



WHAT IS . . .

a Pseudoconvex Domain?

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Pseudoconvexity is a most central concept in modern complex analysis. However, if your training in that area is limited to functions of just *one* complex variable, you probably have never heard of it, since every open subset of the complex plane \mathbb{C} is pseudoconvex. Pseudoconvexity or, better, *nonpseudoconvexity*, is a higher-dimensional phenomenon. Incidentally, this is true also for *Euclidean* convexity: every open connected subset of \mathbb{R} is convex. Life is definitely more interesting in higher dimensions!

Pseudoconvexity is so central because it relates to the very core of holomorphic (i.e., complex analytic) functions, which is intimately intertwined with power series, the identity theorem, and analytic continuation. This concept has its roots in Friedrich Hartogs's surprising discovery in 1906 of a simple domain H in \mathbb{C}^2 with the property that every function that is holomorphic¹ on H has a holomorphic extension to a *strictly larger* open set \hat{H} . In dimension one there is no such thing! In fact, if P is a boundary point of a domain $D \subset \mathbb{C}$, the function $f_P(z) = 1/(z - P)$ is clearly holomorphic

on D , but surely it has no holomorphic extension to any neighborhood of P .²

Hartogs's example is so amazing and historically significant, and yet completely elementary, that it deserves to be presented in any exposition of the subject. Consider the domain $H \subset \mathbb{C}^2 = \{(z, w) : z, w \in \mathbb{C}\}$ defined by

$$H = \{(z, w) : |z| < 1, 1/2 < |w| < 1\} \cup \{|z| < 1/2, |w| < 1\}.$$

Let $f : H \rightarrow \mathbb{C}$ be holomorphic and fix r with $1/2 < r < 1$. The function

$$F(z, w) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(z, \zeta) d\zeta}{\zeta - w}$$

is easily seen to be holomorphic on $G = \{(z, w) : |z| < 1, |w| < r\}$. Observe that for fixed z_0 with $|z_0| < 1/2$ the function $w \rightarrow f(z_0, w)$ is holomorphic on the disc $\{|w| < 1\}$, and hence, by the Cauchy integral formula, $f(z_0, w) = F(z_0, w)$ for $|w| < r$. Thus $f \equiv F$ on $\{(z, w) : |z| < 1/2, |w| < r\}$, which implies $f \equiv F$ on $H \cap G$ by the identity theorem, so that F provides the holomorphic extension of f from H to $\hat{H} = H \cup G$.

Hartogs's discovery immediately raises the fundamental problem of characterizing those domains $D \subset \mathbb{C}^n$ for which holomorphic extension of *all* holomorphic functions on D does NOT hold. Such domains are called *domains of holomorphy*. More precisely, according to this point of view, D is a domain of holomorphy, if for each boundary point $P \in \partial D$ there exists a function f_P holomorphic on D which does not extend

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¹Holomorphic functions in several complex variables are those continuous functions on an open set $D \subset \mathbb{C}^n$ which are holomorphic in each variable separately. Iteration of the Cauchy integral formula for discs readily implies that such functions are then C^∞ in all underlying real variables and have local power series expansions in the complex variables.

²This sort of simple construction does not extend to more than one variable, as the zeroes and singularities of holomorphic functions are not isolated in the case of two or more variables. The reader may find more details in [3].

holomorphically to any neighborhood of P .³ As mentioned earlier, every domain in the complex plane is trivially a domain of holomorphy, and this easily implies that every product domain $D = D_1 \times D_2 \times \cdots \times D_n$ with $D_j \subset \mathbb{C}$, $j = 1, \dots, n$, is also a domain of holomorphy. Furthermore, it is elementary to show that every *Euclidean convex* domain $D \subset \mathbb{C}^n$ is a domain of holomorphy. Hartogs, of course, produced the first example of a domain which is NOT a domain of holomorphy. The reader may consult [2] for other surprising consequences of Hartogs's discovery.

The essence of *pseudoconvexity* is now captured by the following statement.

Pseudoconvexity is that local analytic/geometric property of the boundary of a domain D in \mathbb{C}^n which characterizes domains of holomorphy.

Note that it is not at all clear that the *global* property of being a domain of holomorphy should allow a purely *local* characterization, i.e., something that can be recognized by just looking near each boundary point. In fact, the validation of this statement was the culmination of major efforts over a period of more than forty years.

Just a few years after Hartogs's discovery, E. E. Levi studied domains of holomorphy with *differentiable* boundaries. He found the following simple differential condition, which is remarkably similar to the familiar differential characterization of Euclidean convexity. We assume that $D \cap U = \{z \in U : r(z) < 0\}$, where r is a C^2 real-valued function with $dr \neq 0$ on a neighborhood U of $P \in bD$.

Theorem (Levi, 1910/11). *A) If there exists a holomorphic function on $D \cap U$ which does not extend holomorphically to P (in particular, if D is a domain of holomorphy), then*

$$L_P(r; t) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(P) t_j \bar{t}_k \geq 0 \text{ for all } t \in \mathbb{C}^n$$

$$\text{with } \sum_{j=1}^n \frac{\partial r}{\partial z_j}(P) t_j = 0.$$

B) If $L_P(r; t) > 0$ for all $t \neq 0$ which satisfy $\sum_{j=1}^n \frac{\partial r}{\partial z_j}(P) t_j = 0$, then the neighborhood U can be chosen so that $U \cap D$ is a domain of holomorphy.⁴

³This definition is formally weaker than the one commonly found in the literature; namely, a domain of holomorphy is a domain on which there exists a single holomorphic function which cannot be extended holomorphically to any of its boundary points. However, by a 1932 fundamental theorem of H. Cartan and P. Thullen, the two notions are in fact equivalent.

⁴With $z_j = x_j + i y_j$, $j = 1, \dots, n$, the complex partial differential operators $\partial/\partial z_j$ are defined by $\partial/\partial z_j = (1/2)(\partial/\partial x_j - i \partial/\partial y_j)$; an analogous definition holds for their conjugates $\partial/\partial \bar{z}_j$.

Note that if $D \subset \mathbb{C}$, the restriction on t is satisfied only for $t = 0$, so the conditions in A) and B) trivially hold in this case.

Levi's results made it clear that the "complex Hessian" $L_P(r; t)$ —now universally called the *Levi form*—plays a fundamental role in the characterization of domains of holomorphy. The term "pseudoconvex" was introduced in this context in the influential 1934 "Ergebnisbericht" of H. Behnke and P. Thullen, which summarized the status and principal open questions in multidimensional complex analysis at that time. To distinguish Levi's differential conditions from other formulations of pseudoconvexity, one refers to the condition in A) as *Levi pseudoconvexity*. If the stronger version in B) holds, one says that D is *strictly* or *strongly* pseudoconvex at P .

By Levi's result, if D is strictly pseudoconvex at every boundary point, then D is **locally** a domain of holomorphy. The emphasis on "locally" is critical. Levi himself recognized that his result was far from yielding the wished-for corresponding global version. For many years it remained a central open problem—known as the *Levi problem*—to show that a strictly pseudoconvex domain is indeed a domain of holomorphy. Solutions were finally obtained in the early 1950s by K. Oka, H. Bremermann, and F. Norguet, thereby vindicating the central role of pseudoconvexity.

The extension to arbitrary domains requires an appropriate definition of pseudoconvexity. Many equivalent versions have been introduced over the years. Perhaps most elegant is a formulation that involves the notion of *plurisubharmonic* function introduced by Oka and P. Lelong in the 1940s.⁵ Suffice it to say that a C^2 function r on D is plurisubharmonic precisely when its Levi form satisfies $L_z(r; t) \geq 0$ for all $t \in \mathbb{C}^n$ and $z \in D$ and that general plurisubharmonic functions can be well approximated from above by C^2 or even C^∞ plurisubharmonic functions. Let us denote by $\text{dist}(z, bD)$ the Euclidean distance from z to bD .

Definition. A domain $D \subset \mathbb{C}^n$ is said to be *pseudoconvex* (or Hartogs pseudoconvex) if the function $\varphi(z) = -\log \text{dist}(z, bD)$ is plurisubharmonic on D .

Note that φ is a continuous function which tends to ∞ as $z \rightarrow bD$. One verifies that convex domains are pseudoconvex and that a domain with C^2 boundary is pseudoconvex according to this definition if and only if it is Levi pseudoconvex. Also, any pseudoconvex domain is the increasing

⁵Subharmonic functions were first introduced in one complex variable by F. Hartogs in 1906. That concept was generalized in the obvious way to n real variables in the 1920s. In contrast, plurisubharmonic functions are those functions of n complex variables which are subharmonic on each complex line where defined.

union of strictly (Levi) pseudoconvex domains with C^∞ boundaries. Incidentally, it is known that a domain D is (Euclidean) convex if and only if $-\log \text{dist}(z, bD)$ is a convex function.

The general version of the solution of Levi's problem is then stated as:

A domain in \mathbb{C}^n is a domain of holomorphy if and only if it is pseudoconvex.

To conclude, let me briefly mention two topics involving pseudoconvexity which continue to stimulate important research work.

The first deals with studying the boundary behavior of analytic objects, such as special classes of holomorphic functions, biholomorphic maps between open sets in \mathbb{C}^n , and solutions of the inhomogeneous Cauchy-Riemann equations. Many such questions are pretty well understood in case the boundary of the domain is *strictly* pseudoconvex. (See [3] for some references.) A natural goal then is to extend such results to the general Levi pseudoconvex case, say with C^∞ boundary. Let us emphasize that the problems are genuinely higher dimensional, since in dimension one all smoothly bounded domains are automatically *strictly* pseudoconvex. The situation is quite complicated and very technical. Some results are known to fail in the general case, as evidenced, for example, by the so-called "worm domain" discovered by K. Diederich and J. E. Forneaess in 1976. (See [1].) Other results have been verified assuming additional conditions such as Euclidean convexity and/or "finite type"—an important generalization of strict pseudoconvexity that was introduced by J. J. Kohn in the early 1970s. And other questions still remain unsolved. For example, Charles Fefferman proved in the mid-1970s that every biholomorphic mapping between smoothly bounded strictly pseudoconvex domains in \mathbb{C}^n extends smoothly to the boundary. This result has been extended to the case of pseudoconvex domains of finite type and to some other special situations, but to my knowledge—in spite of many efforts—the problem remains open for arbitrary Levi pseudoconvex domains.

Another natural question centers around our basic understanding of pseudoconvexity and its relationship to Euclidean convexity. The explicit formulations of pseudoconvexity mentioned in this article clearly are complex analogues of corresponding characterizations of convexity. In particular, convexity implies pseudoconvexity. Furthermore, it is elementary, but nontrivial, to show that a domain is *strictly pseudoconvex* near P if and only if it is strictly Euclidean convex (i.e., the relevant matrix of second-order partial derivatives is positive definite) with respect to suitable local holomorphic coordinates centered at P . Stated differently, strict pseudoconvexity is—locally—simply the biholomorphically invariant version of

strict convexity. Unfortunately, this neat characterization breaks down already in the case of simple pseudoconvex domains of finite type, as shown by an example discovered by J. J. Kohn and L. Nirenberg in 1972. However, if one drops all regularity conditions of the coordinates on the boundary, one is left with the following tantalizing question, whose answer is still unknown.⁶

Given a smoothly bounded domain $D \subset \mathbb{C}^n$ and a point $P \in bD$ such that $D \cap U$ is pseudoconvex for some neighborhood U of P , can U be chosen so that $D \cap U$ is biholomorphically equivalent to a Euclidean convex domain $W \subset \mathbb{C}^n$?

Further Reading

- [1] H. P. BOAS, WHAT IS... a Worm? *Notices Amer. Math. Soc.* **50** (2003), 554–555.
- [2] R. M. RANGE, Extension phenomena in multidimensional complex analysis: Correction of the historical record, *Math. Intell.* **24** (2002), 4–12.
- [3] ———, Complex analysis: A brief tour into higher dimensions, *Amer. Math. Monthly* **110** (2003), 89–108. (Includes references to systematic introductions to multidimensional complex analysis.)

⁶The question is purely local. At the global level, it is known that the answer is negative. For example, in 1986 N. Sibony produced a smoothly bounded pseudoconvex domain in \mathbb{C}^2 which cannot even be properly embedded in a convex domain in some \mathbb{C}^N .