W H A T   I S . . .

Essential Dimension?

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Informally speaking, the essential dimension of an algebraic object is the minimal number of independent parameters one needs to describe it. This notion was introduced in [1], where the objects in question were field extensions of finite degree. The general definition below is due to A. Merkurjev.

Essential Dimension of a Functor

Fix a base field k and let $\mathcal{F}$ be a covariant functor from the category of field extensions K/k to the category of sets. We think of $\mathcal{F}$ as specifying the type of algebraic object under consideration and $\mathcal{F}(K)$ as the set of algebraic objects of this type defined over K. For a field extension K/K_0, the natural ("base change") map $\mathcal{F}(K_0) \to \mathcal{F}(K)$ allows us to view an object defined over K_0 as also being defined over the larger field K. Any object $\alpha \in \mathcal{F}(K)$ in the image of this map is said to *descend* to K_0. The *essential dimension* $\mathrm{ed}(\alpha)$ is defined as the minimal transcendence degree of K_0/k, where α descends to K_0.

For simplicity we will assume that $\mathrm{char}(k) = 0$ from now on. Much of what follows remains true in prime characteristic (with some modifications).

Example 1. Let $\mathcal{F}(K)$ be the set of isomorphism classes of nondegenerate n-dimensional quadratic forms defined over a field K. Every quadratic form over K can be diagonalized. That is, q is K-isomorphic to the quadratic form $(x_1, \dots, x_n) \mapsto a_1x_1^2 + \dots + a_nx_n^2$ for some $a_1, \dots, a_n \in K^*$. Hence, q descends to $K_0 = k(a_1, \dots, a_n)$ and $\mathrm{ed}(q) \leqslant n$.

Example 2. Let $\mathcal{F}(K)$ be the set of equivalence classes of K-linear transformations $T \colon K^n \to K^n$. Here, as usual, K-linear transformations are considered equivalent if their matrices are conjugate over K. If T is represented by an $n \times n$ matrix (a_{ij}), then T descends to $K_0 = k(a_{ij} \mid i, j = 1, \dots, n)$, so that a priori $\mathrm{ed}(T) \leqslant n^2$. However, this is not optimal; we can specify T more economically by its rational canonical form R. Recall that R is a block-diagonal matrix $\mathrm{diag}(R_1, \dots, R_m)$, where each R_i is a companion matrix. If $m = 1$ and $R = R_1 = \begin{pmatrix} 0 & \dots & 0 & c_1 \\ 1 & \dots & 0 & c_2 \\ & \ddots & & \vdots \\ 0 & \dots & 1 & c_n \end{pmatrix}$, then T descends to $k(c_1, \dots, c_n)$ and thus $\mathrm{ed}(T) \leqslant n$. A similar argument shows that $\mathrm{ed}(T) \leqslant n$ for any m.

Example 3. Let $\mathcal{F}(K)$ be the set of isomorphism classes of elliptic curves defined over K. Every elliptic curve X over K is isomorphic to the plane curve cut out by a Weierstrass equation $y^2 = x^3 + ax + b$, for some $a, b \in K$. Hence, X descends to $K_0 = k(a, b)$ and $\mathrm{ed}(X) \leqslant 2$.

In many instances one is interested in the "worst case scenario", i.e., in the number of independent parameters that may be required to describe the "most complicated" objects of a particular kind. With this in mind, we define the essential dimension $\mathrm{ed}(\mathcal{F})$ of the functor $\mathcal{F}$ as the supremum of $\mathrm{ed}(\alpha)$ taken over all $\alpha \in \mathcal{F}(K)$ and all K. We have shown that $\mathrm{ed}(\mathcal{F}) \leqslant n$ in Examples 1 and 2 and $\mathrm{ed}(\mathcal{F}) \leqslant 2$ in Example 3. We will later see that, in fact, $\mathrm{ed}(\mathcal{F}) = n$ in Example 1. One can also show that $\mathrm{ed}(\mathcal{F}) = n$ in Example 2 and $\mathrm{ed}(\mathcal{F}) = 2$ in Example 3.

Essential Dimension of an Algebraic Group

Of particular interest are the Galois cohomology functors $\mathcal{F}_G$ given by $K \mapsto H^1(K, G)$, where G is an algebraic group over k. Here, $H^1(K, G)$ denotes the set of isomorphism classes of G-torsors (otherwise known as principal homogeneous spaces)

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over $\mathrm{Spec}(K)$. For many groups G this functor parametrizes interesting algebraic objects. For example, $H^1(K, \mathrm{O}_n)$ is the set of isomorphism classes of n-dimensional quadratic forms over K (in other words, $\mathcal{F}_{\mathrm{O}_n}$ is the functor of Example 1), $H^1(K, \mathrm{PGL}_n)$ is the set of isomorphism classes of central simple algebras of degree n over K, $H^1(K, G_2)$ is the set of isomorphism classes of octonion algebras over K, etc. (On the other hand, the functors $\mathcal{F}$ in Examples 2 and 3 are not of the form $\mathcal{F}_G$ for any algebraic group G.) The essential dimension of $\mathcal{F}_G$ is called *the essential dimension of* G and is denoted by $\mathrm{ed}(G)$.

Algebraic groups of essential dimension zero are precisely the *special groups*, introduced by J-P. Serre in the 1950s. An algebraic group G over k is called special if $H^1(K, G) = \{1\}$ for every field K/k. For example, SL_n and Sp_{2n} are special for every n. Over an algebraically closed field of characteristic zero, special groups were classified by A. Grothendieck. The essential dimension $\mathrm{ed}(G)$ may be viewed as a numerical measure of how much G differs from being special.

Symmetric Groups

Computing $\mathrm{ed}(\mathrm{S}_n)$ is closely linked to the classical problem of simplifying polynomials of degree n in one variable by a Tschirnhaus transformation and may be viewed as an algebraic variant of Hilbert's 13th problem [1]. In his 1884 "Lectures on the Icosahedron", F. Klein classified faithful finite group actions on the projective line $\mathbb{P}^1$ and used this classification to show that, in our terminology, $\mathrm{ed}(\mathrm{S}_5) = 2$. More generally, J. Buhler and I showed (by a different method) that $\mathrm{ed}(\mathrm{S}_n) \geqslant \lfloor n/2 \rfloor$ for any n, and $\mathrm{ed}(\mathrm{S}_n) \leqslant n - 3$ for $n \geqslant 5$; see [1]. Using these inequalities one easily finds $\mathrm{ed}(\mathrm{S}_n)$ for $n \leqslant 6$. For larger n the only additional bit of insight we have is via birational classifications of finite group actions on low-dimensional unirational varieties (over an algebraically closed field), extending Klein's original approach. In dimension 2 this yields $\mathrm{ed}(\mathrm{A}_6) = 3$ (due to Serre) and in dimension 3, $\mathrm{ed}(\mathrm{A}_7) = \mathrm{ed}(\mathrm{S}_7) = 4$ (due to A. Duncan). Serre's argument is based on the Enriques-Manin-Iskovskikh classification of rational G-surfaces, and Duncan's is based on the recent work on rationally connected G-threefolds by Yu. Prokhorov. In higher dimensions this approach appears to be beyond the reach of Mori theory, at least for now. The exact value of $\mathrm{ed}(\mathrm{S}_n)$ remains open for every $n \geqslant 8$.

Projective Linear Groups

The value of $\mathrm{ed}(\mathrm{PGL}_n)$ is intimately connected with the theory of central simple algebras. An important open conjecture, due to A. A. Albert, is that every division algebra of prime degree p is cyclic. Suppose that the base field k contains a primitive pth root of unity. Then the essential dimension of a cyclic division algebra is easily seen to be 2. Thus $\mathrm{ed}(\mathrm{PGL}_p) \geqslant 2$, and if this inequality happens to be strict for some p, then Albert's conjecture fails. The value of $\mathrm{ed}(\mathrm{PGL}_p)$ is 2 for $p = 2$ or 3 and is unknown for any other prime.

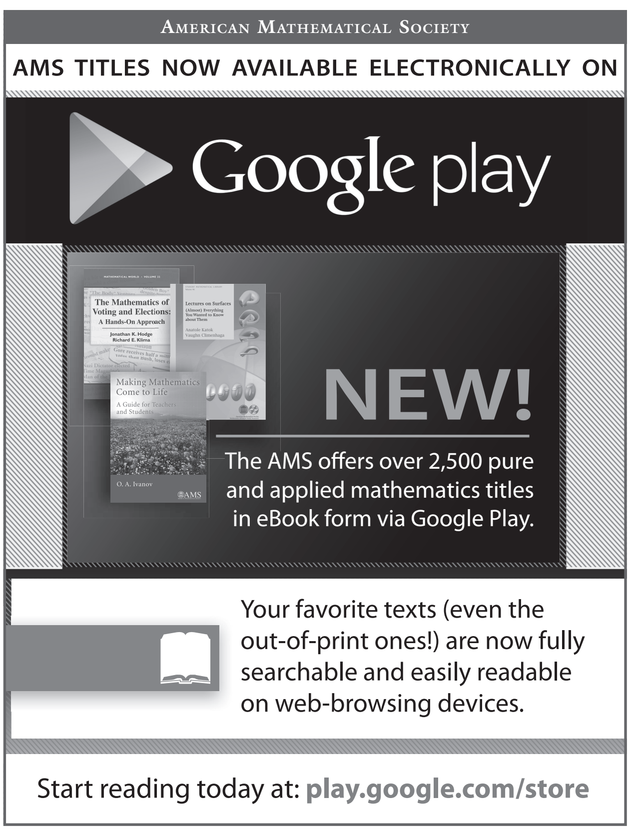
The problem of computing $\mathrm{ed}(\mathrm{PGL}_n)$ first arose in C. Procesi's pioneering work on universal division algebras in the 1960s. The inequality $\mathrm{ed}(\mathrm{PGL}_n) \leqslant n^2$ proved by Procesi has since been strengthened (see [3]), but the new upper bounds are still quadratic in n. In the 1990s B. Kahn asked if $\mathrm{ed}(\mathrm{PGL}_n)$ grows sublinearly in n, i.e., if there exists a $C > 0$ such that $\mathrm{ed}(\mathrm{PGL}_n) \leqslant Cn$ for every n. By the primary decomposition theorem we lose little if we assume that n is a prime power. Until recently, the best known lower bound was $\mathrm{ed}(\mathrm{PGL}_{p^r}) \geqslant 2r$. This has been dramatically improved by A. Merkurjev, who showed that $\mathrm{ed}(\mathrm{PGL}_{p^r}) \geqslant (r-1)p^r + 1$ for any prime p and any $r \geqslant 2$. In particular, this inequality answers Kahn's question in the negative. Surprisingly, essential dimension is not a Brauer invariant; there are central simple algebras A of degree 4 such that $\mathrm{ed}(\mathrm{M}_2(A)) < \mathrm{ed}(A)$. Little is known about essential dimension of Brauer classes.

Cohomological Invariants

Let G be an algebraic group over k. A cohomological invariant of G of degree n is a morphism of functors $\mathcal{F}_G \to H^n$, where $H^n(K)$ is the nth Galois cohomology group. The coefficient module can be arbitrary; in the examples below it will always be $\mathbb{Z}/2\mathbb{Z}$. (Observe that $H^n(K)$ is a group, whereas, in general, $\mathcal{F}_G(K) = H^1(K, G)$ has no group structure.) Serre noted that if k is algebraically closed and G admits a nontrivial cohomological invariant of degree n, then $\mathrm{ed}(G) \geqslant n$. Letting G be the orthogonal group O_n, so that $\mathcal{F}_G$ is the functor considered in Example 1, and applying the above inequality to the cohomological invariant $\mathcal{F}_G \to H^n(K)$ which takes a quadratic form $q = a_1x_1^2 + \cdots + a_nx_n^2$ to its nth Stiefel-Whitney class $(a_1) \cdots (a_n) \in H^n(K)$, we obtain $\mathrm{ed}(\mathrm{O}_n) \geqslant n$ and thus $\mathrm{ed}(\mathrm{O}_n) = n$ (we showed that $\mathrm{ed}(\mathrm{O}_n) \leqslant n$ in Example 1). Another interesting example, also due to Serre, is the degree 5 invariant of the exceptional group F_4, which gives rise to the inequality $\mathrm{ed}(F_4) \geqslant 5$.

Nontoral Finite Abelian Subgroups

Nontoral finite abelian subgroups first arose in the foundational work of A. Borel on the cohomology of classifying spaces for compact Lie groups. Nontoral finite abelian subgroups of algebraic groups were subsequently studied by T. Springer, R. Steinberg, R. Griess, and many



others. In a special group such as SL_n, every finite abelian subgroup is contained in a torus. Ph. Gille, B. Youssin, and I generalized this as follows: if $A \subset G$ is a finite abelian subgroup, then $\mathrm{ed}(G) \geqslant \mathrm{rank}(A) - \mathrm{rank}(Z_G(A)^0)$. Here G is connected and reductive, $\mathrm{rank}(A)$ is the minimal number of generators of A, $Z_G(A)^0$ is the connected component of the centralizer of A in G, and the rank of $Z_G(A)^0$ is the dimension of its maximal torus. (Note that the above inequality is of interest only if A is nontoral; otherwise the right-hand side is $\leqslant 0$.) For example, the exceptional group $G = E_8$ has a self-centralizing subgroup $A \simeq (\mathbb{Z}/2\mathbb{Z})^9$; hence $\mathrm{ed}(E_8) \geqslant 9$. I am not aware of any other proof of the last inequality. In particular, no nontrivial cohomological invariants of E_8 of degree 9 are known.

Spinor Groups

Over the past few years there has been rapid progress in the study of essential dimension, based on an infusion of methods from the theories of algebraic stacks and algebraic cycles. These developments are beyond the scope of the present note; please see the surveys [2, 3] for an overview and further references. I will, however, mention one unexpected result that has come up in my joint work with P. Brosnan and A. Vistoli: it turns out that the essential dimension of the spinor group Spin_n increases exponentially with n. This has led to surprising consequences in the theory of quadratic forms. Note that $\mathcal{F}_{\mathrm{Spin}_n}$ is closely related to n-dimensional quadratic forms with trivial discriminant and Hasse-Witt invariant, very much in the spirit of Example 1, and that there are no high rank finite abelian subgroups in Spin_n to account for the exponentially high value of $\mathrm{ed}(\mathrm{Spin}_n)$. Are there high-degree cohomological invariants of Spin_n? We do not know.

Further Reading

[1] J. Buhler and Z. Reichstein, On the essential dimension of a finite group, *Compositio Math.* **106** (1997), no. 2, 159–179.

[2] A. Merkurjev, Essential dimension, in *Quadratic Forms—Algebra, Arithmetic, and Geometry* (R. Baeza, W. K. Chan, D. W. Hoffmann, and R. Schulze-Pillot, eds.), Contemporary Math., vol. 493, Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.

[3] Z. Reichstein, Essential dimension, *Proceedings of the International Congress of Mathematicians*, Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 162–188.