WHAT IS...
Thompson's Group?

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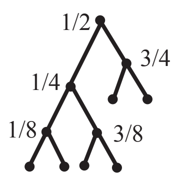
Introduction

Richard J. Thompson's group T is the first known finitely presented infinite simple group. Hundreds of papers have been devoted to this group, to its two finitely presented companions $F \subset T \subset V$, and to related groups.

These groups describe transformations from one rooted binary tree to another. Such trees can be encoded by sequences of rational numbers: The root of the tree is represented by the dyadic fraction $1/2$. Each dyadic fraction $x \cdot (1/2)^k \in (0,1)$ (x odd) is the unique *parent* of two dyadic *children*, namely, $(2x \pm 1) \cdot (1/2)^{k+1}$. A sequence $S = \{a_1 < \cdots < a_n\} \subset (0,1)$ of dyadic fractions is called a *tree sequence of length* $n \geq 0$ if the parent of each $a_i \neq 1/2$ is also in S.



$\{1/8 < 1/4 < 3/8 < 1/2 < 3/4\}$

Exercise. Draw the tree associated with the tree sequence $\{1/8 < 1/4 < 1/2 < 3/4 < 13/16 < 7/8\}$.

One tree sequence $S(1) = \{a_1(1) < \cdots < a_n(1)\}$ is transformed into a second $S(2) = \{a_1(2) < \cdots < a_n(2)\}$ of the same length by the homeomorphism of $[0,1]$ that fixes 0 and 1, takes $a_i(1)$ to $a_i(2)$, and is linear between. These homeomorphisms are the elements of F and are precisely the piecewise linear homeomorphisms with break points at dyadic rationals and slopes powers of 2.

Exercise. [See the section "Manipulating the Elements" in this article for hints to the solution of this nontrivial exercise.] Show that the group F is generated by the two homeomorphisms $X_0, X_1 : [0,1] \to [0,1]$, that linearly extend $\{0 < 1/2 < 3/4 < 1\} \to \{0 < 1/4 < 1/2 < 1\}$ and $\{0 < 1/2 < 3/4 < 7/8 < 1\} \to \{0 < 1/2 < 5/8 < 3/4 < 1\}$.

If we identify the endpoints of $[0,1]$ to form a circle S^1, then F acts on S^1. We obtain T by adding one more generating homeomorphism $Y : S^1 \to S^1$, namely, $x \mapsto x + 1/2$, modulo 1.

The third of the Thompson groups, called V, acts on S^1, contains T, and adds one final *discontinuous* generator Z that fixes the half-open interval $[0, 1/2)$ and interchanges the half-open intervals $[1/2, 3/4)$ and $[3/4, 1)$.

Exercise. In what manner do T and V manipulate trees?

Origins

Matt Brin has called Thompson's groups F, T, and V *chameleons*. The reader may be interested in some of the subjects in which Thompson's groups have arisen: logic (R. J. Thompson), solvable and unsolvable problems in group theory (R. J. Thompson and R. McKenzie), homotopy and category theory (Peter Freyd and Alex Heller), shape theory (Jerzy Dydak and Harold M. Hastings), Teichmüller theory and mapping class groups (Robert C. Penner), dynamic data storage in trees (Daniel D. Sleator, Robert E. Tarjan, and William P. Thurston).

Properties of F

The group F does not fit easily into standard categories. It is non-Abelian and torsion free and contains a free semigroup on two generators, yet it contains no non-Abelian free subgroup. It is not a matrix group. Every subgroup of F is either finite-rank free Abelian or contains an infinite-rank free Abelian subgroup.

Exercise. Find a free Abelian subgroup of F of rank 2. Then find a free Abelian subgroup of F of infinite rank.

[Think of homeomorphisms with disjoint support.]

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For the specialist only, we mention that F is not hyperbolic, is not in the Kropholler hierarchy, and may or may not be automatic.

For the general reader, we will define the notion of *amenability* and explain the most famous unsolved problem about F, first stated by Ross Geoghegan: "Is F amenable?" Amenability arises from the Hausdorff-Banach-Tarski paradox, which states that the unit ball in Euclidean 3-space can be rigidly "torn" into finitely many pieces and then reassembled to form *two* copies of the unit ball. This paradox is possible precisely *because* the group G of rigid symmetries of 3-space is nonamenable.

A group G is amenable if it carries a left-invariant, finitely additive measure of total measure one that is defined on all subsets. Such measures exist in a group G if and only if each finitely generated subgroup H satisfies the following simple geometric condition: H has arbitrarily large finite subsets with relatively small boundary. More precisely, there are arbitrarily large finite subsets X of H for which an arbitrarily small fraction of the elements $x \cdot g$, with $x \in X$ and g a generator, lie outside of X.

Exercise. Show that free Abelian groups satisfy this geometric condition. Show that free non-Abelian groups do not.

[It is known that Thompson's group F contains many of the former and none of the latter. The Hahn-Banach theorem from real analysis was first proved in order to show that Abelian groups are amenable.]

At a recent conference devoted to the group a poll was taken. *Is F amenable?* Twelve participants voted "yes" and twelve voted "no".

Manipulating the Elements of Thompson's Group F

Let $S = \{a_1 < \cdots < a_n\} \subset (0,1)$ be a tree sequence. If $a_i = x \cdot (1/2)^{k_i}$ (x odd), then we call k_i the exponent of a_i.

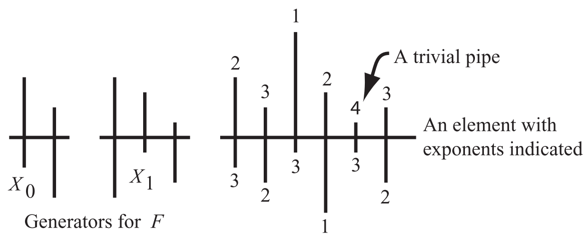
Problem. How can we tell that a sequence of exponents comes from a tree sequence?

Exercise. The sequence $\{k_1, \ldots, k_n\}$ of exponents completely determines the tree sequence.

The exercise is the basis of a geometric representation of the homeomorphism $h \in F$ transforming $S(1)$ into $S(2)$:

Let $I_1, \ldots, I_n$ denote n vertical intervals, called *pipes*, in the xy-plane, each crossing the x axis and ordered from left to right so that I_i represents the pair $(a_i(1), a_i(2))$. In the following diagram the area above the axis represents $S(1)$; the lower half represents $S(2)$. Each parent in $S(1)$ and $S(2)$ should be longer than its children.

Exercise. Relative heights determine exponents, upper and lower, and hence determine $S(1)$ and $S(2)$ and the homeomorphism h.

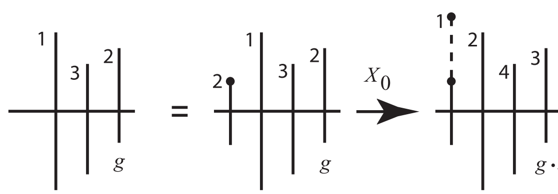


If a pipe is shorter on both ends than its nearest neighbors, then the pipe is called *trivial*.

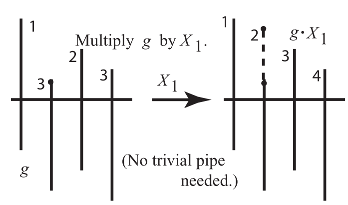
Exercise. The insertion or deletion of a trivial pipe does not change the homeomorphism (element of F) that is represented by the pipe system.

Exercise. A pipe system represents the identity homeomorphism if and only if the upper exponent sequence equals the lower exponent sequence.

A pipe system representing $g \cdot X_0^{\pm 1}$ or $g \cdot X_1^{\pm 1}$ can be formed from a pipe system for g simply by lengthening the upper portion of one pipe: the pipe of exponent 2 to the left of exponent 1 for X_0; the pipe of exponent 3 to the right of exponent 1 and to the left of exponent 2 for X_1. Before the lengthening, insert a trivial pipe in the correct position if necessary. (See the following diagrams.)



Multiply g by X_0: add a trivial pipe, and then promote that pipe to have upper exponent 1.



Problem. How should pipes be lengthened to multiply g on the left by $X_i^{\pm 1}$?

Exercise. Use these moves, their inverses, and the deletion of trivial pipes to reduce an arbitrary pipe system to the empty system (the identity).

Further Reading

See [J. W. Cannon, W. J. Floyd, and W. R. Parry: Introductory notes on Richard Thompson's groups, *L'Enseignement Mathématique*, **42** (1996), 215–256] for references. MathSciNet lists about 180 papers in which Thompson's groups play a major role. The papers of M. G. Brin, V. S. Guba, and M. V. Sapir are good sources for further references.