

Figure 8. Rules of interaction for generalized crossings.

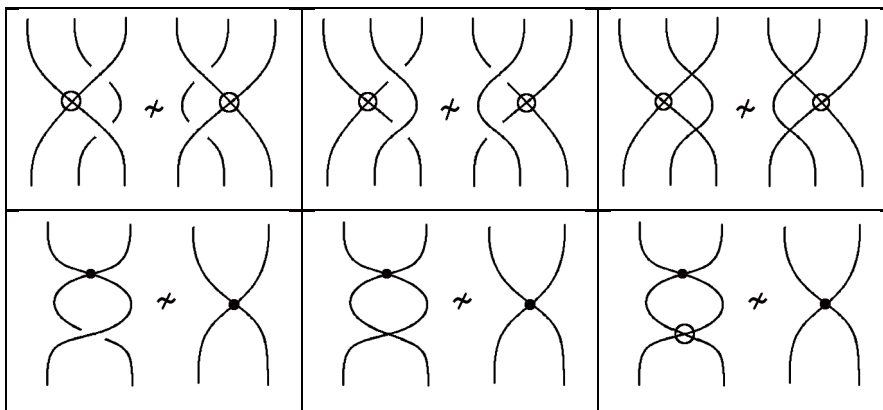


Figure 9. Forbidden moves.

algebra imposes certain *a priori* constraints on the resulting algebraic structures, for example, associativity of multiplication, which seem somehow artificial. More importantly, by insisting on forcing the algebraic structure into a predefined framework, we risk sacrificing useful information.

The combinatorial diagrammatic viewpoint suggests a method for deriving the minimal algebraic structure determined by Reidemeister equivalence of knot diagrams: start by labeling sections of a knot diagram with generators of an algebraic

structure and defining operations where the pieces meet at crossings. The Reidemeister moves then determine axioms for our new algebraic structure.

We can divide the resulting algebraic structures into *arc algebras* where the labels are attached to *arcs*, that is, portions of the knot diagram from one undercrossing point to the next (which can be traced without lifting your pencil), and *semiarc algebras* where the generators are *semiarcs*, that is, portions of the knot diagram obtained by dividing at both over- and undercrossing points.

For example, if we label arcs in a knot diagram with generators and define an operation $x \triangleright y$ to mean “the result of x going under y ”, then the Reidemeister moves tell us the minimal axioms the algebraic structure must satisfy in order to respect the knot structure. The resulting algebraic object, called a *kei* (隼) or *involutory quandle*, was defined in the 1940s by Mituhisa Takasaki [23]. See Figure 12.

Definition. A *kei* is a set X with a map $\triangleright : X \times X \rightarrow X$ satisfying for all $x, y, z \in X$,

- (i) $x \triangleright x = x$,
- (ii) $(x \triangleright y) \triangleright y = x$, and
- (iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

The first axiom says every element is idempotent; the second says that the operation is its own right-inverse, and the third says that in place of associativity, we have self-distributivity. Note the parallel with the group axioms. Examples of kei structures include Abelian groups with $x \triangleright y = 2y - x$ and $\mathbb{Z}[t]/(t^2)$ -modules with $x \triangleright y = tx + (1 - t)y$.

Giving a knot diagram an *orientation* (preferred direction of travel) lets us relax the requirement that the \triangleright operation is its own right-inverse, instead requiring only that \triangleright has a right-inverse operation \triangleright^{-1} . We then think of $x \triangleright y$ as x crossing under y from right to left and $x \triangleright^{-1} y$ as x crossing under y from left to right. The resulting algebraic object is called a *quandle*.

Definition. A *quandle* is a set X with maps $\triangleright, \triangleright^{-1} : X \times X \rightarrow X$ satisfying for all $x, y, z \in X$,

- (i) $x \triangleright x = x$,
- (ii) $(x \triangleright y) \triangleright^{-1} y = (x \triangleright^{-1} y) \triangleright y = x$, and
- (iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

It is not hard to show that in a quandle we have $x \triangleright^{-1} x = x$ for all x , that the inverse operation \triangleright^{-1} is also self-distributive, and that the two operations are mutually distributive. Indeed, these facts can be proved algebraically from the axioms or graphically using Reidemeister moves. Examples of quandle structures include kei, which form a subcategory of the category of quandles, as well as groups, which are quandles under n -fold conjugation $x \triangleright y = y^{-n}xy^n$ for $n \in \mathbb{Z}$, $\mathbb{Z}[t^{\pm 1}]$ -modules

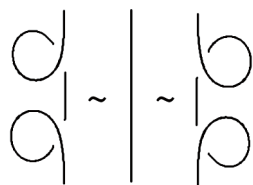


Figure 10. Framed version of Reidemeister I move.

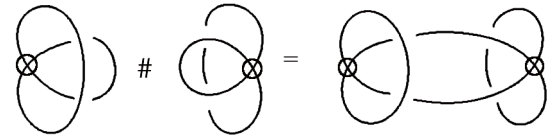


Figure 11. The Kishino virtual knot.

with $x \triangleright y = tx + (1 - t)y$ (called *Alexander quandles*), and symplectic vector spaces with $\mathbf{x} \triangleright \mathbf{y} = \mathbf{x} + \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}$.

The arc algebra arising from framed oriented moves is called a *rack*.

Definition. A *rack* is a set with operations satisfying quandle axioms (ii) and (iii) but not necessarily (i).

The rack axioms are equivalent to the seemingly circular requirement that the functions $f_y : X \rightarrow X$ defined by $f_y(x) = x \triangleright y$ are rack automorphisms. Racks blur the distinction between elements and operators, as every element of a rack is both an element and an automorphism of the algebraic structure. Examples of rack structures include quandles, modules over $\mathbb{Z}[t^{\pm 1}, s]/s(t + s - 1)$ with $x \triangleright y = tx + sy$ (known as (t, s) -racks), and *Coxeter racks*, inner product spaces with $\mathbf{x} \triangleright \mathbf{y}$ given by reflecting \mathbf{x} across \mathbf{y} [9].

To form a more egalitarian algebraic structure, we can divide an oriented knot diagram at both over- and undercrossing points and let the semi-arcs at a crossing act on each other as in Figure 13. The semiarc algebra of an oriented knot is called a *biquandle*; it is defined by a mapping of ordered pairs $B : X \times X \rightarrow X \times X$ satisfying certain invertibility conditions together with the *set-theoretic Yang-Baxter equation*

$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B)$$

where $I : X \rightarrow X$ is the identity map. See [10] for more.

The category of biquandles includes quandles as a subcategory by defining $B(x, y) = (y \triangleright x, x)$. An example of a biquandle which is not a quandle is an *Alexander biquandle*, a module over $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ with $B(x, y) = (ty + (1 - ts)x, sx)$ where $s \neq 1$.

Including new operations at virtual, flat, and singular crossings with axioms determined by the corresponding interaction rules yields a family of related algebraic structures such as *virtual biquandles*, *singular quandles*, *semiquandles*, and more.

Each generalized knot has an associated algebraic object determined by the types of crossings it contains and the equivalence relation defining it. Unoriented knots have fundamental kei; oriented knots have fundamental quandles and biquandles; framed oriented knots have fundamental racks,

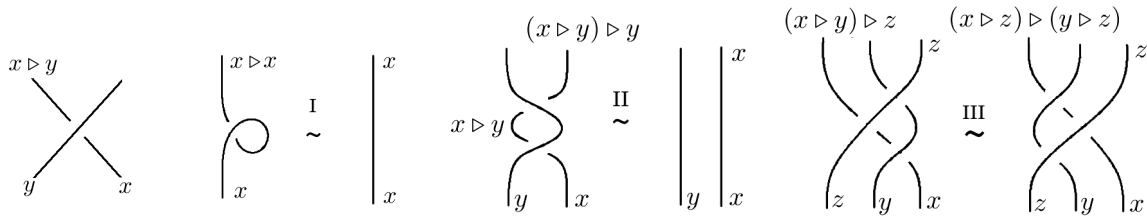


Figure 12. Involutory quandle (kei) operation and axioms.

and so forth. As with generalized knots themselves, our new algebraic objects are equivalence classes of strings of symbols under Reidemeister-style equivalence relations, which we can also understand as algebraic axioms.

Much like groups, examples of kei, quandle, rack, and biquandle structures are found throughout mathematics, lurking just beneath the surface in vector spaces, modules over polynomial rings, Coxeter groups, Hopf algebras, Weyl algebras, permutations, and more. It is important to note that not every kei, quandle, and so forth comes from a specific knot or link; rather, labelings of knots by these algebraic objects are preserved by the Reidemeister moves.

New Knot Invariants

Theorem (Joyce, 1982). *There exists a homeomorphism $f : S^3 \rightarrow S^3$ taking an oriented knot K to another oriented knot K' if and only if the fundamental quandles $Q(K)$ and $Q(K')$ are isomorphic.*

When he introduced the term “quandle” in his 1982 dissertation, David Joyce showed that the fundamental quandle is a *complete* invariant of classical knots up to ambient homeomorphism. Roger Fenn and Colin Rourke later showed that the fundamental rack classifies irreducible framed oriented links in certain three-manifolds. Despite these powerful results, perhaps due to the impracticality of comparing algebraic structures described by generators and relations, most knot theorists shunned arc algebras in favor of other invariants. Indeed, kei, quandles, and racks have been independently rediscovered sufficiently often

to have accumulated an impressive collection of alternative names, such as “crystals”, “distributive groupoids”, and “automorphic sets” [16, 20, 3].

As a complete invariant up to ambient homeomorphism, the knot quandle determines many other classical knot invariants. Indeed, many well-known invariants can be easily derived from the knot quandle: the fundamental group of the knot complement and the Alexander invariants, for instance, can be computed from a presentation of the knot quandle. The hyperbolic volume of a knot has recently been shown to be a quandle cocycle invariant [12]. Even the celebrated Jones polynomial can be understood in terms of deformations of matrix representations of arc and semiarc algebras [7].

With the combinatorial revolution in knot theory, interest in arc and semiarc algebras and their knot invariants has been reinvigorated. One useful method for getting computable invariants from arc algebras is to compute the set of homomorphisms from the knot’s fundamental arc algebra into a target arc algebra T . Such a homomorphism assigns an element of the target object to each arc in a diagram of the knot, and such an assignment determines a unique homomorphism provided the crossing relations are satisfied. These homomorphisms can be pictured as “colorings” of the knot diagram by elements of T .

If the target object T is finite, then the set of homomorphisms will likewise be finite, and we can simply count homomorphisms to get a computable integer-valued invariant known as a *counting invariant*. Figure 14 shows all colorings of the trefoil knot by the three-element kei R_3 with operation table given by

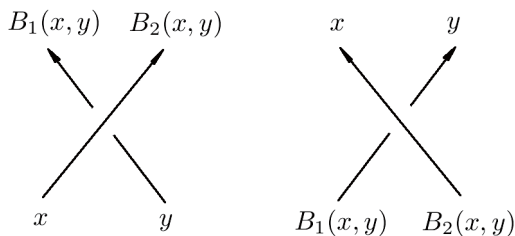


Figure 13. Biquandle operations.

\triangleright	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

The finiteness condition for coloring objects is not required; if the target kei is infinite but has a topology, for instance, then the set of homomorphisms itself is a topological space whose topological properties then become knot invariants [22].

Counting invariants are only the beginning; any invariant of algebraically labeled knot diagrams

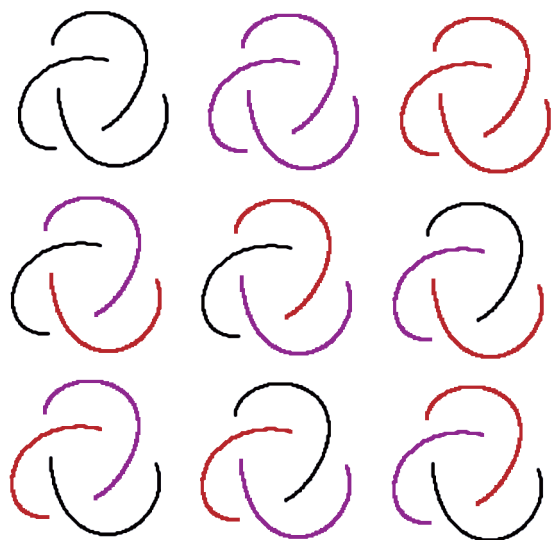


Figure 14. Kei colorings of the trefoil by R_3 .

defines an *enhancement* of the counting invariant by taking the multiset of invariant values over the set of colorings of the knot. One simple example uses the cardinality of the image subkei of a kei homomorphism; instead of counting “1” for each kei homomorphism $f : Q(K) \rightarrow T$, we record $t^{|\text{Im}(f)|}$, obtaining a polynomial invariant

$$p(t) = \sum_{f:Q(K) \rightarrow T} t^{|\text{Im}(f)|},$$

which has the original counting invariant as $p(1)$. For example, the trefoil in Figure 14 has counting invariant 9 with respect to R_3 and enhanced invariant $p(t) = 3t + 6t^3$. A more sophisticated example of an enhancement is the family of CJKLS quandle cocycle invariants which associate a *Boltzmann weight* $\phi(f)$ with each quandle coloring f , determined by a cocycle in the second cohomology of T [5]. In true combinatorial-revolutionary spirit, such two-cocycles have a geometric interpretation as virtual link diagrams [4].

Enhancement of counting invariants in which we replace a cardinality with a set is a basic example of a more general phenomenon known as *categorification*, in which simpler structures are replaced with higher-powered algebraic structures. Pioneered by mathematical physicists such as Louis Crane and John Baez, categorification involves replacing sets with categories, operations with functors, and equalities with isomorphisms. A prime example is *Khovanov homology*, in which a combinatorial algorithm for computing the Jones polynomial from a knot diagram is turned into a graded chain complex whose Euler characteristic is the Jones polynomial, with the homology

groups forming a new, stronger invariant [18]. Similar methods have been applied to the HOMFLYPT polynomial and various other quantum knot invariants, resulting in new, stronger categorified knot invariants. This remarkable idea has sparked a firestorm of new research too vast to adequately address in this space. Nonetheless, we once again see the combinatorial revolution in action, as what was previously merely notation has itself become a mathematical object of interest.

Not Just for Knot Theorists

While arc and semiarc algebras such as quandles and racks have obvious utility in defining knot invariants, fundamentally they are basic algebraic structures analogous to groups or vector spaces whose potential for applications elsewhere in mathematics is still largely unexplored. The fact that groups are first encountered as sets of symmetries does not limit their applications to geometric rotations and reflections; similarly, despite their knotty origin, quandles, racks, and biquandles are likely to have many applications not tied to knots and links.

Starting with a symmetry group, if we take the subset consisting of only rotations, we get a subgroup; taking the subset consisting of only reflections, however, yields not a subgroup but a subkei. Conjugation in a group is a quandle operation; the resulting *conjugation quandles* quantify the failure of commutativity by turning it into an algebraic structure, analogously to commutators turning associative algebras into Lie algebras. Indeed, kei, quandles, and racks can be found lurking wherever operations are noncommutative.

The applications of generalized knots and knot-inspired algebraic structures are just starting to be explored, and many open questions remain. One rich source of project ideas comes from a simple question: anywhere groups are found, we can ask what results when we replace the group with a kei, quandle, or rack. Replacing the knot group with the knot quandle eliminates the need to worry about the peripheral structure, for example, and replacing groups with quandles simplifies monodromy computations analogously [3, 25]. Work is currently under way on the *Dehn quandle* of a surface (analogous to the mapping class group), as well as on quandle and rack homology, quandle Galois theory, and much more [26, 6]. What arc algebra structures might be lurking in elliptic curves, dynamical systems, or tensor categories?

Knot diagrams have now come full circle, from schematic representations of geometric curves in space to interesting mathematical objects in their own right. The shift from thinking of knots as topological to typographical objects gives us new flexibility and opens the door to new discoveries

and applications. As was the case with complex numbers, there has been some resistance to virtual knot theory from the old guard, though this author's subjective impression is that the tide is turning as more knot theorists embrace the new generalized knot theory and its related algebraic structures. In addition to providing new technical tools for knot theorists and other mathematicians, generalized knots provide a novel perspective on what it means to be a knot.

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