An Eloquent Formula for the Perimeter of an Ellipse

Semjon Adlaj

The values of complete elliptic integrals of the first and the second kind are expressible via power series representations of the hypergeometric function (with corresponding arguments). The complete elliptic integral of the first kind is also known to be eloquently expressible via an arithmetic-geometric mean, whereas (before now) the complete elliptic integral of the second kind has been deprived such an expression (of supreme power and simplicity). With this paper, the quest for a concise formula giving rise to an exact iterative swiftly convergent method permitting the calculation of the perimeter of an ellipse is over!

Instead of an Introduction

A recent survey [16] of formulae (approximate and exact) for calculating the perimeter of an ellipse is erroneously resuméd:

> There is no simple exact formula: There are simple formulas but they are not exact, and there are exact formulas but they are not simple.

No breakthrough will be required for a refutation, since most (if not everything!) had long been done by Gauss, merely awaiting a (last) clarification.

The Arithmetic-Geometric Mean and a Modification Thereof

Introduce a sequence of pairs $\{x_n,\ y_n\}_{n=0}^{\infty}$:

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := \sqrt{x_n y_n}.$$

Define *the arithmetic-geometric mean* (which we shall abbreviate as AGM) of two positive numbers x and y as the (common) limit of the (descending) sequence $\{x_n\}_{n=1}^{\infty}$ and the (ascending) sequence $\{y_n\}_{n=1}^{\infty}$ with $x_0 = x$, $y_0 = y$.[1]

The convergence of the two indicated sequences is said to be quadratic [7, p. 588]. Indeed, one might readily infer that (and more) by putting

$$r_n := \frac{x_n - y_n}{x_n + y_n},\ n \in \mathbb{N},$$

and observing that

$$r_{n+1} = \left(\frac{\sqrt{x_n} - \sqrt{y_n}}{\sqrt{x_n} + \sqrt{y_n}}\right)^2 = \left(\frac{\sqrt{1+r_n} - \sqrt{1-r_n}}{\sqrt{1+r_n} + \sqrt{1-r_n}}\right)^2$$
$$= \left(\frac{1 - \sqrt{1 - r_n^2}}{r_n}\right)^2 \approx \frac{r_n^2}{4},$$

where the sign for approximate equality $\approx$ might be interpreted here as an asymptotic (as r_n tends to zero) equality.

Next, introduce a sequence of triples $\{x_n,\ y_n,\ z_n\}_{n=0}^{\infty}$:

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := z_n + \sqrt{(x_n - z_n)(y_n - z_n)},$$
$$z_{n+1} := z_n - \sqrt{(x_n - z_n)(y_n - z_n)}.$$

Define *the modified arithmetic-geometric mean* (which we abbreviate as MAGM) of two positive numbers x and y as the (common) limit of the (descending) sequence $\{x_n\}_{n=1}^{\infty}$ and the (ascending) sequence $\{y_n\}_{n=1}^{\infty}$ with $x_0 = x$, $y_0 = y$ and $z_0 = 0$.

Put

$$\xi_n := x_n - z_n,\ \eta_n := y_n - z_n,\ \rho_n := \frac{\xi_n + \eta_n}{x_n + y_n},\ n \in \mathbb{N}.$$

Each iteration for the AGM requires an addition, a division, a multiplication, and taking the square root. The first iteration for the MAGM coincides with the first iteration for the AGM. Each subsequent iteration for the MAGM requires three more (than an iteration for the AGM requires) additions,

Semjon Adlaj is professor of mathematics at the Computing Centre of the Russian Academy of Sciences. His email address is `semjonadlaj@gmail.com`.

[1] *We need not impose the assumption that* $x_0 \geq y_0$.

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but with each iteration the speed of convergence for the MAGM (as compared with the speed of convergence, at the corresponding iteration, for the AGM) is greater by a ratio asymptotically coinciding with the ratio ρ_n. The latter claim is clarified by observing that

$$r_{n+1} = \frac{\xi_{n+1} - \eta_{n+1}}{x_{n+1} + y_{n+1}} = \frac{\xi_{n+1}}{x_{n+1} + y_{n+1}} \left(\frac{\sqrt{\xi_n} - \sqrt{\eta_n}}{\sqrt{\xi_n} + \sqrt{\eta_n}} \right)^2 \approx \frac{r_n^2}{4\rho_n}.$$

The ratio ρ_n is eventually (that is, asymptotically) doubled with each iteration.

An example considered (accurately) by Gauss [12] and (sloppily) provided in [7, p. 587] for demonstrating the convergence for the AGM uses the initial values $x = 1$ and $y = 0.8$. We list (chopping off digits) approximations corresponding to four consecutive iterations:

$$\begin{aligned}
x_1 &= 0.9,\\
r_1 &\approx 0.0031056200151418585394958513 48,\\
y_1 &\approx 0.894427190999915878563669467 4,\\
4r_1/r_0^2 &\approx 1.0062208849059621667966558367 8,\\
x_2 &\approx 0.8972135954999579392818347337,\\
r_2 &\approx 0.0000024112305476358803359566 69,\\
y_2 &\approx 0.8972092687327323251471393964,\\
4r_2/r_1^2 &\approx 1.0000048224669093045145243407 28,\\
x_3 &\approx 0.8972114321163451322144870651,\\
r_3 &\approx 0.0000000000014535081884673322 19,\\
y_3 &\approx 0.8972114321137369238877556369,\\
4r_3/r_2^2 &\approx 1.0000000000029070163769367771 2,\\
x_4 &\approx 0.8972114321150410280511213510,\\
r_4 &\approx 0.0000000000000000000000000528171,\\
y_4 &\approx 0.8972114321150410280511204032,\\
4r_4/r_3^2 &\approx 1.00000000000000000000000105634.
\end{aligned}$$

The values at the first iteration coincide, of course, with those for the (introduced) MAGM. Now we list approximate values at the second, third, and fourth iterations for the MAGM:

$$\begin{aligned}
x_2 &\approx 0.8972135954999579392818347337,\\
r_2 &\approx 0.0000012074866419162234506275 40,\\
y_2 &\approx 0.8972114287557112303660562524,\\
4r_2\rho_1/r_1^2 &\approx 1.0000018121692852069077586436 74,\\
x_3 &\approx 0.8972125121278345848239454930,\\
r_3 &\approx 0.0000000000000912681941855433 08,\\
y_3 &\approx 0.8972125121276708108923803433 5,\\
4r_3\rho_2/r_2^2 &\approx 1.0000000000001141207215093744 4,\\
x_4 &\approx 0.8972125121277526978581629182,\\
r_4 &\approx 0.0000000000000000000000000000260,\\
y_4 &\approx 0.8972125121277526978581629177,\\
4r_4\rho_3/r_3^2 &\approx 1.0000000000000000000000000000293,
\end{aligned}$$

along with the (tending to 2 as they ought to) ratios:

$$\begin{aligned}
\rho_1 &\approx 1.996894379984858141460504148 65,\\
\rho_2/\rho_1 &\approx 2.0000006009264508817034611282 2,\\
\rho_3/\rho_2 &\approx 2.0000000000001139788063995947 6,\\
\rho_4/\rho_3 &\approx 2.0000000000000000000000000000042.
\end{aligned}$$

Fix $\beta > 1$, let $\{x_n\}$ and $\{y_n\}$ denote the sequences converging to the AGM of $x_0 = \beta$ and $y_0 = 1$, and let $\{\xi_n\}$ denote the descending sequence converging to the MAGM of β^2 and 1 with $\xi_0 = \beta^2$. The following equalities hold:

$$x_n = \beta - \sum_{m=0}^{n-1} \frac{x_m - y_m}{2}, \quad \xi_n = \beta^2 - \sum_{m=0}^{n-1} 2^m \, \frac{x_m^2 - y_m^2}{2}.$$

Proceeding with another example of Gauss, considered in [17] as well, where $\beta = \sqrt{2}$, we write, for $1 \le n \le 4$, approximations for x_n:

$$x_1 \approx 1.2, \; x_2 \approx 1.19815, \; x_3 \approx 1.19814023479,$$
$$x_4 \approx 1.19814023473559220744,$$

and, moving on, we supply approximations for ξ_n:

$$\xi_1 = 1.5, \; \xi_2 \approx 1.457, \; \xi_3 \approx 1.456946582,$$
$$\xi_4 \approx 1.45694658104446362 54.$$

Efficient Calculations of Complete Elliptic Integrals

Unfix β and assume, unless indicated otherwise, that β and γ are two positive numbers whose squares sum to one: $\beta^2 + \gamma^2 = 1$.

Gauss discovered a highly efficient (unsurpassable) method for calculating complete elliptic integrals of the first kind:

$$\int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - \gamma^2 x^2)}} = \frac{\pi}{2M(\beta)}, \tag{1}$$

where $M(x)$ is the arithmetic-geometric mean of 1 and x. In particular, equality (1) holds if (in violation of the assumption, otherwise imposed) $\gamma^2 = -1$:

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{\pi}{2M(\sqrt{2})} \approx 1.31102877714605990523.$$

The integral on the left-hand side of the latter equation is referred to as the lemniscate integral and is interpreted as the quarter length of the lemniscate of Bernoulli whose focal distance is $\sqrt{2}$. The precision of the numerical approximation given (assuming π is known with sufficient precision) is attained after four iterations, that is, at $\pi/(2x_4)$. The reciprocal of $M(\sqrt{2})$ is called the Gauss constant; Gauss, having calculated it to eleven decimal places, wrote in his diary [14] on May 30, 1799, that the discovery "opens an entirely new field of analysis." Thereby, the beautiful field of elliptic functions and elliptic curves[2] intertwining analysis, algebra, and geometry was born.

The formula given via equation (1) signified a qualitative transition in connecting the study of elliptic integrals of the first kind with studying elliptic functions. Yet, a formula, analogous[3] to (1), for calculating elliptic integrals of the second kind had defied all subsequent efforts at attaining it, awaiting December 16, 2011, to be discovered. In fact (yet arguably), searching for an (unsurpassable) formula for calculating elliptic integrals of the second kind has been a great (often hidden) motivator and a driving force behind much of the genuine research on elliptic functions and elliptic curves. As is evidenced by the adjective "elliptic" tagging this field, which Gauss's discovery had once ignited, the problem of calculating the arc length of an ellipse has been its (most) central problem. The formula for calculating complete elliptic integrals of the second kind be now known:

$$\int_0^1 \sqrt{\frac{1-\gamma^2x^2}{1-x^2}}\,dx = \frac{\pi N(\beta^2)}{2M(\beta)}, \tag{2}$$

where $N(x)$ is the modified arithmetic-geometric mean of 1 and x. The integral on the left-hand side of equation (2) is interpreted as the quarter length of an ellipse with a semi-major axis of unit length and a semi-minor axis of length β (and eccentricity γ), whereas the swiftly converging ratio on the right-hand side is elementary enough to be presented in high or, perhaps, elementary school. This formula, unlike any other preceding formula for computing complete elliptic integrals of the second kind, aside from offering a calculating algorithm possessing both (sought-for features) iterativity and fast convergence, is lucent. On the other hand, the Euler formula and the (so-called) Gauss-Kummer series, which, in fact, is due to Ivory [13], aside from lacking simplicity, converge only linearly and particularly slowly for large eccentricities (near one).

The Legendre relation [10], relating complementary complete elliptic integrals of the first and the second kind to each other, might now be rewritten, yielding a parametric (uncountably infinite) family of identities for π:

$$\pi = \frac{2M(\beta)M(\gamma)}{N(\beta^2)+N(\gamma^2)-1} \tag{3}$$

and, in particular, yielding a countably infinite family of identities (where the ratio of $M(\gamma)$ to $M(\beta)$ is an integer power of $\sqrt{2}$), from which, setting $c := \sqrt{2}-1$, we list a few:

$$\pi = \frac{M(\sqrt{2})^2}{N(2)-1} = \frac{M\left(2\sqrt{\sqrt{2}\,c}\right)^2/2}{N\left(4\sqrt{2}\,c^2\right)-2c} = \frac{M\left(\sqrt{2c}\right)^2}{\sqrt{2}N(2c)-1}$$
$$= \frac{2M(c)^2}{\sqrt{2}N\left(c^2\right)-c} = \frac{2M\left(c^2\right)^2}{N\left(c^4\right)-c^2},$$

where the first of the latter chain of identities for π might be inferred from a special case (where $\beta = \gamma$) of the Legendre relation discovered by Euler [11]. Iteratively calculating (for $\beta = \sqrt{2}$) the sequences $\{x_n\}$ and $\{\xi_n\}$, of which we have already calculated the terms up to those whose indices do not exceed $n = 4$, one arrives at the (so-called) Brent-Salamin algorithm for computing π [18].[4] Setting

$$\pi_n := \frac{x_n^2}{\xi_{n+1}-1}$$
$$= \frac{\left(\sqrt{2}+1-\sum_{m=1}^{n-1} x_m - y_m\right)^2}{2\sqrt{2}-1-\sum_{m=1}^{n-1} 2^m (x_m-y_m)^2},\ n \in \mathbb{N},$$

we enlist, for $n \le 4$, approximations for the ratios π_n (descendingly and quadratically converging to π):

$$\pi_1 \approx 3.18,\ \pi_2 \approx 3.1416,\ \pi_3 \approx 3.1415926538,$$
$$\pi_4 \approx 3.14159265358979323846 6.$$

A Few Examples

Although we aim to provide several applications of the formula attained for complete elliptic integrals of the second kind, we can hardly skip a classical demonstration from mechanics providing a shining example of the Gauss formula for calculating complete elliptic integrals of the first kind.

The Period of a Simple Pendulum

Had Appell known of the Gauss method (for calculating complete elliptic integrals of the first kind) he would not have had to "discover" a mechanical interpretation of the "imaginary period" [8] of a simple pendulum [9]. The (two-valued) period T of a simple pendulum[5] might be clearly and

[2] *We need not adhere to the rather common (and ridiculous) separation of the study of elliptic curves from the study of elliptic functions.*

[3] *The sought-for formula, aside from its desired simplicity, must give rise to an iterational and rapidly (faster than linearly) convergent algorithm.*

[4] *Evidently, "Gauss-Euler algorithm" would be a naming less exotic, yet restoring the credit to whom it rightfully belongs.*

[5] *T regarded as a function of $|g|$ – the modulus of g. The choice of the positive direction along the "vertical" is, after all, arbitrarily made, so, regardless of the choice, both signs of g must be accounted for.*

succinctly expressed as

$$T = 2\pi k\sqrt{\frac{l}{g}},\ k := k(\theta)$$
$$= \begin{cases} 1/M\,(\cos(\theta/2)) & \text{if } g > 0,\\ \sqrt{-1}/M\,(\sin|\theta/2|) & \text{if } g < 0, \end{cases}$$

where l is the length of the pendulum, g is the acceleration (due to gravity), θ is the angle of the maximal inclination from the pointing (in the positive direction) downwards vertical (as shown in Figure 1), $0 < |\theta| < \pi$.

The configuration space of the pendulum upon which an external force of constant magnitude and direction (presumably acting along the vertical) is being exerted is a circle. In other words, the weight of the pendulum is (holonomically) constrained to lie on a circle so that its radial component is counterbalanced by pivot reaction force.

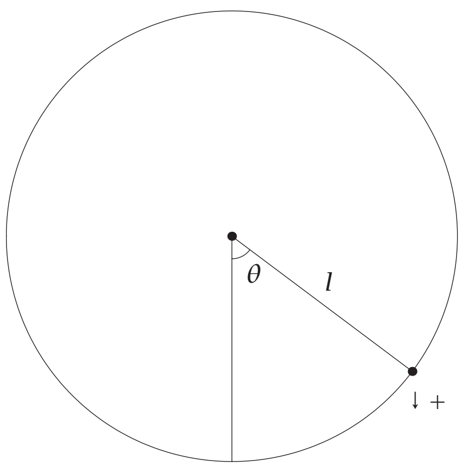

Figure 1. The pendulum.

The period corresponding to the (upper) value of k with g being positive corresponds to gravity pointing downwards (as is customarily assumed). The complementary period, corresponding to the (lower) value of k with g being negative, is then readily seen to correspond to reversing the direction of gravity. Surprisingly, too many (if not all!) "popular" references on elliptic functions, such as [15, pp. 59, 77], and "authoritative" references on mechanics, such as [19, p. 73], have missed (up to this day) these elegant and powerful expressions (for which Gauss must be solely credited), routinely providing, instead, either unfinished calculations or cumbersome power series representation (lacking iterativity and convergence expediency), hardly enabling an understanding of the double-valuedness of T. A particular (self-complementary) case to be pointed out corresponds to the (middle) value $\theta = \pi/2$, for which $|k| = \sqrt{2}/M(\sqrt{2})$. For a full appreciation of the Gauss formula, one must employ it for values of θ approaching π when traditional calculations of T via its power series representation eventually fail to converge at any reasonable time!

Perimeters of Ellipses for Five Values of Eccentricity

Denote by $l(\gamma)$ the ratio of the length of an ellipse of eccentricity γ to its major axis. Let L denote the semilength[6] of the lemniscate of Bernoulli whose focal distance is $\sqrt{2}$, whereas we use the letter M to denote, for brevity, $M(\sqrt{2})$, the reciprocal of the Gauss constant. Thus $L = \pi/M$, as calculated. We shall say that two ellipses are *complementary* if the squares of their eccentricities add up to one.

As defined, $l(0) = \pi$ is the aforementione d ratio for an ellipse whose eccentricity is zero, that is, a circle. The complementary ellipse, being an ellipse with eccentricity 1, is seen to be the (degenerate) ellipse whose semilength coincides with its major axis, so $l(1) = 2$. The latter equality might be alternatively expressed as a limiting equality:

$$\lim_{\beta\to 0}\frac{M(\beta)}{N\,(\beta^2)} = \frac{\pi}{2},$$

which one could have also attained as the limiting case of Legendre relation (3).

Supplementing formula (2) with formula (3), we shall calculate the perimeters of three more ellipses.

The self-complementary ellipse is confocal with the lemniscate, cocentered with the superscribing circle (Figure 2). It is the case to be considered first following the two preceding cases. Here, we have

$$l\left(1/\sqrt{2}\right) = \frac{L+M}{\sqrt{2}} \approx 2.7012877620953510050.$$

The latter equation might be viewed as, the discovered by Euler, special case of the Legendre relation (somewhat) disguised.[7] The precision of the numerical approximation given is attained at $(\pi/x_4 + x_4)/\sqrt{2}$ (with x_4 already calculated).

The two complementary ellipses (Figure 3) for which the eccentricities are c^2 (small) and $2\sqrt{\sqrt{2}}\,c$ (large) are:

$$l\left(c^2\right) = L + cM \approx 3.1183434891444857 7623,$$

$$l\left(2\sqrt{\sqrt{2}}\,c\right) = c\,(L + 2M)$$
$$\approx 2.0786636700153559 5794.$$

[6] *The reader might be cautioned to observe that, according to the definition being given here, the constant L is twice the so-called lemniscate constant.*

[7] *One might also observe that the length of the "sine" curve over half a period, that is, the length of the graph of the function* $t \mapsto \sin(t)$ *from the point where* $t = 0$ *to the point where* $t = \pi$, *is* $\sqrt{2}\,l\,(1/\sqrt{2}) = L + M$.

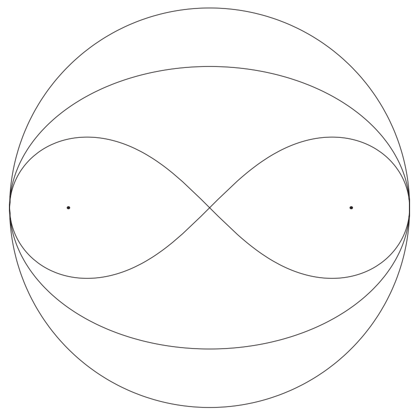

Figure 2. Confocal self-complementary ellipse and lemniscate inscribed in cocentered circle.

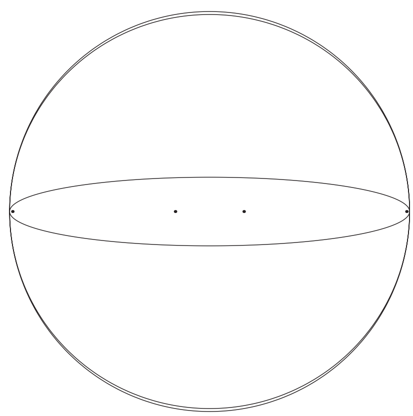

Figure 3. Two complementary ellipses and a circle sharing a single diameter.

The precision of the numerical approximation is attained at $\pi/x_4 + c\,x_4$ for the first ellipse (whose eccentricity is small) and at $c\,(\pi/x_4 + 2\,x_4)$ for the second (whose eccentricity is large).

We now exploit the latter approximation (of the perimeter of the elongated ellipse with $\beta = c^2$) in order to compare formula (2) with two well-known formulae. Put

$$F(a,b,x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2}\, x^n,$$

where $(\cdot)_n$ is the Pochhammer symbol. The ratio $l(\gamma)/\pi$ might be calculated via either one of two formulae, due to Euler and Ivory, respectively:

$$\frac{l(\gamma)}{\pi} = \sqrt{\frac{2-\gamma^2}{2}}\, F\left(-\frac{1}{4}, \frac{1}{4}, \left(\frac{\gamma^2}{2-\gamma^2}\right)^2\right)$$
$$= \frac{1+\beta}{2}\, F\left(-\frac{1}{2}, -\frac{1}{2}, \left(\frac{1-\beta}{1+\beta}\right)^2\right).$$

Assuming exact arithmetic, 337 terms of the first (Euler) power series $\sqrt{3}\,c\,F(-1/4, 1/4, 8/9)$ are required to achieve the precision of the approximation given for the latter perimeter, whereas 55 terms of the second (Ivory) power series $\sqrt{2}\,c\,F(-1/2, -1/2, 1/2)$ are still necessary to achieve that precision. The Ivory formula, although faster than the Euler formula, is (still) linearly convergent and is (particularly) slow for large eccentricities when compared with formula (2), which, being quadratically convergent, is quite indifferent[8] to larger eccentricities. As was the case with the Gauss formula for complete elliptic integrals of the first kind, the presented formula, for complete elliptic integrals of the second kind must be employed for critical values of the elliptic modulus γ (nearing one), as all conventional power series representations fail to converge at any reasonable time before a fuller appreciation evolves. As γ approaches one, the corresponding value on the right-hand side of formula (2) remains bounded, unlike the corresponding value on the right-hand side of formula (1). Thereby, the convergence of formula (2) as traditional calculations fail makes it even more convincingly superior, being the only formula applicable for practically viable calculations at critical range.

We emphasize that Cayley's formula [10]:

$$l(\gamma) = 2 + \left(\ln\left(\frac{4}{\beta}\right) - \frac{1}{1\cdot 2}\right)\beta^2$$
$$+\frac{1\cdot 3}{2\cdot 4}\left(\ln\left(\frac{4}{\beta}\right) - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)\beta^4$$
$$+\frac{1\cdot 3^2\cdot 5}{2\cdot 4^2\cdot 6}\left(\ln\left(\frac{4}{\beta}\right) - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)\beta^6$$
$$+\frac{1\cdot 3^2\cdot 5^2\cdot 7}{2\cdot 4^2\cdot 6^2\cdot 8}\left(\ln\left(\frac{4}{\beta}\right) - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{2}{5\cdot 6} - \frac{1}{7\cdot 8}\right)\beta^8 + \cdots,$$

although traditionally regarded as the remedy for calculating the perimeters of elongated ellipses, does not truly eliminate the convergence problem of Ivory's formula, replacing it with a convergence problem for calculating values of the (transcendental) logarithmic function over an unbounded domain (or, equivalently, in a neighborhood of zero). Incidentally, if the precision of the approximation for the latter ellipse (with $\beta = c^2$) is required, then (presuming the difference $\ln(4) - \ln(\beta)$ is known with sufficient precision) all terms up to (and including) the term involving β^{26} from Cayley's formula become necessary. Many more terms are needed if higher precision is desired, with Cayley's formula being, again, another power series representation for the perimeter, with the power (of β) growing only linearly.[9]

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[8] *Another formal definition is compelling here. Yet, avoiding digression, readers are urged to come up with one of their own.*

[9] *Even a polynomial growth (of the power of β) of high order would not suffice for matching the speed with which formula (2) converges; nothing less than an exponential growth would!*

Broad Concepts behind the Formula and an Epilogue

Fundamental research involving Tethered Satellite Systems at the Division of Stability of Motion and Mechanics of Controlled Systems at the CCRAS,[10] conducted by a team led by S. Ya. Stepanov,[11] has required extensive use of the elliptic functions apparatus. Two traditional approaches to studying elliptic functions (due to Jacobi and Weierstrass) were naturally united by adopting Sophus Lie's (algebraic) methodology for solving differential equations. The two groups of (linear fractional) transformations respectively fixing the differential equations, satisfied by the Weierstrass elliptic function and the Jacobi sine function, turn out to be isomorphic with each other (both being isomorphic with the Klein four-group). An essential elliptic function for which the corresponding transformations acquire the simplest form might then be (canonically) defined [3]. Halving values for such a function is far less cumbersome than halving the values of either Weierstrass or Jacobi elliptic functions, thus permitting, in particular, an attainment of exact values (expressed in quadratic radicals) at all eighth lattice points [4]. Exact special values of the modular invariant at the boundary of the fundamental domain were also (most) efficiently calculated. The formidable search for an explicit inverse of the modular invariant, initiated by Abel [1] and adamantly (yet unsuccessfully) pursued by Ramanujan, had then reached its destination on (the 212th anniversary of the Gauss discovery) May 30, 2011 [5]. Moreover, a canonical formula for halving points on elliptic curves (via efficiently inverting the doubling formula) and yielding an iterative algorithm for computing (incomplete) elliptic integrals (qualitatively revising the traditional view of the constructability of inverses for such integrals) was attained [6]. The formula for calculating the perimeter of an ellipse, presented in this paper, turned out to be next. The chapter "Elliptic Integrals" by L. M. Milne-Thomson [2, ch. 17] is highly recommended for an essential overview, and a quotation from [7, p. 591], referring to Landen transformations, seems appropriate for a conclusion: "Indeed, Landen himself evidently never realized the importance of his idea."

Agreeing, we must say that it took Gauss and over two centuries to be properly conveyed!

[10] *The Computing Centre of the Russian Academy of Sciences, Moscow, Russia.*

[11] *Sergey Yakovlevich Stepanov is one of the pioneering (in the 1960s) researchers of gyrostatic stability and stabilization of satellites.*

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