WHAT IS...

an Octopus Decomposition?

Larry Conlon

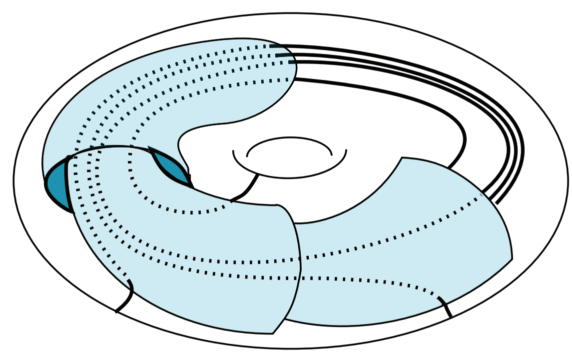

Figure 1. A Reeb foliated solid torus.

This essay concerns foliated n-manifolds of codimension one. To keep things intuitive and relatively uncomplicated, M will be an oriented, compact 3-manifold without boundary and $\mathcal{F}$ a transversely oriented foliation of M by surfaces. The restriction to the case $n = 3$ is not essential, however, and the interested reader will easily carry out the generalization. Since no background in foliation theory is presumed, some brief explanations are in order.

Foliations of 3-manifolds

The foliation $\mathcal{F}$ is given by a finite covering of M by "flowboxes" $\{B_\alpha\}_{\alpha=1}^m$, where $B_\alpha \cong J \times J \times J$, $J = [-1, 1]$. The factors $J \times J \times \{t\} = P_\alpha \times \{t\}$ are called "$\mathcal{F}$-plaques", and the factors $\{(u, v)\} \times J = \{(u, v)\} \times J_\alpha$ will be called "$\mathcal{J}$-plaques". It is assumed that, if $B_\alpha \cap B_\beta \neq \emptyset$, then these boxes properly overlap (i.e., their interiors intersect) and that each $\mathcal{F}$-plaque of one meets at most one $\mathcal{F}$-plaque of the other, properly overlapping it in a connected set, and that each $\mathcal{J}$-plaque of one meets at most one $\mathcal{J}$-plaque of the other, again properly overlapping it in a connected set. Think of the $\mathcal{F}$-plaques as "steppingstones"—standing on one of them, you are allowed to step onto any overlapping one, thence onto any which overlaps that one, etc. All of the allowable walks you can take trace out a one-to-one immersed surface L in M called a *leaf of the foliation* $\mathcal{F}$. The leaves can be globally very complicated; for instance, a leaf might be dense in M, but locally they stack up like the pages (leaves) of a book. The term "foliation" (*feuilletage*), however, does not refer to books (let alone foliage) but was coined by G. Reeb with reference to a flaky dough often used in France to make pastries.

Similarly, the $\mathcal{J}$-plaques define a 1-dimensional foliation $\mathcal{J}$ everywhere transverse to $\mathcal{F}$. This foliation is a tool for working with $\mathcal{F}$ and is somewhat arbitrary.

We understand that both sets of plaques come with orientations that are coherent in each flowbox and that the orientations of overlapping plaques coincide on the overlaps. Thus M is oriented and $\mathcal{F}$ is transversely oriented by the orientation of $\mathcal{J}$.

Larry Conlon is emeritus professor of mathematics at Washington University in St. Louis. His email address is `lc@math.wustl.edu`.

DOI: http://dx.doi.org/10.1090/noti872

The Reeb Foliation of the 3-Sphere

A standard and, in fact, quite important example is given by the Reeb foliation of the 3-sphere S^3 by surfaces, one of which is a torus $T = T^2$, the remaining leaves being cup-shaped copies of $\mathbb{R}^2$. The key fact here is that $S^3 = D^2 \times S^1 \cup S^1 \times D^2$, the union of two solid tori, glued together along their common boundary $T^2 = S^1 \times S^1$ so that the meridians $\partial(D^2) \times \{x\}$ of the first match up with the longitudes $S^1 \times \{z\}$ of the second, $\|z\| = 1$, and vice versa. Each solid torus is foliated, as indicated in Figure 1, the cup-shaped planes spiraling out on the boundary torus asymptotically. (These leaves are sometimes described as infinite snakes repeatedly swallowing their tails.) A natural choice of the transverse flow $\mathcal{J}$ is also indicated in the figure. Reportedly, this example was created by Reeb in response to the suggestion by C. Ehresmann, his thesis director, that he prove that S^3 cannot be foliated by surfaces!

A subset $W \subseteq M$ is said to be "$\mathcal{F}$-saturated" (or simply "saturated") if it is a union of leaves of $\mathcal{F}$. In studying the topological structure of codimension-one foliations, the open, connected, saturated sets play a key role. As a simple example, consider the compact set $X \subset S^3$, which is the union of the compact leaf T of the Reeb foliation and one of the noncompact leaves L. The complement of this set is open and saturated and has two components. One component is just the interior of the solid torus not containing L, the other component V being slightly more interesting. We can choose the set $\{B_\alpha\}_{\alpha=1}^m$ of flowboxes in such a way that, for some $p < m$ and $1 \leq \alpha \leq p$, the corresponding flowboxes are contained in the solid torus containing L and cover that solid torus, and the plaques $P_\alpha \times \{\pm 1\}$ lie in $L \cup T$. Thus, $V \cap B_\alpha$ is a disjoint union of at most countably many open "sandwiches" $P_\alpha \times (t_i, s_i)$, where the plaques $P_\alpha \times \{t_i, s_i\}$ lie in L. Notice that, if you are inside V, you see L twice—once as the floor and once as the roof of your world. The intervals $J_x = \{x\} \times [t_i, s_i]$, $x \in L$, are "poles" joining the floor to the roof. One forms the abstract manifold $\hat{V}$, called the transverse completion of V, by adjoining to V its floor and roof. In $\hat{V}$, the boundary consists of two distinct copies of L, but the natural immersion $\hat{\imath} : \hat{V} \looparrowright M$ identifies the floor and roof with the single leaf L. Our foliations induce foliations $\hat{\mathcal{F}} = \hat{\imath}^{-1}(\mathcal{F})$ and $\hat{\mathcal{J}} = \hat{\imath}^{-1}(\mathcal{J})$. Notice that $\hat{V} \cong L \times I$, the interval fibers being the leaves of $\hat{\mathcal{J}}$ and $\hat{\mathcal{F}}$ being a trivial foliation transverse to these fibers. While overly simple, this example will be useful to keep in mind in what follows.

The Octopus Decomposition

The term "octopus decomposition" refers to an elementary but incredibly useful structure theorem for open, connected, saturated sets $W \subset M$. To the best of my knowledge, this decomposition, without its colorful name, was first described by P. Dippolito in his seminal *Annals* paper (1978) on foliations, but the honor may actually belong to many authors (e.g., G. Hector). It is a concept that was "in the air" around that time. As above, note that the intersection $W \cap B_\alpha$ is a disjoint union of sandwiches of the form $P_\alpha \times (t_i, s_i)$ with the added possibility that there will be a component $P_\alpha \times [-1, s_\alpha)$ and/or $P_\alpha \times (t_\alpha, 1]$ (the floor and/or the roof of B_α is engulfed by W) and there is also the possibility that $W \cap B_\alpha = B_\alpha$ (the entire flowbox is engulfed by W). One can still form the transverse completion $\widehat{W}$ by assembling boundary leaves from $P_\alpha \times \{t_i, s_i\}$ and $P_\alpha \times \{t_\alpha, s_\alpha\}$. It is an exercise, using the finiteness of the flowbox cover and the fact that W is connected, to show that $\partial\widehat{W}$ has finitely many components. Again, the natural immersion $\hat{\imath} : \widehat{W} \looparrowright M$ may identify some of these leaves pairwise. A compact "nucleus" $K \subset \widehat{W}$ may not be fibered by intervals. It will contain those flowboxes engulfed by W and the half-flowboxes $P_\alpha \times [-1, s_\alpha)$ and/or $P_\alpha \times (t_\alpha, 1]$ also engulfed by W. There is some freedom in the choice of nucleus, and we can suppose that K is a compact, connected manifold with boundary and corners. The corners will come in pairs of circles cutting off annuli $A_i \subset \partial K$, $1 \leq i \leq r$, which are transverse to $\hat{\mathcal{F}} = \hat{\imath}^{-1}(\mathcal{F})$, making K a sutured manifold in the sense of D. Gabai. The boundary of K splits up into these finitely many transverse annuli and pieces of boundary leaves. We denote the set $K \setminus \bigcup_i A_i$ by K° and remark that the components of the complement $\widehat{W} \setminus K^\circ$ are trivial interval bundles $V_i = B_i \times I$, the fibers being leaves of $\hat{\mathcal{J}}$ and B_i being a noncompact, connected subsurface of $\partial\widehat{W}$ with ∂B_i a finite union of circles. These are called the "arms" of $\widehat{W}$ and, with a little care in the choice of K, one guarantees that each arm V_i attaches to K precisely along one annulus A_i. While the foliation $\hat{\mathcal{F}}|V_i$ will not generally be a product, it will be transverse to the interval fibers. The structure of foliations transverse to the fibers of an interval bundle is very well understood and easy to control, so the challenge in understanding the topology of $\hat{\mathcal{F}}$ is entirely localized in the compact nucleus K.

And so there we have it.

$$\widehat{W} = K \cup V_1 \cup V_2 \cup \cdots \cup V_r$$

is the octopus decomposition of $\widehat{W}$, the nucleus K being the head and the arms V_i being the tentacles. Of course these are mutant octopi, since generally $r \neq 8$, but the more precise term "polypus" just doesn't have the same ring to it. Some of these octopi, such as $\widehat{V}$ in the Reeb foliation, are so badly mutated that they have only one arm and no head at all!

Laminations

Finally, an analogous concept occurs in the theory of surface laminations of 3-manifolds. These are compact partial foliations of M with lots of empty space, introduced as tools in 3-manifold topology by D. Gabai and U. Oertel. The connected components W of the complement have a structure very analogous to that of open, connected, saturated sets. Not to be outdone by foliators in the colorful language department, laminators refer to the nucleus as the "guts" of $\widehat{W}$. Any topological obstruction to extending the lamination to a full foliation of M will be contained in the guts. Those laminations that do not so extend are called "genuine laminations" and play a key role in 3-manifold theory.

Further Reading

[1] A. Candel and L. Conlon, *Foliations I*, American Mathematical Society, 1999, Chapter 5.
[2] D. Calegari, *Foliations and the Geometry of 3-Manifolds*, Oxford University Press, 2007, Chapter 5.