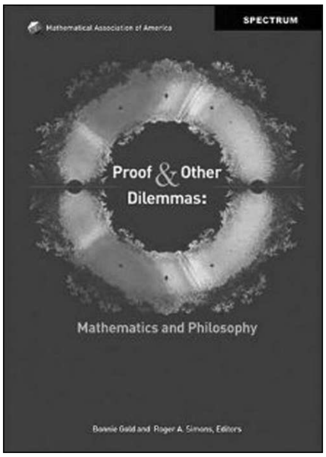
Book Review



Proof and Other Dilemmas: Mathematics and Philosophy

Reviewed by Jeremy Avigad

Proof and Other Dilemmas: Mathematics and Philosophy
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We mathematicians are curious people, in both senses of the word. Whether we are writing up results or talking shop with a colleague over beer, our language is utterly opaque to our nonmathematical friends, even the most intelligent and well educated among them. Nor can these friends appreciate the visceral pleasure we experience when we see the effects of a clever definition play out in a crisp, well-engineered proof. We are keenly aware that what we do is both wonderful and strange. And so we often reflect on what we do and wonder: What sorts of things are these exotic abstractions that govern our lives so thoroughly? Are we making proper use of them—or, perhaps, are they making proper use of us?

Such curiosity has a venerable tradition. Plato and Aristotle sought to understand the nature of abstract mathematical objects and the methods by which we come to know their properties. That agenda is still central to two of the main branches of the western philosophical tradition: *metaphysics* (or *ontology*) aims to give an overarching account of the sorts of things that make up the world and the relationships between them; and *epistemology* is the general study of knowledge and reliable ways of obtaining it. Rocks, governments, numbers, and beliefs are all *things*, but they are very different sorts of things. Metaphysics tries to explain exactly what sorts of things they are and how they interact; epistemology tries to explain how we come to know about them.

Throughout the centuries, there has been considerable interaction between philosophy and mathematics, with no sharp line dividing the two. René Descartes encouraged a fundamental mathematization of the sciences and laid the philosophical groundwork to support it, thereby launching modern science and modern philosophy in one fell swoop. In his time, Leibniz was best known for metaphysical views that he derived from his unpublished work in logic. Seventeenth-century scientists were known as natural philosophers; Newton's theory of gravitation, positing action at a distance, upended Boyle's

Jeremy Avigad is professor of philosophy and mathematical sciences at Carnegie Mellon University. His email address is `avigad@cmu.edu`.

mechanical philosophy; and early modern philosophy, and philosophy ever since, has had to deal with the problem of how, and to what extent, mathematical models can explain physical phenomena. Statistics emerged as a response to skeptical concerns raised by the philosopher David Hume as to how we draw reliable conclusions from regularities that we observe. Laplace's *Essai philosophique sur la probabilités*, a philosophical exploration of the nature of probability, served as an introduction to his monumental mathematical work, *Théorie analytique des probabilités*.

In these examples, the influence runs in both directions, with mathematical and scientific advances informing philosophical work, and the converse. Riemann's revolutionary *Habilitation* lecture of 1854, *Über die Hypothesen welche der Geometrie zu Grunde liegen* ("On the hypotheses that lie at the foundations of geometry"), was influenced by his reading of the neo-Kantian philosopher Herbart. Gottlob Frege, the founder of analytic philosophy, was a professor of mathematics in Jena who wrote his doctoral dissertation on the representation of ideal elements in projective geometry. Late nineteenth-century mathematical developments, which came to a head in the early twentieth-century crisis of foundations, provoked strong reactions from all the leading figures in mathematics: Dedekind, Kronecker, Cantor, Hilbert, Poincaré, Hadamard, Borel, Lebesgue, Brouwer, Weyl, and von Neumann all weighed in on the sweeping changes that were taking place, drawing on fundamentally philosophical positions to support their views. Bertrand Russell and G. H. Hardy exchanged letters on logic, set theory, and the foundations of mathematics. F. P. Ramsey's contributions to combinatorics, probability, and economics played a part in his philosophical theories of knowledge, rationality, and the foundations of mathematics. Alan Turing was an active participant in Wittgenstein's 1939 lectures on the foundations of mathematics and brought his theory of computability to bear on problems in the philosophy of mind and the foundations of mathematics.

These stories, however, now seem quaint, evoking reminiscences of an intellectual climate that is now long gone. Most contemporary mathematicians have no interest in academic philosophy journals, in which the jargon and technical discussion renders contemporary debates inaccessible to them. The disciplinary specialization cuts both ways: cutting-edge mathematical developments are equally inaccessible to working philosophers, who tend to retreat to more familiar ground. The unfortunate result is that the channels of communication between the mathematical and philosophical communities have run dry.

Proof and Other Dilemmas: Mathematics and Philosophy aims to change that. Bonnie Gold and Roger Simons, mathematicians at Monmouth University and Rhode Island College, respectively, have gathered essays by philosophers and mathematicians alike and have woven them together with copious editorial notes and a thoughtful introduction. What started as a sabbatical project for Gold grew into a three-year effort that seems to have involved no small amount of coaxing and encouragement on their part. The result reflects their patent appreciation of the mathematical and philosophical traditions and a sincere interest in bringing them closer together.

The book consists of four sections. The first, "Proof and How It Is Changing", opens with an engaging article by Michael Detlefsen, "Proof: Its nature and significance". Detlefsen addresses a range of topics, including the role of proof, the role of empirical methods in mathematics, the role of formalization, and questions as to whether computer-assisted proofs and diagrammatic proofs can play a justificatory role. He marshals a rich historical context, for example, to fill out the distinction between methods of discovery and methods of justification. In the next essay, Jonathan Borwein, a leading figure in the burgeoning field of experimental mathematics, takes the issue of empirical methods head on. He provides a view from the trenches, surveying a number of striking uses of computational methods in discovering and confirming mathematical assertions. Without repudiating the distinction between discovery and justification, or that between computational evidence and conventional proof, his examples make it clear that experimental methods can provide important forms of insight and understanding. Joseph Auslander rounds out the section with a thoughtful reflection on conventional mathematical proofs and what we gain from them. Bolstered by concrete examples, Auslander argues that although proofs do serve to certify their results, the nature of this certification is more complex than is commonly acknowledged and that they serve other purposes as well, such as providing mathematical explanations and supporting exploration.

The second section is titled "Social Constructivist Views of Mathematics". As portrayed in this collection, social constructivism is the view that mathematical objects are created by, and in some sense dependent on, mathematical communities. Now, few will deny that social, economic, and political forces have shaped the development of mathematics and that it is important to understand how such forces influence mathematical thought. At issue, rather, is the extent to which this should play a part in the metaphysical story. The three essays in this section represent a range of responses. In "When is a problem solved?" Philip J. Davis generally steers clear of ontology while he explores the nature of mathematical problems and the ways and senses in which we take them to be solved. Reuben Hersh is more insistent that social aspects of mathematical practices be incorporated into our account of mathematical objects, because, he argues, this will draw our attention to important aspects of mathematics. Julian Cole, a young philosopher, focuses instead on conventional metaphysical questions. Social constructivism seems to commit one to saying that there were no numbers while dinosaurs walked the planet and that affine sheaves came into existence some time after the French Revolution; Cole aims to explain why a social constructivist need not accept these claims.

The third section, "The Nature of Mathematical Objects and Mathematical Knowledge", is dominated by philosophers, and some background will be helpful here. Early in the twentieth century, the term "metaphysics" briefly fell into disrepute as the logical positivists tried to distinguish between meaningful scientific questions, asked within a precise methodological framework, and "metaphysical" questions, which they took to be empty of content. Later logical positivists were more open to the possibility that "external" or "pragmatic" questions regarding the choice of framework are subject to rational debate, but it was really W. V. O. Quine who made metaphysics respectable again by rejecting the internal/external distinction altogether. According to Quine, philosophy should be viewed as continuous with the sciences, part of an all-encompassing "web of beliefs" that we continually update and revise in light of our experiences. From that perspective, questions as to which fundamental objects should be granted ontological status are on a par with questions about the life cycle of the drosophila or the boiling point of water. Of course, different sciences rely on different vocabularies and methods of justification, and philosophical questions invoke specifically philosophical considerations; but, for Quine, it is all a matter of doing science. This attitude is known as "naturalism", as it subjugates philosophy to the natural sciences.

What it takes to be a card-carrying naturalist is subject to debate, but most philosophers are sympathetic to the claim that our metaphysical theories should square with our best scientific understanding. Fitting abstract mathematical objects into the picture then poses a number of problems. We learn things about physical objects through our interactions with them; how do we come to have reliable knowledge about mathematical objects? The causal interaction of one billiard ball with another can explain why the second one has moved; how can a mathematical theory explain an experimental outcome? For reasons like these, there has been a recent trend in the philosophy of mathematics toward "irrealism". Put simply, realism is the view that mathematical objects exist and have objective properties, and irrealism is the view to the contrary. According to the latter, it may be perfectly rational and correct for a working scientist to assert "3 and 5 are prime numbers," and hence "there are prime numbers," but at the same time to hold that natural numbers do not really exist; or to assert "gravitational acceleration at the Earth's surface is a constant," while maintaining that there aren't really any constants.

The essays in the third section survey contemporary metaphysical views. In "Mathematical objects", Stewart Shapiro describes a number of positions on either side of the realist/irrealist divide, as well as two positions that cut across these categories: neologicism, which holds that mathematical truths are analytic, which is to say true by stipulation or by virtue of essentially linguistic norms; and structuralism, which aims to draw ontological conclusions from the fact that mathematicians care only about properties of mathematical structures that are invariant up to isomorphism. In "The existence of mathematical objects", Charles Chihara provides a friendly overview of the metaphysical enterprise and then sketches his own structuralist position, according to which mathematical objects do not exist. In "Mathematical Platonism", Mark Balaguer, like Shapiro, surveys realist and irrealist positions and considers arguments in favor of each. In "The nature of mathematical objects", Øystein Linnebo invokes broad considerations with respect to the goals of a semantic theory of reference to justify a certain realist stance. The only nonphilosopher to contribute to this section is Barry Mazur, who provides a lucid overview of category-theoretic methodology and explains how it effectively supports a mathematical focus on structures and their isomorphism-invariant properties.

The fourth section, "The Nature of Mathematics and Applications", marks a return to the real world. In "Extreme science: Mathematics as the science of relations and such", Robert Thomas aims to characterize the subject matter of mathematics in such a way as to locate mathematics in relation to other sciences, do justice to mathematical practice, and explain how it is that mathematical theories can have applications to the physical world. Guershon Harel, who works in mathematics education, has written an essay titled "What is mathematics? A pedagogical answer to a philosophical question". He is broadly concerned with making sense of "mental acts such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving" in a way that can inform not just our practices of teaching and learning but our understanding of mathematics itself. Keith Devlin, a well-known popularizer of mathematics, asks "What will count as mathematics in 2100?" In his bold but carefully reasoned essay, he predicts that statistical methods and applications to social sciences like economics and linguistics will play an increasing role. In "Mathematics applied: The case of addition", Mark Steiner explores the way that mathematical concepts are introduced and evolve to fit the needs of the empirical sciences, such as particle physics. In "Probability—A philosophical overview", Alan Hájek provides an informative survey of standard philosophical approaches to the theory of probability.

When all is said and done, what is most striking about this collection is the gap between mathematical and philosophical concerns. Most of the contributions that were written by professional philosophers deal with metaphysics: Linnebo argues that mathematical objects exist and have objective properties; Cole agrees, but maintains that they are socially dependent; Chihara argues that mathematical objects do not exist; Shapiro and Balaguer marshal arguments for both sides, without choosing between them. In fact, Balaguer has argued that not only is there no principled reason to favor one side over the other but that there is no *fact of the matter* as to whether or not mathematical objects exist. This conclusion is not surprising; it indicates that, when one asks questions like these at a sufficiently high level of abstraction, there isn't enough left hanging on the conclusion to fuel a meaningful argument.

This points to a regrettable aspect of that particular style of philosophy, namely, that it ultimately has little to say about ordinary mathematical practice. Faced with the question, "What particular mathematical objects exist?" the answer one gets is either "all of them" or "none of them". This reminds me of a quip I have heard attributed to Sydney Morgenbesser that philosophers are people who know something about everything but nothing about anything. What is common to the essays in this collection that were *not* written by philosophers is an intense concern as to how we should go about living our mathematical lives: the kinds of mathematics we should do, the kinds of questions we should ask, the kinds of objects we should consider, the kinds of answers we should seek, and the kinds of guidance we should give our students. Of the articles written by philosophers, only those by Detlefsen and Steiner address any of these concerns.

The tension is palpable. In his essay, Hersh proclaims that conventional philosophical debates simply miss the point:

> The trouble with Platonism is not so much that it's wrong. The trouble is, it's an easy answer, it avoids looking for scientific answers … when you step back, and look at what you and your colleagues are doing, you can recognize another fascinating problem: to understand mathematics as a special aspect of human thought and culture. [p. 98]

Similarly, Mazur locates the principal benefit of a category-theoretic approach in its avoidance of philosophical issues:

> A stark alternative—the viewpoint of categories—is precisely to *dim* the lights where standard mathematical foundations shines them the brightest. Instead of focusing on the question of modes of justification, and instead of making any explicit choice of set theory, the genius of categories is to provide a vocabulary that keeps these issues at bay. [p. 240]

Thomas goes out of his way to emphasize that his interest is not in conventional philosophy, but in coming up with a satisfactory understanding of the mathematics we actually do:

> My concern is not to solve philosophical problems but rather to have philosophical problems that purport to be about mathematics actually be about what I can recognize as mathematics. [p. 255]

And Harel sees the philosophy of mathematics as disjoint from anything having to do with the way mathematics should be taught and learned:

> It is an open, empirical question whether mathematicians' ontological stances on the nature of mathematical practice have any bearing on their views of how mathematics is learned and, consequently, how it should be taught. I conjecture that teachers' approaches to the learning and teaching of mathematics are not determined by their ontological stance on the being and existence of mathematics. [pp. 275–276]

While the tone of these remarks ranges from annoyance to benign acceptance, the consensus seems to be that the philosophy of mathematics has almost nothing to do with the practice of mathematics itself.

This is a shame. The mathematicians who have contributed to this collection have raised compelling questions about the nature and proper functioning of their craft. Their essays are rich with ideas and insights, though not always expressed in a manner that is precise enough to support rigorous analysis. If there is one thing that philosophers are traditionally good at, it is working with vague ideas and introducing the kind of conceptual clarity that can lead to philosophical and scientific progress. Imagine the interesting things we might learn, if only we could get mathematicians and philosophers talking again.

There are encouraging signs that the philosophical community is starting to come around. A small but growing number of contemporary philosophers are beginning to ask questions that aim to shed light on ordinary mathematical standards and goals: Why do some proofs provide better explanations than others? Why do we sometimes prefer an elementary proof over a more high-powered one, and vice versa? What does it mean to understand a theorem or proof, beyond recognizing its correctness? What makes a concept fruitful, or the "right" one to use for a particular purpose? Why are certain historical developments so important? Should we be content with a proof that involves a computation that is too lengthy to be carried out by hand? It is still not clear how best to go about answering questions like these, but progress has been made, in fits and starts. It is disappointing that the present collection does not contain more work of this sort.

Nonetheless, this book goes a long way toward improving communication between the mathematical and philosophical communities. As any good marriage counselor will tell you, reconciliation has to be preceded by an airing of differences, and each party needs to understand the other's goals and desires before they can begin to seek common ground. What is endearing about the articles in this collection is that they are honest and heartfelt and written in the spirit of openness and communication. That Gold and Simons were able to elicit such candor and good will is no small accomplishment, and the result is a promising start on mending the schism.