WHAT IS...

a Scissors Congruence?

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Two polyhedra in Euclidean 3-space are called *scissors congruent* (s.c.) if they can be subdivided into the same finite number of smaller polyhedra such that each piece in the first polyhedron is congruent to one in the second. If two polyhedra are s.c., then they clearly have the same volume; and for the analogous notion of s.c. in the plane, it was probably known already by the Greeks that two polygons are s.c. if and only if they have the same area. However, based on some remarks in a letter by C. F. Gauss (1844), D. Hilbert included on his famous list of mathematical problems (1900) the question of finding two polyhedra of the same volume that could be proven *not* to be s.c. This is Hilbert's 3rd problem, and it was solved by M. Dehn (1901), who found a necessary condition for two polyhedra to be s.c. that he showed was not satisfied for the cube and the regular tetrahedron of the same volume. Finally, J. P. Sydler (1965) showed that equal volume together with Dehn's condition are also sufficient for s.c. of two polyhedra in Euclidean 3-space.

The notion of s.c. of *polytopes* makes sense in all dimensions, as well as in *spherical* or even *hyperbolic* geometry. As a model for hyperbolic n-space $\mathcal{H}^n$ we use the upper half space in $\mathbb{R}^n$, consisting of points whose last coordinate is positive, and in this model hyperbolic *lines*, *planes*, etc., are Euclidean half circles (or lines), half spheres (or half planes), etc., perpendicular to the boundary. Now a polytope in Euclidean ($\mathbb{R}^n$), spherical (S^n), or hyperbolic n-space ($\mathcal{H}^n$) is a compact body that can be decomposed into finitely many *simplices*. In turn, a simplex is the intersection of $n+1$ half spaces whose boundaries are in general position. Thus we have the Generalized Hilbert 3rd Problem (GH3) of finding necessary and sufficient conditions for two polytopes in $X = \mathbb{R}^n, S^n$ or $\mathcal{H}^n$ to be s.c. For this problem one introduces an abelian group, the s.c. group $\mathcal{P}(X)$ consisting of finite formal sums of symbols $+[P]$ or $-[P]$ for P any polytope of X, subject to the relations (i) $[P] = [P'] + [P'']$ if P decomposes into P' and P'', (ii) $[P] = [gP]$ if g is an isometry of X. In these terms the above problem is to find (computable) invariants like the volume and to show that two polytopes are equivalent in $\mathcal{P}(X)$ if and only if they have the same invariants. For example, "volume" (or "area" in dimension 2) gives rise to a homomorphism V_X from $\mathcal{P}(X)$ to the reals $\mathbb{R}$. Dehn's condition can be expressed by another homomorphism, the *Dehn invariant* from $\mathcal{P}(\mathbb{R}^3)$ to another abelian group, the *tensor product*, consisting of finite formal sums of symbols $l \otimes \nu$, where l and ν are respectively a real number and a real number modulo 1 and where the tensor symbol is additive in both variables l and ν. Then the Dehn invariant for a polyhedron P is defined by the sum of tensors $l \otimes \nu$ for all edges of P, where l, respectively ν, is the *length* of the edge, respectively the *dihedral angle* at the edge, divided by 2π. Thus Sydler's theorem just states that two polyhedra in $\mathbb{R}^3$ are s.c. if and only if they have the same volume and Dehn invariant. A similar result in dimension 4 was shown by B. Jessen (1972). But in higher-dimensional Euclidean spaces the GH3 is still open.

The interest in s.c. has gradually increased since it was noticed about thirty years ago that

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the s.c. group $\mathcal{P}(X)$ is closely related to the *homology groups* of the isometry group for the geometry of X as a discrete group. In fact, GH3 is related to difficult problems in homological algebra and algebraic K-theory. In the following we shall concentrate on the cases $X = S^3$, respectively $\mathcal{H}^3$. Again, in these cases the Dehn invariant is a well-defined homomorphism, and its kernel is the 3rd homology group of the orthogonal groups O(4), respectively O(1,3). Hence GH3 in these cases is equivalent to calculating the homology groups of these Lie groups as discrete groups. Taken together, the s.c. groups for S^3 and $\mathcal{H}^3$ have a simple algebraic description as follows: Let $\mathcal{P}_{\mathbb{C}}$ be the abelian group consisting of formal sums of symbols $+[z]$ or $-[z]$, where z is a complex number different from 0 and 1 and subject to a certain 5-term relation for each pair of distinct such numbers. Now consider the involution t given by complex conjugation of the symbols. Then $\mathcal{P}(\mathcal{H}^3)$ is isomorphic to the subgroup $\mathcal{P}^-$ of elements x satisfying $t(x) = -x$, and $\mathcal{P}(S^3)$ is essentially isomorphic to the subgroup $\mathcal{P}^+$ of elements x satisfying $t(x) = x$. Geometrically, if we use the upper half space model for hyperbolic 3-space with the Riemann sphere $\mathbb{C}\cup\infty$ as boundary, then each symbol $[z]$ above corresponds to a hyperbolic tetrahedron with vertices $(\infty, 0, 1, z)$. In fact, it is well known that any 4-tuple of points on the Riemann sphere can be brought into this form by the action of a *Möbius transformation*, i.e., a map that extends to an isometry of hyperbolic space. In these terms the above-mentioned 5-term identity corresponds to the following s.c.: Consider a 5-tuple (u, v, w, x, y) of points on the Riemann sphere. Then the union of the two hyperbolic tetrahedra (u, w, x, y) and (u, v, w, y) can be decomposed into the three tetrahedra $(v, w, x, y), (u, v, x, y), (u, v, w, x)$ (cf. the schematic description in the figure). The relation to spherical geometry is more indirect, and we shall not go into that. Now, if we consider $\mathcal{P}_{\mathbb{C}}$ as the sum of $\mathcal{P}^+$ and $\mathcal{P}^-$, then the two Dehn-invariants together define an invariant for $\mathcal{P}_{\mathbb{C}}$, which has a simple algebraic description in terms of the symbols $[z]$, and the kernel is the 3rd homology group of the group $\mathrm{Sl}(2, \mathbb{C})$ of 2×2 matrices of determinant 1, i.e., the group of Möbius transformations mentioned above.

Using the algebra of $\mathcal{P}_{\mathbb{C}}$, one can prove that it is a *divisible group*; that is, for every integer n and element x there is a y such that $x = ny$. Hence the same thing is true for the s.c. groups; i.e., any spherical or hyperbolic polyhedron can be subdivided into n s.c. pieces. For $n = 2$, this is true in all dimensions and all three geometries by a direct geometric construction often attributed to C. L. Gerling, a contemporary of Gauss. Another

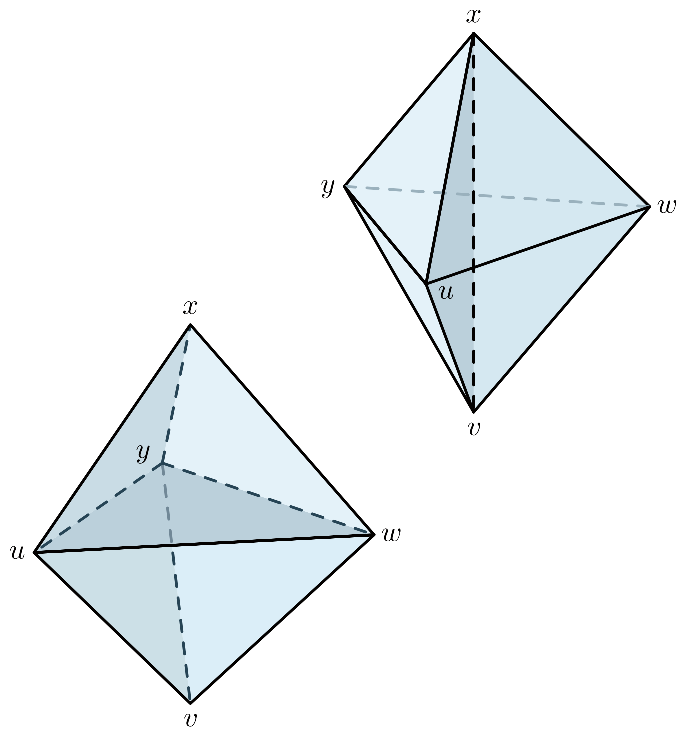

Figure 1. Two schematic decompositions of the polytope *uvwxy*.

consequence is that the 3rd homology group of $\mathrm{Sl}(2, \mathbb{C})$ as a discrete group is also divisible. This originally gave the first nontrivial example of the so-called Friedlander-Milnor conjecture, which essentially determines the homology of a Lie group as a discrete group except for components that are rational vector spaces. The latter, however, can be rather large. Now this conjecture has been proved in many cases and is a subject of current investigation.

Furthermore, if we let V_S, respectively V_H, denote the volume homomorphism for S^3, respectively $\mathcal{H}^3$, then $C = V_S + iV_H$ gives rise to a well-defined homomorphism of the homology group to the complex numbers modulo the integers: the *characteristic homomorphism* in the sense of Cheeger-Chern-Simons. In terms of the symbols $[z]$ above, this is essentially given by the *dilogarithmic function*, which is an integral of $\log(1-z)/z + \log z/(1-z)$ and which respects the 5-term defining relation for $\mathcal{P}_{\mathbb{C}}$. Some general further relations for this function involving a pair of Dynkin diagrams, which were conjectured by the physicist A. B. Zamolodchikov, have recently been proven by B. Keller and others using quivers in representation theory. Thus questions related to s.c. occur in many different areas of mathematics, particularly in algebra, geometry, and number theory.

As an example of an application of the theory of arithmetic groups to s.c., let us mention that, by some theorems of A. Borel (1977), it follows that if one restricts to spherical or hyperbolic polyhedra with vertices whose coordinates are algebraic numbers, then two such P and P' are s.c. if and only if (i) P and P' have the same volume and Dehn invariant, and furthermore (ii) $V_H \circ A_*([P]-[P']) = 0$ for all field automorphisms A of the complex field, where A_* is the induced map on $\mathcal{P}_{\mathbb{C}}$ given by applying A to each symbol. This result is of particular interest in the theory of hyperbolic manifolds, since a *fundamental domain* in this connection is a hyperbolic polyhedron with algebraic vertices. Surprisingly, it also has applications in the spherical case. Thus one can give infinitely many concrete examples (see [1, Thm. 11.19]) from which it follows that one of the following natural conjectures cannot be true: (1) (Jessen) Volume and Dehn invariant determine s.c. for spherical polyhedra, (2) (Schläffli), (Cheeger-Simons) A 3-dimensional spherical simplex with all dihedral angles being rational multiples of π has volume a rational multiple of the volume of S^3 (i.e., $2\pi^2$). The first conjecture is motivated by the Euclidean case, the second by the analogous fact for areas of spherical triangles and by a few cases for which the volume is actually known and which satisfy the conjecture for trivial reasons. However, it follows that at most only one of these conjectures can be true.

References for Further Reading

[1] J. L. Dupont, *Scissors Congruences, Group Homology and Characteristic Classes*, Nankai Tracts in Mathematics, vol. 1, World Scientific, Singapore, 2001.

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