us for coffee on the north side. He would pay. (In fact, many years later, when he had already retired and I was fully employed, he still insisted on paying!) At one of those meetings I discovered a gap in my thesis (which involved locally compact abelian groups and so every statement had a dual). I told him I would try to fill it for the following week. When I presented this, he asked why didn't I do the dual and proceeded to tell me the story of a Jewish mother who gave her son two ties for his birthday. The next time he visited he dutifully wore one of them. His mother asked "Didn't you like the other one?" My colleague, Ray Hoobler, who was a student of Gerhard a few years later, told me that when Gerhard first took him on, he gave him a ride home and had Ray terrified as he frequently took both hands off the wheel, gesticulating wildly as he enthusiastically told Ray about possible thesis topics.

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When I completed my dissertation in 1964, Gerhard simply informed me that I was going to Chicago. He said the weather wasn't good, but the mathematics was, so that's where I was going. Many years later he observed that, in the 1960s, we were in a kind of golden age when research was very active and universally respected. It was the only time in his memory when mathematics wasn't underfunded.

When I returned to Berkeley some years later, he mentioned that he had been a member of Bourbaki. He claimed that, when he arrived for his first session in Paris, he asked the cab driver to take him to the Eiffel Tower "because that was the only thing I knew how to say in French". Later I learned that he spoke excellent French.

He also mentioned that, as a young man, he was at the Institute for Advanced Study working on class field theory. There would be tea every day at 3:00, and he would meet Andre Weil, who would ask about his progress. After several months he solved the problem he was working on and rushed to Weil's office to tell him about it. As he opened the door, Weil said, "What you are about to tell me is in Nakayama's paper of 1941." Gerhard then wryly remarked "if it doesn't kill you it will make you stronger" (I believe this is, more or less, a quote from Nietzsche).

At some point in the 1970s Gerhard was put on the committee to select the new instructors at Berkeley and given literature from the administration for guidance, which included affirmative action policies. He called the dean and told him he couldn't do this because ethnic and racial criteria have no place in mathematics. Having seen Nazis up close and personal and having had the difference between "Jewish" and "Aryan" mathematics made clear to him, he could not, in turn, consider racial/ethnic criteria in selecting new instructors. The dean then argued that there is a difference between "negative racism", which the Nazis practiced, and the current policy, whose purpose was to have a positive outcome. Gerhard did not regard this as adequate and resigned from the committee.

I was in Berkeley and participated in a Festschrift for Gerhard on the occasion of his sixty-fifth birthday. As it happened, that was exactly the time he was made a member of the National Academy of Sciences. He asked me "Is there any way to get out of this?" and I responded, "the only way is to die."

In 1998 I visited Berkeley on sabbatical. Once there I asked the department for Internet access and was told this would cost \$50 for the semester, which I paid. Then I went down to the math library and asked about library use. I was told this would cost \$100 for the semester, but that it also included Internet access. I naively said fine, I just paid \$50 for that and so I would be happy to pay another \$50 and have both, but the answer was no! The first fifty was paid to the math department and the hundred goes to the library. When I mentioned this to Gerhard, he offered to send them a check for \$100 "to shame them into relenting." I told him not to bother because one can't shame people who have no shame.

I'm personally directly knowledgeable about only a few of Gerhard's many and excellent papers. Among them are a series with Dan Mostow on faithful representations of real and complex Lie groups, which globalize Ado's theorem and which, because of their importance and trenchant character, I mention here. The generalities concerning representative functions are dealt with in Andy Magid's discussion in the present article.

Theorem. Let G be a connected real or complex Lie group.

In the real case G has a faithful finite dimensional smooth representation if and only if its radical and a Levi factor have such a representation. In the complex case G has a faithful finite-dimensional holomorphic representation if and only if the radical does (since a complex semisimple group always has a faithful representation). If G is either real or complex with a faithfully represented Levi factor and a simply connected radical, then G has a faithful finite-dimensional smooth (resp. holomorphic) that is unipotent on the nilradical.

Gerhard's other articles that I'm familiar with are *Automorphisms of Lie algebras* and his paper on faithfully representing a Lie group together with its automorphism group (1978 *Pacific Journal*, his last published paper). This played a role in something I did with Fred Greenleaf, which appeared in his Festschrift (1980 *Pacific Journal*). There is also an unpublished manuscript on lattices of the early 1980s which, although the methods are quite different, had some influence on my paper in *Math Zeit.* 1999.

After his retirement, Gerhard engaged heavily in photography, taking beautiful images during his many trips to the deserts and forests in Canada, Arizona, New Mexico, and elsewhere. He also got involved in computer-generated images. Among these were fractals. He gave my wife and me several of these stunning images in color which we framed and hung. Finally, Gerhard's work has had an enduring effect on many younger mathematicians. I give two examples. Hossein Abbaspour, my coauthor on our book, *Basic Lie Theory*, (2007 and dedicated to Gerhard), who works in low-dimensional topology, remarked to me that Hochschild cohomology is pervasive in his subject and that "Hochschild is all over my mathematical life." Moreover, in a 2009 dissertation entitled *Cohomological Aspects of Complete Reducibility of Representations*, Yannis Farmakis, âŁ a student of mine, proved, among other things, the following theorem: Let G be a locally compact group, H a closed subgroup, and ρ a continuous representation of G on a real or complex Banach space V. If G/H is compact and has finite volume and $\rho|H$ is completely reducible, then ρ itself is completely reducible. The proof is based on ideas of "injectivity" and "injective resolution" of a continuous module V developed by Hochschild-Mostow in their seminal paper *Cohomology of Lie groups*, which was needed in order to take into account the additional structure (differentiability and integrability) present in the G module V.

Nazih Nahlus

Gerhard Hochschild as My Advisor and Friend

As an advisor, Hochschild was always a wonderful friend. Modest and sincere as a mathematician and as a person, he has had an enormous effect on my life which I will never forget.

I took Hochschild's year-long course on Lie groups in 1980–81. It was a pleasure to follow his beautiful lectures. I decided to ask him whether he would agree to be my advisor. Hochschild replied that, since he was about to retire, it would be preferable for me to work with a younger person, but fortunately for me he agreed to consider it. However, he would make no promises until he saw the results of the qualifying examination. Prior to taking the exam I told Hochschild that I guessed and then proved the cancellation laws for finite-dimensional Lie algebras over a field. At first, he seemed doubtful about this result, but the following day, he told me that my observation follows from the theory of groups with operators.

Although I could have written my thesis exclusively on Lie groups and Lie algebras, Hochschild advised me to learn algebraic groups and Hopf algebras as well. When I needed a recommendation of my teaching for job applications, I told Hochschild that a senior faculty colleague had lost my file after I had been his teaching assistant for three quarters. To my delight, Hochschild immediately made a phone call and in a very deep voice "requested" him to "find his file now!" In 1986 I asked Hochschild whether he was satisfied with the results of my thesis (since I had freed it from the restriction "up to coverings"), he joked by saying that only proving the Riemann hypothesis would impress him. He also advised that, after the Ph.D., one should find one's own path in research.

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Students of Gerhard Hochschild

- George Leger Jr., University of Illinois at Urbana-Champaign (1951)
- Kung Shih, University of Illinois at Urbana-Champaign (1953)
- Ronald Macauley, University of Illinois at Urbana-Champaign (1955)
- Andrzej Białynicki-Birula, University of California, Berkeley (1960)
- Donald Osteberg, University of California, Berkeley (1960)
- Byoung-Song Chwe, University of California, Berkeley (1961)
- James Ax, University of California, Berkeley (1961)
- William Giles, University of California, Berkeley (1962)
- George Rinehart, University of California, Berkeley (1962)
- Martin Moskowitz, University of California, Berkeley (1964)
- Leonard Ross, University of California, Berkeley (1964)
- Theodore Tracewell, University of California, Berkeley (1964)
- Siegfried Grosser, University of California, Berkeley (1965)
- Bostwick Wyman, University of California, Berkeley (1966)
- Gert Almkvist, University of California, Berkeley (1966)
- Raymond Hoobler, University of California, Berkeley (1966)
- Richard Mateosian, University of California, Berkeley (1969)
- Howard Stauffer, University of California, Berkeley (1969)
- David Johnson, University of California, Berkeley (1971)
- Farshid Minbashian, University of California, Berkeley (1972)
- John Reinoehl, University of California, Berkeley (1975)
- Brian Peterson, University of California, Berkeley (1976)
- Walter Ferrer Santos, University of California, Berkeley (1980)
- Rolf Farnsteiner, Universität Hamburg (1982)
- John Ryan, University of California, Berkeley (1984)
- Nazih Nahlus, University of California, Berkeley (1986)

However, I felt at that time that I needed one more bit of direction. This was graciously given by Andy Magid, who encouraged me to work on pro-affine algebraic groups by suggesting a very interesting problem related to the Hochschild-Mostow theory. Consequently, when I visited Hochschild in the summer of 1993, he was very happy with my research progress.

On my sabbatical visit to Berkeley in 1998–99, Hochschild told me that my visit helped him to keep his office another year. Although by then he was out of mathematics, it was greatly inspiring, and I am grateful for those weekly meetings during which he would listen to my ideas and make very general comments.

Hochschild also had a great sense of humor. For instance, when I told him about my concerns about cholesterol, he replied: "do not worry, it will eventually go to zero!" He was also interested in reading books on theoretical physics, even though, as he commented to me, the underlying mathematics in such books for "the general reader" is always brought up in the maximally denigrating way!

Concerning his books, Hochschild's GTM book in 1981, *Basic Theory of Algebraic Groups and Lie Algebras*, is impressive on many levels. It is entirely self-contained, assuming no knowledge beyond the first year of graduate study in algebra. His treatment of commutative algebra, algebraic geometry (leading to coset varieties), and Lie algebras could serve as excellent introductions to such topics. This book covers a great range of material in about 260 pages. Moreover, it is written with maximal clarity by one of the great authorities of the century. Similar comments can be made concerning his book *The Structure of Lie Groups*, starting with Tannaka duality for compact groups, basics of covering spaces and manifolds, and many other topics.