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Notices

of the American Mathematical Society

January 2013 **Volume 60, Number 1**

AMS Staff Photos (see page 83)

125th Anniversary

American Mathematical Society

AMS

Advancing Mathematics Since 1888

AMERICAN MATHEMATICAL SOCIETY

CURRENT EVENTS BULLETIN

Friday, January 11, 2013, 1:00 PM to 5:00 PM

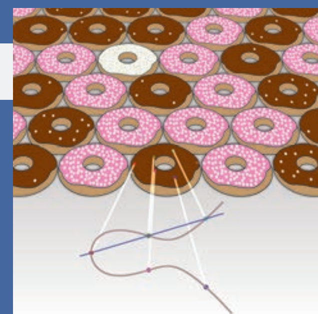
Room 6F, Upper Level, San Diego Convention Center

Joint Mathematics Meetings, San Diego, CA

1:00 PM Wei Ho

How many rational points does a random curve have?

Bhargava and his school have turned classical questions about cubic polynomial equations in two variables over the integers in a whole new direction.



2:00 PM Sam Payne

Topology of nonarchimedean analytic spaces

"Tropical" algebra and geometry is a burgeoning field, suggesting interesting paths into geometry over fields like the *p*-adic numbers.

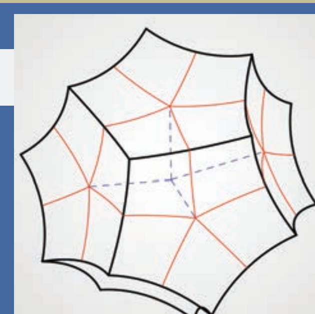
Image courtesy of Matthew Baker, Georgia Institute of Technology.

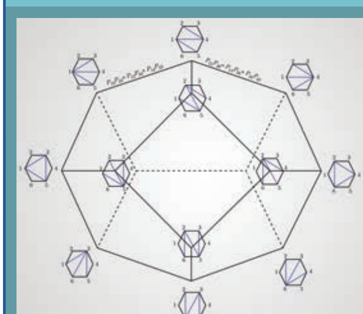
3:00 PM Mladen Bestvina

Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston's vision for three-manifolds

Another vindication of Thurston's fabulous insights into three-dimensional geometry comes with a wonderful group-theoretic construction.

Image created by Silvio Levy.





4:00 PM Lauren Williams

Cluster algebras

An exotic but simple structure that depends on a finite directed graph was brought to light in representation theory by Fomin and Zelevinsky. It has now been shown to play an important role in subjects as diverse as Poisson geometry and the theory of triangulations of surfaces as well.

Organized by David Eisenbud, University of California, Berkeley

125th Anniversary
AMERICAN MATHEMATICAL SOCIETY
AMS
Advancing Mathematics Since 1888

GOALS: to highlight the excellence of mathematical achievements in the Americas within the context of the international arena, and foster collaborations among researchers, students, institutions and mathematical societies in the Americas.

Guanajuato, Mexico
August 5 - 9, 2013

Plenary Speakers
James Arthur (Toronto, Canada)
Artur Avila (IMPA, Brazil and CNRS, France)
Manjul Bhargava (Princeton, USA)
Luis Caffarelli (Texas, USA)
Ingrid Daubechies (Duke, USA)

Invited Speakers
Martin Barlow (British Columbia, Canada)
Mónica Clapp (UNAM, Mexico)
Fernando Codá Marques (IMPA, Rio de Janeiro, Brazil)
Gonzalo Contreras (CIMAT, Mexico)
Octav Cornea (Montréal, Canada)
Alicia Dickenstein (Buenos Aires, Argentina)
Patricio Felmer (CMM, Chile)
Clovis Gonzaga (Florianópolis, Brazil)
Christof Geiss (UNAM, Mexico)
Onésimo Hernandez-Lerma (CINVESTAV, Mexico)
Yoshiharu Kohayakawa (Sao Paulo, Brazil)
Mark Lewis (Alberta, Canada)
Alejandro Maas (CMM, Santiago, Chile)
Robert McCann (Toronto, Canada)
Roberto Miatello (Cordoba, Argentina)
Fernando Rodriguez-Villegas (Austin, Texas, US)
Martin Sambarino (Montevideo, Uruguay)
Eva Tardos (Cornell, US)
Eduardo Teixeira (Fortaleza, Brazil)
Gunther Uhlman (Washington and UC Irving, US)
Amie Wilkinson (Northwestern and Chicago, US)

PRIZES
Nominations should be sent by e-mail to
mca2013.prizes@gmail.com
The deadline for nominations is January 31, 2013.

MCA Prize:
Five prizes of $1000 each will be awarded to mathematicians who are no more than 12 years past their PhD and either received their graduate education or currently hold a position in one or more countries in the Americas.

Americas Prize:
One prize of $5000 will be awarded to an individual or a group in recognition of their work to enhance collaboration and the development of research that links mathematicians in several countries in the Americas.

Solomon Lefschetz Medal
Two medals with an award of $5000 will be given to mathematicians in recognition of their excellence in research and their contributions to the development of mathematics in a country or countries in the Americas.

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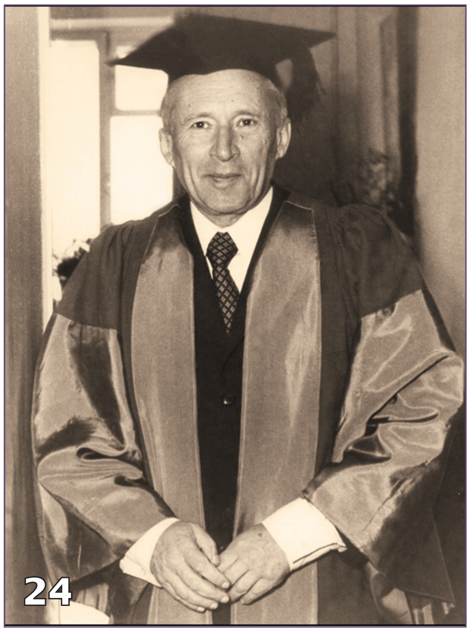
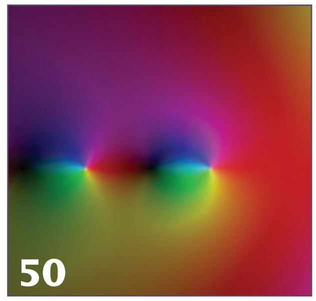
Notices
of the American Mathematical Society

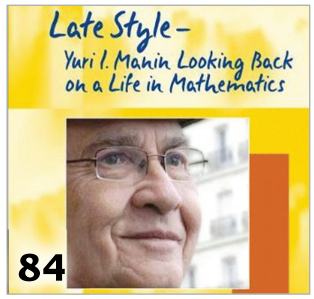
January 2013

Communications

Commentary







January offers a rich variety of material. We have a feature article on wavelets by three of the world's experts. We have the second part of the article celebrating Ramanujan's125th birthday. We have a remarkable and insightful article about closed forms in mathematics. There is also a memorial article for the astonishing and versatile mathematician I. M. Gelfand. Finally, a student gives her impression of the Joint Meetings in Boston in January of 2012.

—Steven G. Krantz, Editor

Features

Notices
of the American Mathematical Society

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Departments

Opinion

My Year on Capitol Hill: 5 Lessons I Have Learned

This past academic year, I had the privilege of being the 2011–2012 AMS Congressional Fellow. I want to take this opportunity to give my personal perspective of what I have learned about the culture and legislative process on Capitol Hill. Here are five of the lessons I learned.

1) **Scaling Helps Put Big Numbers into Perspective**: Today, out of every one dollar the government spends, roughly sixty-two cents are devoted to mandatory spending on entitlement programs (including Social Security, Medicare, Medicaid, and interest on federal debt). Another twenty cents are spent on defense appropriations. That leaves just eighteen cents for discretionary nondefense items, including research funding within the NSF, NIST, NASA, and the NIH. Most budgetary debates are over these eighteen cents. Often one hears of a "$100 million dollar cut here", or "a $100 million dollar increase there". For the average constituent, such a figure becomes "just another number" with no reference to everyday life.

There is a better way to drive home the point. The average American family makes roughly $50,000 a year. Assume that the entire U.S. budget is scaled to this hypothetical family's income of $50K. A $100 million cut in a federal program would be analogous, in our family's budget, to cutting $1.40 or roughly twelve cents a month: for example, saving twelve cents per month on a gallon of milk. In the meantime there is a $3,400 monthly mortgage payment (the other eighty-two percent).

If nothing is done about our current fiscal situation, the discretionary nondefense portion will become a smaller piece of the federal budget, with interest payments on debt increasing substantially.

2) **Congress was Designed to Prevent the Enactment of Legislation**: The process of introducing and passing legislation is messy, with no clear path and no known algorithm proven to lead to a solution. Most legislation that is proposed is a combination of political kabuki dashed with media theatrics, resulting in no concrete action. Legislation is introduced to raise awareness of an issue, to keep debate on a subject alive, or to make core constituents happy. Serious legislation, however, involves dialogue with all stakeholders, with extensive input and communication from all interested parties. I learned this lesson during my involvement with legislation aimed at improving standards and methodology in forensic science. The current status (at the time of my submission of this piece) of this specific bill remains unsettled. However, the long and complicated process of negotiating the legislation is evidence that serious and deliberative debate is a prerequisite for enacting legislation.

3) **Budget is Policy**: The best way to understand anyone's priorities is to look at their checkbook. The same is true of the Federal Government. The priorities of the administration and of the Congress will always be reflected in their budgets. For most people, reading the federal budget is akin to reading a telephone book. Nevertheless, knowing the budget brings a thorough understanding of programs and initiatives in federal agencies. Equally important is understanding the process of negotiating a budget and how "friction points" are resolved. An intricate knowledge of the budget and its process will enable a legislator to work nimbly and gain the respect of his/her fellow colleagues and staffers.

4) **Framing Scientific Research as "Need to Have" not "Nice to Have"**: Science research in the United States is predominately funded by the American taxpayer. From my observation, science funding enjoys wide bipartisan support. However, what is true today is not necessarily true tomorrow. We are now entering a period of tight budgets. Science funding will not be immune to deep and debilitating cuts, so our community must avoid complacency. We must frame science not as a luxury but as something essential, with documented and significant return on investment for our nation. The case for basic science funding is even more imperative, given the tendency by lawmakers to put a lower priority on basic research than on applied research, which they believe will translate into immediate economic growth. A balanced approach must be emphasized. In addition, the scientific community must be willing to change the existing culture of science to adapt to the new fiscal reality that emphasizes spreading funding opportunities for younger investigators, increased scrutiny for established investigators with significant grant support, and the quality rather than the quantity of publications.

5) **The Greatest Threat to Democracy is**…**?**: Contrary to popular opinion, the greatest threat to democracy is NOT the numerous cable TV and radio talk shows, nor is it the growth of political action committees and third-party interest groups. No! The greatest threat to our democracy is the absence of our participation. This truism is the most obvious yet the most under-appreciated lesson I learned. I encourage everyone reading this article to take seriously the responsibility of citizenship. Constituents must continually communicate with their representatives, offer cogent suggestions, and thank them for their support on an issue. As my senator's chief of staff once told me: "We have no idea what is on people's minds until they contact our office." A serious legislator may not agree with your viewpoints, but he/she will make sure to respond appropriately. Our system is "broken" only when citizens no longer take the time to participate in democracy and engage on the issues.

I want to close by thanking the American Mathematical Society for their sponsorship of my Congressional Fellowship this year. This experience has truly been both life-changing and informative and I will always be grateful for this opportunity. Thank you!

—Richard Yamada
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`yry@duke.edu`

DOI: http://dx.doi.org/10.1090/noti938

Letters to the Editor

Old and New Algorithms for π

This letter concerns Semjon Adlaj's article "An eloquent formula for the perimeter of an ellipse" (*Notices* **59**, September 2012, 1094–1099). In his comments on the "(so-called) Brent-Salamin algorithm" for computing π, Professor Adlaj misses some important points.

First, both Brent and Salamin acknowledged their debt to Gauss and Legendre. That the names "Brent-Salamin" or "Salamin-Brent" are widely used is probably due to the ambiguity of calling something new after Gauss and Legendre, e.g., a Google search for "Gauss-Legendre" gives many hits on Gauss-Legendre quadrature.

Second, although Euler discovered the special case of Legendre's relation that is used in the simplest Brent-Salamin algorithm ($k=k'=1/\sqrt{2}$), the more general form of Legendre's relation is needed for the members of the family of algorithms that arise from choosing $k \neq k'$. Since Legendre's relation is not attributed to Euler, it would be uninformative to use the name "Gauss-Euler" as Professor Adlaj suggests [footnote 4]. A Google search for "Gauss-Euler" gives even more hits than one for "Gauss-Legendre", but they are almost all irrelevant.

Third, and more important, none of those three great mathematicians of the past would have appreciated the significance of such an algorithm, because they lived in the days before electronic computers and fast algorithms, such as the Schönhage-Strassen algorithm, for multiplication of large integers. Without such technology and modern algorithms, the Brent-Salamin algorithm is a relatively poor algorithm for computing π-algorithms based on the Maclaurin series for arctan$(1/n)$, such as Machin's $\pi/4=4\arctan(1/5)-\arctan(1/239)$, are far superior (even today, they are competitive if combined with binary splitting and fast multiplication algorithms). Indeed, on reading Gauss's unpublished notebook entry of May 1809, it seems probable that he did not regard his discovery as an algorithm for computing π, since π only appears in the denominator of the right-hand side of the crucial equation. More likely Gauss regarded this equation as an interesting identity involving elliptic integrals, only incidentally involving the known constant π. (The relevant notebook entry is reproduced on page 99 of the book *Pi: Algorithmen, Computer, Arithmetik* by Arndt and Haenel.)

Finally, perhaps this emphasis on the computation of a single constant is unwarranted. Brent's 1975 and 1976 papers, not referenced by Professor Adlaj, showed that *all* elementary functions can be evaluated to given accuracy just as fast as π, up to a constant factor, by using the arithmetic-geometric mean. This of course includes the computation of an infinite set of constants such as e^{π} and π/e. No doubt this fact would have been of more interest to Euler, Legendre, and Gauss than yet another formula or algorithm for π.

—Richard Brent
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Yet Another Remark on Algebra Education

In the October 2012 issue of the *Notices*, Peter Johnson argued, citing studies, that algebra does not necessarily transfer critical thinking skills over to other areas.

Just as Dr. Johnson's article is not meant to discredit algebra but to induce skepticism, so is this point: it seems likely that, testing for transference, a study in educational psychology would consider only introductory college algebra classes, and not more rigorous upper-levels.

Such lower-level courses often eschew thoughtful development of algebraic structures, given concerns about time and the students' prerequisites, for rote application of methods. While this may transfer a little bit of critical thinking, certainly algebra's biggest contribution would come from classes like linear algebra and group theory, which are unlikely to be studied psychologically.

All of this is to say that, while a lack of documentation of algebra's role in improving critical thinking encourages skepticism, so does the fact that higher algebra is rarely if ever considered.

—Douglas Weathers
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(Received October 5, 2012)

In Response to Rob Kirby's "Whither Journals?"

I was saddened to read Rob Kirby's article "Whither Journals?" in the October 2012 issue of the *Notices*.

It seems that it is becoming a trend in the mathematical community to accept and endorse journals that collect "author processing charges" (APC). This trend has grown out of the frustration of the community with the fact that we have to pay a third party in order to read our papers; it has been decided that it makes more sense to pay a third party in order to publish our papers. I fail to see the logic.

One possible explanation for this trend is that moving to APC publication will make all papers "open-access". But that cannot be the explanation, because almost all papers are already published on people's websites or on the arXiv (and if some researchers refuse to put their publicly funded research on their website or on the arXiv, the answer for that might be legislation).

The only explanation I can find for this trend is that some of its proponents are people of such good intentions that they simply cannot see the obvious corruption and other twisted results that embracing APC publication will result in. For example, Rob Kirby suggested that his hypothetical journals XJM, YJM and ZJM "might correspond to A+, A, and A- papers…" and later, in parenthesis: "Papers graded C or lower could appear in journals in which no money changes hands and only volunteer work is done; B papers would fall somewhere in between." Note the educational message encapsulated here ("nothing is any good if no money changes hands"). From a more practical point of view, what I read is this: commercial publishers will charge higher prices from authors submitting to the more prestigious journals. I do not need to spell out the possible consequences.

If academic publishing is funded by public money, we must search for low-cost publishing models. Volunteer work and community effort are not bad words. Over the last two decades, many have "stepped up to the plate," and several excellent journals come to mind. Anyone can find them (and read them) online.

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(Received October 17, 2012)

Srinivasa Ramanujan: Going Strong at 125, Part II

Krishnaswami Alladi, Editor

Ramanujan passport photo.

The 125th anniversary of the birth of Srinivasa Ramanujan was on December 22, 2012. To mark the occasion, the Notices *is publishing a feature article of which this is the second and final installment. This installment contains pieces by Ken Ono, Kannan Soundararajan, Robert Vaughan, and Ole Warnaar on various aspects of Ramanujan's work. The first installment appeared in the December 2012 issue, and contained an introductory piece by Krishnaswami Alladi plus pieces by George Andrews, Bruce Berndt, and Jonathan Borwein on Ramanujan's work.*

Krishnaswami Alladi is professor of mathematics at the University of Florida. His email address is `alladik@ufl.edu`.

DOI: http://dx.doi.org/10.1090/noti926

Ken Ono

Modular Forms, Partitions, and Mock Theta Functions

Ramanujan's work on modular forms, partitions, and mock theta functions is rich with tantalizing examples of deeper mathematical structures. Indeed, his tau-function, his partition congruences, and his mock theta functions are prototypes that have helped to shape the modern mathematical landscape.

Ramanujan was fascinated by the coefficients of the function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24} \tag{1}$$
$$= q - 24q^2 + 252q^3 - 1472q^4 + \cdots,$$

where $q := e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$. This function is a weight 12 *modular form*. In other words, $\Delta(z)$ is a function on the upper half of the complex plane such that

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for every matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. He conjectured (see p. 153 of [12]) that

$$\tau(nm) = \tau(n)\tau(m),$$

for every pair of coprime positive integers n and m, and that

$$\tau(p)\tau(p^s) = \tau(p^{s+1}) + p^{11}\tau(p^{s-1}),$$

for primes p and positive integers s. Mordell proved these conjectures, and in the 1930s Hecke later

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developed a framework for the theory of modular forms and L-functions in which such properties play a fundamental role.

Ramanujan was also very interested in the size of the numbers $\tau(n)$, and for primes p he conjectured (see pp. 153–154 of [12]), but could not prove, that

$$|\tau(p)| \le 2p^{\frac{11}{2}}.$$

This speculation is the first example of the Ramanujan-Petersson Conjectures, among the deepest problems in the analytic theory of automorphic forms. This inequality was triumphantly confirmed by Deligne [8] as a corollary to his proof of the Weil Conjectures, work which won him a Fields Medal in 1978. Although Ramanujan did not anticipate the Weil Conjectures (the Riemann Hypothesis for varieties over finite fields), he correctly anticipated the importance of optimally bounding coefficients of modular forms.

Ramanujan is also well known for his work on the congruence properties of his tau-numbers. For example, he proved that (see page 159 of [12])

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}. \tag{2}$$

About forty years ago, Serre [13] and Swinnerton-Dyer [14] wrote beautiful papers interpreting such congruences in terms of certain two-dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the Galois group of the algebraic closure of $\mathbb{Q}$. Deligne had just proved that such representations encode the coefficients of certain modular forms as "traces of the images of Frobenius elements". Serre and Swinnerton-Dyer interpreted Ramanujan's tau-congruences, such as (2), as the first nontrivial examples of certain exceptional representations. For the prime $\ell = 691$, there is a (residual) Galois representation

$$\rho_{\Delta,691} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/691\mathbb{Z})$$

which, for primes $p \ne 691$, satisfies

$$\rho_{\Delta,691}(\mathrm{Frob}(p)) = \begin{pmatrix} 1 & * \\ 0 & p^{11} \end{pmatrix},$$

where $\mathrm{Frob}(p) \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the "Frobenius element at p". Congruence (2) is then implied by Deligne's prescription that, for primes $p \ne 691$, one has

$$\tau(p) = \mathrm{Tr}(\rho_{\Delta,691}(\mathrm{Frob}(p))) \equiv 1+p^{11} \pmod{691}.$$

The theory of modular ℓ-adic Galois representations has subsequently flourished and famously is the "language" of Wiles's proof of Fermat's Last Theorem.

Ramanujan's work on tau-congruences is intimately related to his work on partition congruences. A *partition* of the natural number n is any nonincreasing sequence of natural numbers whose sum is n, and the number of partitions of n is denoted by $p(n)$. The generating function for $p(n)$ has two convenient representations:

$$\tag{3} \begin{aligned} \sum_{n=0}^{\infty} p(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}. \end{aligned}$$

The infinite product representation is analogous to the infinite product for $\Delta(z)$ in (1), which explains the important role that modular forms play in the study of $p(n)$, while the other representation as a "basic hypergeometric series" foreshadows Ramanujan's last work: his discovery of "mock theta functions".

In groundbreaking work, Ramanujan proved that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

He also conjectured corresponding congruences, known as Ramanujan's congruences, modulo powers of 5, 7, and 11. Extending his ideas, Atkin and Watson proved [3], [15]:

If $\delta = 5^a 7^b 11^c$ *and* $24\lambda \equiv 1 \pmod{\delta}$,

then $p(\delta n + \lambda) \equiv 0 \pmod{5^a 7^{\lfloor \frac{b}{2} \rfloor + 1} 11^c}$.

These congruences have inspired much further work.

In the 1940s Dyson, seeking a combinatorial explanation for the congruences, defined [9] the *rank* of a partition to be the largest summand minus the number of summands. He conjectured that the partitions of $5n+4$ (resp. $7n+5$) are divided into 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7), thereby providing a combinatorial explanation for the congruences mod 5 and 7. This conjecture was proved in the 1950s by Atkin and Swinnerton-Dyer [5]. Dyson was unable to offer a combinatorial explanation for the mod 11 congruence, and he conjectured the existence of a statistic, which he referred to as the "crank", which would suitably explain the congruences mod 5, 7, and 11. In the 1980s Andrews and Garvan [2] finally found the elusive crank.

In addition to finding a combinatorial explanation for Ramanujan's original congruences, researchers have looked for further congruences. In the 1960s Atkin [4] found congruences, though not so systematic, with modulus 13, 17, 19, 23, 29, and 31. About ten years ago the author and Ahlgren [1], [10] found that there are partition

Photo courtesy of Krishnaswami Alladi.

The lovely colonial style architecture is evident in the Town High School in Kumbakonam that Ramanujan attended.

congruences modulo every integer Q relatively prime to 6. For example, we have

$$p(59^4 \cdot 13n + 111247) \equiv 0 \pmod{13}.$$

These more recent results arise from the fact that (3) is essentially a modular form. This fact, combined with deep work of Shimura, makes it possible to apply the methods of Deligne and Serre, which explained Ramanujan's tau-congruences, to the partition numbers.

During his last year of life, when he was seeking a return to good health in south India, Ramanujan discovered functions he called *mock theta functions*. In his last letter to Hardy, dated January 12, 1920, Ramanujan shared hints (see p. 220 of [6]) of his last theory. The letter, roughly four typewritten pages, consists of formulas for seventeen strange power series and a discussion of their asymptotics. It contained no proofs of any kind. By changing signs in (3), we obtain a typical example of a mock theta function:

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} a_f(n) q^n \\ &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \end{aligned}$$

For followers of Ramanujan, the main questions were: What did Ramanujan mean by a mock theta function, and what roles do these functions play in mathematics? These questions would remain open for eighty years. In the meantime, such researchers as Andrews, Choi, Dragonette, Dyson, Gordon, Hickerson, McIntosh, Selberg, and Watson, to name a few, proved beautiful theorems about these strange q-series. Then in 2002, in his Ph.D. thesis [17] written under Zagier, Zwegers made sense of the mock theta functions. He discovered how to "complete" them by adding a nonholomorphic function, a so-called "period integral", to obtain a *real analytic modular form*.

In the case of $f(q)$ and its companion $\omega(q)$, Zwegers defined the vector-valued functions

$$\begin{aligned} F(z) &= (F_0(z),\ F_1(z),\ F_2(z))^T \\ &:= (q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T, \\ G(z) &= (G_0(z),\ G_1(z),\ G_2(z))^T \\ &:= 2i\sqrt{3}\int_{-\overline{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau, \end{aligned}$$

where the $g_i(z)$ are theta functions, and he proved that $H(z) := F(z) - G(z)$ satisfies

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z)$$

and

$$H(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $\zeta_n := e^{\frac{2\pi i}{n}}$, making it a *vector-valued real analytic modular form*.

All of Ramanujan's mock theta functions turn out to be holomorphic parts of special weight 1/2 real analytic modular forms, which Bruinier and Funke [7] call *weak harmonic Maass forms*. Loosely speaking, a *weight k harmonic Maass form* is a smooth function $M(z)$ on $\mathbb{H}$ which transforms as does a weight k modular form and which also satisfies $\Delta_k(M) = 0$. Here the *hyperbolic Laplacian* Δ_k, where $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Since modular forms appear prominently in mathematics, I think that one expects these functions to have far-reaching implications. One might expect the functions themselves to play many roles. This has already turned out to be the case. These forms have appeared prominently in the following subjects [11], [16]: partitions and q-series, moonshine, Donaldson invariants, probability theory, Borcherds products and elliptic curves, and many others.

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Photo courtesy of Krishnaswami Alladi.

Kannan Soundarajan receiving the First SASTRA Ramanujan Prize from Dr. Aurobinda Mitra, director of the Department of Science and Technology, India, on December 20, 2005. Manjul Bhargava (second from left), also a winner of the prize that year, looks on. Also in the picture are Krishnaswami Alladi, chair of the prize committee (next to Soundararajan), and R. Sethuraman, vice-chancellor of SASTRA (between Bhargava and Mitra).

K. Soundararajan

Ramanujan and the Anatomy of Integers

The story of Hardy and Ramanujan and the taxicab number 1729 is well known. It may be less familiar that taxicab numbers also formed the motivation for a famous paper of Hardy and Ramanujan on the number of prime factors of integers (see paper 35 and announcement 32 in [22]). They note that "anybody who will make a practice of factorising numbers, such as the numbers of taxi-cabs" would verify that "round numbers"—integers composed of a considerable number of comparatively small factors—"are exceedingly rare." Hardy and Ramanujan explain this phenomenon by showing that almost all numbers n below x (in the sense of density) are formed of about $\log\log n$ prime factors. More precisely, if $\omega(n)$ denotes the number of distinct prime factors of n, then $|\omega(n) - \log\log n| \le \Phi(n)\sqrt{\log\log n}$ for almost all $n \le x$, where $\Phi(n)$ is any function tending to infinity with n, and the same result holds also for $\Omega(n)$, which counts the prime factors of n with multiplicity.

The Hardy-Ramanujan result was refined by Erdős and Kac [9], who showed that $(\omega(n) - \log\log n)/\sqrt{\log\log n}$ has an approximately normal distribution with mean 0 and variance 1. The Erdős-Kac theorem is in relation to the Hardy-Ramanujan theorem as the Central Limit Theorem is to the Law of Large Numbers. These two results formed the impetus for the development of probabilistic ideas in number theory, and we refer the reader to Elliott [7], Tenenbaum [27], and Kubilius [17] for accounts of related work.

Hardy and Ramanujan proved their theorem by means of an ingenious induction argument that establishes a uniform upper bound for $\pi_k(x)$, the number of integers below x with exactly k prime factors. The prime number theorem tells us that $\pi(x) = \pi_1(x)$, the number of primes below x, is approximately $x/\log x$. Landau generalized this to show that, for any fixed k and large x, $\pi_k(x)$ is approximately $\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}$. In probabilistic terms, this suggests that the number of prime factors of an integer has an approximately Poisson distribution with parameter $\log\log x$. Since a Poisson distribution with large parameter λ approximates a normal distribution with mean and variance equal to λ, this interpretation is in keeping with the Hardy-Ramanujan and Erdős-Kac theorems. It is therefore natural to wonder whether the uniform upper bound of Hardy and Ramanujan could be refined to an asymptotic formula, which would be a considerable strengthening of the Erdős-Kac theorem. This problem was highlighted by both

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Hardy and Ramanujan and inspired a great deal of work over the years by many authors (notably Pillai [20], Erdős [8], Sathe [25], and Selberg [26]), culminating in the definitive work of Hildebrand and Tenenbaum [15].

The Hardy-Ramanujan theorem, furthered by the insight of Erdős, led to a host of problems (see, for example, [13]) designed to understand the multiplicative structure of a random integer—or as Granville [4] has termed it, "the anatomy of (random) integers". Interestingly, the anatomy of a random integer is strikingly similar to the anatomy of random polynomials over finite fields and to the anatomy of the cycle decompositions of random permutations; see [2] and the forthcoming graphic novel(!) [12] for accounts of these similarities. Moreover, understanding the anatomy of integers has proven valuable in many problems of algorithmic number theory, for example, in the development of factorization algorithms such as the quadratic sieve [21].

Out of the many beautiful theorems in the area, let us mention the striking recent work of Ford [10] in understanding the distribution of integers having a divisor of a given size. Ford's work resolved a longstanding problem of Erdős: How many distinct integers are there in the $N \times N$ multiplication table? The Hardy-Ramanujan theorem, Erdős observed, gives a quick proof that the number of distinct integers in the multiplication table is $o(N^2)$. A typical integer of size N has about $\log\log N$ prime factors, and so the product of two such integers has about $2\log\log N$ prime factors. Thus a typical element appearing in the multiplication table has $2\log\log N$ prime factors, whereas a typical integer of size N^2 has only about $\log\log(N^2) = \log\log N + \log 2$ prime factors. In other words, elements in the multiplication table are quite atypical and thus few in number. Refining a result of Tenenbaum [28], Ford's work establishes that true order of magnitude for the number of integers in the multiplication table is $N^2/((\log N)^{\delta}(\log\log N)^{\frac{3}{2}})$, with δ being the esoteric constant $1 - (1 + \log\log 2)/\log 2 = 0.086\ldots$. In the study of random permutations, there arises an analogous problem: Given n and $1 \le k \le n-1$, the permutation group S_n acts naturally on the k-element subsets of $\{1, \ldots, n\}$. Now given a random element in S_n, what is the probability that there exists some k-element subset left fixed by this permutation? Partial results on this problem were established by Luczak and Pyber [19] and have interesting applications [5]. Using Ford's work on integers as a guide to the anatomy of permutations, one can obtain a more refined understanding of this question [6].

We now turn to Ramanujan's work on "highly composite numbers". These are numbers n such that $d(n) > d(k)$ for all $1 \le k < n$, where $d(n)$ counts the number of divisors of n. Ramanujan described (see paper 15 in [22]) the anatomy of such highly composite integers and also made a careful study of the extremal order of $d(n)$. Ramanujan was also interested in finding large values of the iterated functions $d(d(n))$ (whose exact order is determined in the surprisingly recent paper [3]) and $d(d(d(n)))$. Closely related to highly composite numbers are the *superabundant* numbers: these are the integers n for which $\sigma(n)/n$ is larger than $\sigma(k)/k$ for all $1 \le k \le n-1$, and where $\sigma(n)$ denotes the sum of the divisors of n. Superabundant numbers were studied extensively by Alaoglu and Erdős [1], who were directly motivated by Ramanujan's work. Interestingly, Ramanujan himself had studied these classes of numbers, in notes related to his long paper on highly composite numbers, but owing to space considerations these were not included in the published work. These manuscript notes were discovered as part of Ramanujan's "Lost Notebook" [23] and were published with annotations by Nicolas and Robin in [24]. While interesting, these problems seem far from the mainstream, and so it may come as a surprise that the Riemann Hypothesis is equivalent to the estimate $\sigma(n) \le H_n + \exp(H_n)\log H_n$ (where H_n is the "harmonic number" $H_n = \sum_{j=1}^{n} 1/j$), and the difficult cases of this inequality are precisely when n is a superabundant number; see Lagarias [18] for a charming account of this connection.

We end by mentioning an unpublished and largely unknown fragment of Ramanujan on a central topic in the anatomy of integers, namely, on "smooth" (or "friable") numbers. A number is called y-smooth if all its prime factors are at most y. Smooth numbers appear prominently in algorithmic number theory (e.g., in factoring algorithms) and are also useful in seemingly unrelated questions such as Waring's problem; see [16] and [11] for excellent surveys on these integers without large prime factors. A basic question is to estimate $\Psi(x, y)$, which counts the number of y-smooth integers below x; a precise understanding of $\Psi(x, y)$ uniformly in a wide range is equivalent to the Riemann Hypothesis [14]. The first published work on smooth numbers in 1930 was by Dickman, who showed that if $x = y^u$, $u > 0$ is considered fixed, and y tends to infinity, then $\Psi(x, y) \sim \rho(u)x$, where ρ (the Dickman function) is defined as the unique continuous solution to the differential-difference equation $u\rho'(u) = -\rho(u-1)$ for $u > 1$, and with initial condition $\rho(u) = 1$ for $0 \le u \le 1$. Given the significance of smooth numbers, it may be interesting to record that Ramanujan appears to have made a study of this problem. On page 337 of [23] one finds formulas for $\Psi(y^u, y)$ in the range $1 \le u \le 5$, which give inclusion-exclusion type expressions for $\rho(u)$ in terms of iterated integrals and with a clear indication of how the pattern is to be continued.

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R. C. Vaughan

The Hardy–Littlewood–Ramanujan Method

In modern analytic number theory there are three very powerful standard techniques that are central to many investigations. One is the use of the theory of Dirichlet series, including the Riemann zeta function and Dirichlet L-functions; a second is the application of sieve theory in all its manifestations; and the third is the subject of this survey. It has its genesis in two papers of Ramanujan. The most famous of these is the celebrated paper [Hardy, G. H. & Ramanujan, S. 1918] (there are also announcements in [Hardy, G. H. & Ramanujan, S. 1917a] and [Hardy, G. H. & Ramanujan, S. 1917b]), which is concerned mostly with the partition function, but the paper [Ramanujan, S. 1918] also describes the same fundamental idea. This is that interesting number theoretic functions have representations, or close approximations, as infinite series indexed by the positive rational numbers—that is, the a/q with $1 \le a \le q$ and $(a,q) = 1$—and moreover that such expressions can be obtained by consideration of generating functions that have the unit circle as a natural boundary and the terms in the aforementioned series arise from the neighborhood of singularities

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at $e(a/q)$. A very simple example of this is the interesting formula

$$\sigma(n) = \frac{\pi^2}{6} n \sum_{q=1}^{\infty} q^{-2} c_q(n), \tag{1}$$

which first appears in the second of these papers. Here $c_q(n)$ is the eponymous sum

$$c_q(n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e(an/q). \tag{2}$$

In addition to the partition function, Hardy and Ramanujan also observe that the same ideas can be applied to the number of representations of a number as a sum of a fixed number of squares.

One might speculate how this could have developed if Ramanujan had not been struck down by illness. Fortunately, Hardy and Littlewood realized the very great importance and generality of the underlying conceptions and were able to develop them so as to apply to the Goldbach and Waring, and cognate, problems. There was also the realization of the predictive possibilities of the method in those cases where the arguments could not be otherwise pursued to a satisfactory conclusion. Later this led, through further important ideas of I. M. Vinogradov and Davenport, in the 1930s to the essential resolution of the ternary Goldbach problem (namely, that every large odd integer is the sum of three primes) and substantial progress in the known upper bounds for $G(k)$ (the smallest s such that every large natural number is the sum of at most s kth powers) in Waring's problem (see [Vaughan, R. C. 1997]). Davenport, along with Birch and Lewis, embarked on the question of applying the method to nondiagonal questions, for example, the nontrivial representation of zero by cubic forms (see [Davenport, H. 1963] and [Davenport, H. 1962, 2005]). [Davenport, H. & Lewis, D. J. 1963] had also established the Artin conjecture concerning the local-to-global principle for diagonal forms for all but a finite set of degrees. When I was a postgraduate student in the late 1960s, I became interested in the method. However, I was warned by a number of distinguished mathematicians that the method was already played out and it would be better to work on other things! Nevertheless, in [Vaughan, R. C. 1977] I did manage to use the method to deal with the cases not dealt with by Davenport and Lewis, and in a different direction [Montgomery, H. L. & Vaughan, R. C. 1975] we were able to establish that there is a positive number δ such that the number $E(x)$ of even numbers not exceeding x which are not the sum of two primes satisfies

$$E(x) \ll x^{1-\delta}.$$

In the last twenty-five years there has been a huge explosion of activity. This note is too short to do more than mention a few highlights, with the hope that the reader may be pointed in an interesting direction. On Waring's problem the development of efficient differencing, building on the earlier work, especially that of Davenport, led to significant further progress on the upper bound for $G(k)$, and there is a survey of this and some other connected results in [Vaughan, R. C. & Wooley, T. D. 2002]. Very recently, the sensational work of [Wooley, T. D. to appear] now means that the asymptotic formula for the number of representations of a natural number as the sum of s kth powers is known to hold when $s \geq 2k^2 - 2k - 8$. On questions involving forms there has been impressive progress on applications of the method on a number of fronts, especially by [Schmidt, W. M. 1979] and [Wooley, T. D. 1997] on small zeros; [Heath–Brown, D. R. 1983], [Hooley, C., 1988, 1991, 1994], and [Hooley, C., 1988, 1991, 1994] on cubic forms; [Heath-Brown, D. R. & Browning, T. D. 2009] on cubic polynomials; and [Brüdern, J & Wooley, T. D. 1998] and [Brüdern, J & Wooley, T. D. 2003] on a variety of topics.

In a different direction, [Heath-Brown, D. R. 1981] has ingeniously combined the circle method with sieve theory, and this was taken up by [Brüdern, J. 1988] in his thesis and has been exploited extensively by [Brüdern, J. 1995], [Brüdern, J. & Kawada, K. 2002], [Kawada, K. 1997], and [Kawada, K. 2005]. This has spawned a whole new industry of ongoing work on sums of powers of primes and almost primes.

In the original series of papers "On some problems of partitio numerorum" by Hardy and Littlewood there was an unpublished manuscript on small differences between consecutive primes, and it was this manuscript which, when combined with developments in the large sieve, led to the work of [Bombieri, E. & Davenport, H. 1966] and the later refinements of [Pil'tjaĭ, G. Z. 1972], [Huxley, M. N. 1977], and [Maier, H. 1985] on

$$\delta = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n}. \tag{3}$$

The later proof by [Goldston, D. A., Pintz, J. & Yıldırım 2009] that $\delta = 0$ rests mostly on sieve theory, yet the motivation for the shape of some of the main terms comes from heuristics arising from the Hardy-Littlewood method. In many ways the motivation for the large sieve also comes from the Hardy–Littlewood method, and perhaps this can be seen most clearly in [Gallagher, P. X. 1967]'s proof of the large sieve inequality.

An unexpected development was the adaptation of the method to deal with questions involving arithmetic progressions in general sets. This began with the seminal paper of [Roth, K. F. 1953] dealing with sets having no three elements in arithmetic progression and led to [Gowers, W. T. 1998]'s proof of Szemerédi's theorem and the theorem of [Green, B. J. & Tao, T. 2008] that the primes contain arbitrarily long arithmetic progressions. This is another very active area; see, for example, [Green, B. J. 2007] or [Tao, T. & Vu, Van, 2006]. A cognate area concerns the mean square distribution of sets of integers in arithmetic progressions. The prototype is the more precise version of the Barban-Davenport-Halberstam theorem concerning the set of primes due to [Montgomery, H. L. 1970] and refined by [Hooley, C. 1975]. Montgomery used a variant of the Hardy-Littlewood method, and more recently this was revived by [Goldston, D. A. & Vaughan, R. C. 1996] for the set of primes and by [Vaughan, R. C. 1998] to treat general dense sets. This has been taken a step further by [Brüdern, J. 2009], which extends the ideas of the latter paper to treat general binary additive problems.

Another aspect of the method is that the idea of large peaks near "rational points" $e(a/q)$ with q relatively small (the major arcs) and lesser peaks near $e(a/q)$ with q relatively large (the minor arcs) has suggested divisions of arguments in, apparently, otherwise quite unrelated topics. An important example of this is the work of Bombieri, Iwaniec, Huxley, and Watt on exponential sums which led to the currently best-known bounds for the Riemann zeta function on the $\frac{1}{2}$-line, and the error terms in the Dirichlet divisor problem and Gauss's estimation of the number of lattice points in a large disc centered at the origin. For an exposition of this, see [Huxley, M. N. 1996].

For standard accounts of the Hardy-Littlewood method in its more classical forms, see [Davenport, H. 1962, 2005] and [Vaughan, R. C. 1997]. The foreword to the second edition of [Davenport, H. 1962, 2005] gives a good survey of a number of the applications as they stood in 2005.

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Ramanujan's ${}_1\psi_1$ Summation

Notation. It is impossible to give an account of the ${}_1\psi_1$ summation without introducing some q-series notation. To keep the presentation as simple as possible, we assume that $0 < q < 1$. Suppressing q-dependence, we define two q-shifted factorials: $(a)_\infty := \prod_{k=0}^{\infty}(1 - aq^k)$ and $(a)_z := (a)_\infty/(aq^z)_\infty$ for $z \in \mathbb{C}$. Note that $1/(q)_n = 0$ if n is a negative integer. For $x \in \mathbb{C} - \{0, -1, \dots\}$, the q-gamma function is defined as $\Gamma_q(x) := (q)_{x-1}/(1-q)^{x-1}$.

Ramanujan recorded his now famous ${}_1\psi_1$ summation as item 17 of Chapter 16 in the second of his three notebooks [13, p. 32], [46]. It was brought to the attention of the wider mathematical community in 1940 by Hardy, who included it in his twelfth and final lecture on Ramanujan's work [31]. Hardy remarked that the result constituted "a remarkable formula with many parameters." Instead of presenting the ${}_1\psi_1$ sum as given by Ramanujan and Hardy, we will state its modern form:

$$(1) \qquad \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (q)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty}, \qquad |b/a| < |z| < 1,$$

where it is understood that $a, q/b \notin \{q, q^2, \dots\}$. Characteristically, Ramanujan did not provide a proof of (1). Neither did Hardy, who, however, remarked that it could be "deduced from one which is familiar and probably goes back to Euler." The result to which Hardy was referring is another famous identity—known as the q-binomial theorem—corresponding to (1) with $b = q$: $\sum_{n=0}^{\infty} z^n (a)_n/(q)_n = (az)_\infty/(z)_\infty$ and valid for $|z| < 1$. Although not actually due to Euler, the q-binomial theorem is certainly classic. It seems to have appeared first and without proof (for $a = q^{-N}$) in Rothe's 1811 book *Systematisches Lehrbuch der Arithmetik*, and in the 1840s many mathematicians of note, such as Cauchy (1843), Eisenstein (1846), Heine (1847), and Jacobi (1847), published proofs. The first proof of the ${}_1\psi_1$ sum is due to Hahn in 1949 [30] and, as hinted by Hardy, uses the q-binomial theorem. After Hahn, a large number of alternative proofs of (1) were found, including one probabilistic and three combinatorial proofs [2], [3], [5], [16], [17], [19],

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[20], [18], [23], [24], [32], [34], [35], [44], [48], [53], [50]. The proof from the book, which again relies on the q-binomial theorem, was discovered by Ismail [32] and is short enough to include here. Assuming $|z| < 1$ and $|b| < \min\{1, |az|\}$, both sides are analytic functions of b. Moreover, they coincide when $b = q^{k+1}$ with $k = 0, 1, 2, \ldots$ by the q-binomial theorem with $a \mapsto aq^{-k}$. Since 0 is the accumulation point of this sequence of b's, the proof is complete.

Apart from the q-binomial theorem, the ${}_1\psi_1$ sum generalizes another classic identity, known as the Jacobi triple-product identity: $\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\binom{n}{2}} = (z)_\infty (q/z)_\infty (q)_\infty =: \theta(z)$. This result plays a central role in the theory of theta and elliptic functions.

The ${}_1\psi_1$ Sum as Discrete Beta Integral

As pointed out by Askey [8], [9], the ${}_1\psi_1$ summation may be viewed as a discrete analogue of Euler's beta integral. First, define the Jackson or q-integral $\int_0^{c\cdot\infty} f(t)\,\mathrm{d}_q t := (1-q)\sum_{n=-\infty}^{\infty} f(cq^n)cq^n$. Replacing $(a, b, z) \mapsto (-c, -cq^{\alpha+\beta}, q^\alpha)$ in (1) then gives

$$(2)\quad \int_0^{c\cdot\infty} \frac{t^{\alpha-1}}{(-t)_{\alpha+\beta}}\,\mathrm{d}_q t = c^\alpha \frac{\theta(-cq^\alpha)}{\theta(-c)} \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. For real, positive c the limit $q \to 1$ can be taken, resulting in the beta integral modulo the substitution $t \mapsto t/(1-t)$. Askey further noted in [8] that the specialization $(\alpha, \beta) \mapsto (x, 1-x)$ in (2) (so that $0 < \mathrm{Re}(x) < 1$) may be viewed as a q-analogue of Euler's reflection formula.

Simple Applications of the ${}_1\psi_1$ Sum

There are numerous easy applications of the ${}_1\psi_1$ sum. For example, Jacobi's well-known four- and six-square theorems as well as a number of similar results readily follow from (1); see e.g., [1], [14], [15], [21], [22], [25]. To give a flavor of how the ${}_1\psi_1$ implies these types of results, we shall sketch a proof of the four-square theorem. Let $r_s(n)$ be the number of representations of n as the sum of s squares. The generating function $R_s(q) := \sum_{n\ge 0} r_s(n)(-q)^n$ is given by $\bigl(\sum_{m=-\infty}^{\infty}(-1)^m q^{m^2}\bigr)^s$. By the triple-product identity this is also $\bigl((q)_\infty/(-q)_\infty\bigr)^s$. Any identity that allows the extraction of the coefficient of $(-q)^n$ results in an explicit formula for $r_s(n)$. Back to (1): replace $(b, z) \mapsto (aq, b)$ and multiply both sides by $(1-b)/(1-ab)$. By the geometric series this yields

$$(3)\quad 1 + \frac{(1-a)(1-b)}{1-ab} \sum_{k,n=1}^{\infty} q^{kn}(a^k b^n - a^{-k} b^{-n}) = \frac{(abq)_\infty (q/ab)_\infty (q)_\infty^2}{(aq)_\infty (q/a)_\infty (bq)_\infty (q/b)_\infty},$$

which may also be found in Kronecker's 1881 paper "Zur Theorie der elliptischen Functionen". For $a, b \to -1$ the right side gives $R_4(q)$, whereas the left side becomes

$$\begin{aligned} 1 - 8 \sum_{m=1}^{\infty} q^m \sum_{\substack{n,k=1 \\ nk=m}}^{\infty} n(-1)^{n+k} \\ = 1 + 8 \sum_{m=1}^{\infty} (-q)^m \sum_{\substack{d\ge 1 \\ 4\nmid d|m}} d. \end{aligned}$$

Hence $r_4(n) = 8\sum_{d\ge 1;\ 4\nmid d|n} d$. This result of Jacobi implies Lagrange's theorem that every positive integer is a sum of four squares. By taking $a, b^2 \to -1$ in (3) the reader will have little trouble showing that $r_2(n) = 4(d_1(n) - d_3(n))$, with $d_k(n)$ the number of divisors of n of the form $4m + k$. This is a result of Gauss and Lagrange which implies Fermat's two-square theorem.

Other simple but important applications of the ${}_1\psi_1$ sum concern orthogonal polynomials. In [11] it was employed by Askey and Wilson to compute a special case—corresponding to the continuous q-Jacobi polynomials—of the Askey-Wilson integral, and in [10] Askey gave an elementary proof of the full Askey-Wilson integral using the ${}_1\psi_1$ sum. The sum also implies the norm evaluation of the weight functions of the q-Laguerre polynomials [45]. These form a family of orthogonal polynomials with discrete measure μ on $[0, c\cdot\infty)$ given by $\mathrm{d}_q\mu(t) = t^\alpha/(-t)_\infty\,\mathrm{d}_q t$. The normalization $\int \mathrm{d}_q\mu(t)$ thus follows from the q-beta integral (2) in the limit of large β.

Generalizations in One Dimension

There exist several generalizations of Ramanujan's sum containing one additional parameter. In his work on partial theta functions, Andrews [4] obtained a generalization in which each product of four infinite products on the right-hand side is replaced by six such products. Another example is the curious identity of Guo and Schlosser, which is no longer hypergeometric in nature [27]:

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k(1-ac_kq^k)(c_kq)_\infty(b/ac_k)_\infty}{(b)_k(1-azq^k)(ac_k)_\infty(q/ac_k)_\infty} c_k^k = \frac{1}{(1-z)} \frac{(q)_\infty(b/a)_\infty}{(q/a)_\infty(b)_\infty},$$

where $c_k := z(1-aczq^k)/(1-azq^k)$ and $|b/ac| < |z| < 1$. For $c = 1$ this is (1).

As discovered by Schlosser [49], a quite different extension of the ${}_1\psi_1$ sum arises by considering noncommutative variables. Let R be a unital Banach algebra with identity 1, central elements b and q, and norm $\|\cdot\|$. Write a^{-1} for the inverse of an invertible element $a \in R$. Let $\prod_{i=m}^{n} a_i$ stand for 1 if $n = m-1$, $a_m \cdots a_n$ if $n \ge m$ and $a_{m-1}^{-1} \cdots a_{n+1}^{-1}$ if $n < m-1$, and define

$$\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; z\right)_k^{\pm} := \prod_i \left[z \prod_{s=1}^{r} (1 - a_s q^{i-1})(1 - b_s q^{i-1})^{-1} \right],$$

where $k \in \mathbb{Z} \cup \{\infty\}$, $a_1, \dots, a_r, b_1, \dots, b_r \in R$, $\prod_i = \prod_{i=1}^{k}$ in the $+$ case and $\prod_i = \prod_{i=k}^{1}$ in the $-$ case. Subject to $\max\{\|q\|, \|z\|, \|ba^{-1}z^{-1}\|\} < 1$, the following noncommutative ${}_1\psi_1$ sum holds:

$$\sum_{k=-\infty}^{\infty} \left(\begin{matrix} a \\ b \end{matrix}; z\right)_k^{+} = \left(\begin{matrix} za \\ z \end{matrix}; 1\right)_\infty^{-} \left(\begin{matrix} qa^{-1}z^{-1} \\ qza^{-1}z^{-1} \end{matrix}; 1\right)_\infty^{+} \left(\begin{matrix} bza^{-1}z^{-1}, q \\ ba^{-1}z^{-1}, b \end{matrix}; 1\right)_\infty^{-}.$$

Higher-Dimensional Generalizations

Various authors have generalized (1) to multiple ${}_1\psi_1$ sums. Below we state a generalization due to Gustafson and Milne [28], [41] which is labelled by the A-type root system. Similar such ${}_1\psi_1$ sums are given in [6], [7], [29], [43], [47]. More involved multiple ${}_1\psi_1$ sums with a Schur or Macdonald polynomial argument can be found in [12], [36], [42], [52]. For $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$ denote $|r| := r_1 + \cdots + r_n$. Then

$$\sum_{r \in \mathbb{Z}^n} z^{|r|} \prod_{1 \le i < j \le n} \frac{x_i q^{r_i} - x_j q^{r_j}}{x_i - x_j} \prod_{i,j=1}^{n} \frac{(a_j x_{ij})_{r_i}}{(b_j x_{ij})_{r_i}} = \frac{(az)_\infty (q/az)_\infty}{(z)_\infty (b/az)_\infty} \prod_{i,j=1}^{n} \frac{(b_j x_{ij}/a_i)_\infty (q x_{ij})_\infty}{(q x_{ij}/a_i)_\infty (b_j x_{ij})_\infty},$$

where $a := a_1 \cdots a_n$, $b := q^{1-n} b_1 \cdots b_n$, $x_{ij} := x_i/x_j$ and $|b/a| < |z| < 1$. Milne first proved this for $b_1 = \cdots = b_n$ [41], and shortly thereafter Gustafson established the full result [28]. We have already seen that the ${}_1\psi_1$ sum implies the Jacobi triple-product identity. The latter is the $\mathrm{A}_1^{(1)}$ case of Macdonald's generalized Weyl denominator identities for affine root systems [38]. To obtain further Macdonald identities from the Gustafson-Milne sum, one replaces $z \to z/a$ before letting $a_1, \dots, a_n \to \infty$ and $b_1, \dots, b_n \to 0$. Extracting the coefficient of z^0 (on the right this requires the triple-product identity) results in the Macdonald identity for $\mathrm{A}_{n-1}^{(1)}$.

Higher-dimensional generalizations of a special case of the ${}_1\psi_1$ sum can be given for all affine root systems. A full description is beyond our scope here, so we will sketch only the simplest case. The reader is referred to [26], [38], [39], [40] for the full details. In [39] Macdonald gave the following multivariable extension of the product formula for the Poincaré polynomial of a Coxeter group:

(4)
$$W(\boldsymbol{t}) := \sum_{w \in W} \prod_{\alpha \in R^+} \frac{1 - t_\alpha \mathrm{e}^{w(\alpha)}}{1 - \mathrm{e}^{w(\alpha)}} = \prod_{\alpha \in R^+} \frac{1 - t_\alpha \boldsymbol{t}^{\mathrm{ht}(\alpha)}}{1 - \boldsymbol{t}^{\mathrm{ht}(\alpha)}}.$$

Here R is a reduced, irreducible finite root system in a Euclidean space V, R^+ the set of positive roots, W the Weyl group, and t_α for $\alpha \in R^+$ a set of formal variables constant along Weyl orbits. The symbol $\boldsymbol{t}^{\mathrm{ht}(\alpha)}$ stands for $\prod_{\beta \in R^+} t_\beta^{(\beta,\alpha)/\|\alpha\|^2}$ with $(\cdot,\cdot)$ the W-invariant positive definite bilinear form on V. If all t_α are set to t, then $\boldsymbol{t}^{\mathrm{ht}(\alpha)} = t^{\mathrm{ht}(\alpha)}$ with $\mathrm{ht}(\alpha)$ the usual height function on R, in which case $W(\boldsymbol{t})$ reduces to the classical Poincaré polynomial $W(t)$. Now let S be a reduced, irreducible affine root system of type $S = S(R)$ [38]. In analogy with the finite case, assume that t_a for $a \in S$ is constant along orbits of the affine Weyl group W of S. Then Macdonald generalized (4) to [40]

(5)
$$\sum_{w \in W} \prod_{a \in S^+} \frac{1 - t_a \mathrm{e}^{w(a)}}{1 - \mathrm{e}^{w(a)}} = \prod_{\alpha \in R^+} \frac{(t_\alpha \boldsymbol{t}^{\mathrm{ht}(\alpha)})_\infty (\boldsymbol{t}^{\mathrm{ht}(\alpha)} q^{\chi(\alpha \in B)}/t_\alpha)_\infty}{(\boldsymbol{t}^{\mathrm{ht}(\alpha)})_\infty^2},$$

where B is a base for R. The parameter q on the right is fixed by $q = \prod_{a \in B(S)} \exp(n_a a)$, where $B(S)$ is a basis for S and the n_a are the labels of the extended Dynkin diagrams given in [38]. If R is simply-laced, then $t_a = t$. In the case of $S(R) = \mathrm{A}_1^{(1)}$, $q = \exp(a_0 + a_1)$ so that after replacing $\exp(a_1)$ by x we obtain the ${}_1\psi_1$ sum (1) with $(a, b, z) \to (x/t, tx, t)$. This is not the end of the story concerning root systems and the ${}_1\psi_1$ sum. Identity (5) can be rewritten as [40]

(6)
$$\sum_{\gamma \in Q^\vee} \prod_{\alpha \in R} \frac{(q\,\mathrm{e}^\alpha)_{(\alpha,\gamma)}}{(t_\alpha q\,\mathrm{e}^\alpha)_{(\alpha,\gamma)}} = \prod_{\alpha \in R^+} \frac{(t_\alpha \boldsymbol{t}^{\mathrm{ht}(\alpha)} q)_\infty (\boldsymbol{t}^{\mathrm{ht}(\alpha)} q^{\chi(\alpha \in B)}/t_\alpha)_\infty}{(\boldsymbol{t}^{\mathrm{ht}(\alpha)} q)_\infty (\boldsymbol{t}^{\mathrm{ht}(\alpha)})_\infty} \times \frac{(q\,\mathrm{e}^\alpha)_\infty (q\,\mathrm{e}^{-\alpha})_\infty}{(t_\alpha q\,\mathrm{e}^\alpha)_\infty (t_\alpha q\,\mathrm{e}^{-\alpha})_\infty},$$

where $Q^\vee$ is the coroot lattice. Interestingly, for $t_\alpha = t$ this was also found by Fishel, Grojnowski and Teleman [26] by computing the generating function of the q-weighted Euler characteristics of certain Dolbeault cohomologies. For $R = \mathrm{A}_{n-1}$, $Q^\vee = Q = \sum_{i=1}^{n} r_i \epsilon_i$ with $|r| = 0$, $R = \{\epsilon_i - \epsilon_j : 1 \le i \ne j \le n\}$, and $t^{\mathrm{ht}(\epsilon_i - \epsilon_j)} = t^{j-i}$. By fairly elementary manipulations, the identity (6) may then be transformed into the multiple ${}_1\psi_1$ sum

$$\sum_{r\in\mathbb{Z}^n} z^{|r|}\frac{(a)_{|r|}}{(b)_{|r|}}\prod_{1\le i<j\le n}\frac{x_iq^{r_i}-x_jq^{r_j}}{x_i-x_j}\,\frac{(t^{-1}x_{ij})_{r_i-r_j}}{(tqx_{ij})_{r_i-r_j}}\,t^{r_i-r_j}q^{-r_j}$$
$$=\frac{(az)_\infty(q/az)_\infty(b/a)_\infty(tq)_\infty}{(z)_\infty(b/az)_\infty(q/a)_\infty(b)_\infty}\prod_{i=1}^{n-1}\frac{(t^{i+1}q)_\infty}{(t^i)_\infty}\prod_{i,j=1}^{n}\frac{(qx_{ij})_\infty}{(tqx_{ij})_\infty},$$

for $|b/a| < |z| < 1$ and $|t| < 1$. This is the only result in this survey that is new.

We finally remark that all higher-dimensional ${}_1\psi_1$ sums admit representations as discrete Selberg-type integrals. The most important such integrals are due to Aomoto [6], [7] and Ito [33], and are closely related to (5). Further examples may be found in [37], [51].

Photo courtesy of Krishnaswami Alladi.

The museum at SASTRA University in Kumbakonam, where several manuscripts, letters, papers, documents, photographs, etc., relating to Ramanujan are displayed.

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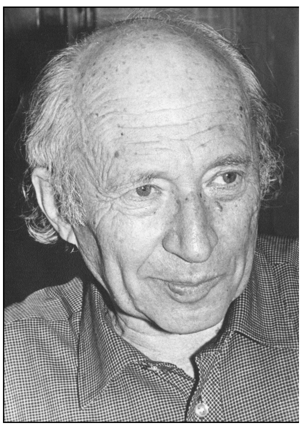
Israel Moiseevich Gelfand, Part I

Vladimir Retakh, Coordinating Editor



I. M. Gelfand

Israel Moiseevich Gelfand, a mathematician compared by Henri Cartan to Poincaré and Hilbert, was born on September 2, 1913, in the small town of Okny (later Red Okny) near Odessa in the Ukraine and died in New Brunswick, New Jersey, USA, on October 5, 2009.

Nobody guided Gelfand in his studies. He attended the only school in town, and his mathematics teacher could offer him nothing except encouragement—and this was very important. In Gelfand's own words: "Offering encouragement is a teacher's most important job." In 1923 the family moved to another place and Gelfand entered a vocational school for chemistry lab technicians. However, he was expelled in the ninth grade as a son of a "bourgeois element" ("*netrudovoi* element" in Soviet parlance)—his father was a mill manager. After that Gelfand (he was sixteen and a half at that time) decided to go to Moscow, where he had some distant relatives.

Until his move to Moscow in 1930, Gelfand lived in total mathematical isolation. The only books available to him were secondary school texts and several community college textbooks. The most advanced of these books claimed that there are three kinds of functions: analytical, defined by formulas; empirical, defined by tables; and correlational. Like Ramanujan, he was experimenting a lot. Around age twelve Gelfand understood that some problems in geometry cannot be solved algebraically and drew a table of ratios of the length of the chord to the length of the arc. Much later it became clear for him that in fact he was drawing trigonometric tables.

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From this period came his Mozartean style and his belief in the unity and harmony of mathematics (including applied mathematics)—the unity determined not by rigid and loudly proclaimed programs, but rather by invisible and sometimes hidden ties connecting seemingly different areas. Gelfand described his school years and mathematical studies in an interview published in *Quantum*, a science magazine for high school students [1]. In Gelfand's own words: "It is my deep conviction that mathematical ability in most future professional mathematicians appears…when they are 13 to 16 years old…. This period formed my style of doing mathematics. I studied different subjects, but the artistic form of mathematics that took root at this time became the basis of my taste in choosing problems that continue to attract me to this day. Without understanding this motivation, I think it is impossible to make heads or tails of the seeming illogicality of my ways in working and the choice of themes in my work. Because of this motivating force, however, they actually come together sequentially and logically."

The interview also shows how a small provincial boy was jumping over centuries in his mathematical discoveries. At the age of fifteen Gelfand learned of a series for calculating the sine. He described this moment in the "*Quantum*" interview: "Before this I thought there were two types of mathematics, algebraic and geometric…When I discovered that the sine can be expressed algebraically as a series, the barriers came tumbling down, and mathematics became one. To this day I see various branches of mathematics, together with mathematical physics, as parts of a united whole."

After arriving in Moscow, Gelfand did not have steady work and lived on earnings from occasional odd jobs. At some point he had the good fortune to work at the checkout counter at the State (Lenin's) Library. This gave Gelfand a rare opportunity to talk with mathematics students from Moscow University. He also started to attend university seminars, where he found himself under intense psychological stress: new breezes were blowing in mathematics with the new demands for rigorous proofs. It was so different from his "homemade" experiments and his romantic views of mathematics. He also learned that none of his discoveries were new. Neither this nor other circumstances of his life deterred him, and his interest in mathematics continued to grow.

Just as his abrupt expulsion from school, the next twist in Gelfand's fate was also one of many paradoxes of life in the Soviet Union. On one hand, as a son of a "bourgeois element" he could not be a university student. On the other hand, at eighteen he was able to obtain a teaching position at one of many newly created technical colleges and at nineteen to enter the Ph.D. program at Moscow University. The reasons were simple: the Soviet state needed knowledgeable instructors to educate its future engineers and scientists of the proper "proletarian origin". But at that time the system was not rigid enough to purge or even strictly regulate graduate schools. As a result, a talented boy was able to enter a Ph.D. program without a college or even a high school diploma.

At the beginning of his career Gelfand was influenced by several Moscow mathematicians, especially his thesis adviser, A. N. Kolmogorov. In the *Quantum* interview Gelfand said that from Kolmogorov he learned "that a true mathematician must be a philosopher of nature." Another influence was the brilliant L. G. Shnirelman. In 1935 Gelfand defended his "candidate" (Ph.D.) thesis and in 1940 obtained the higher degree of Doctor of Science. In 1933 he began teaching at Moscow University, where he became a full professor in 1943 and started his influential seminar. He lost this position temporarily in 1952 during the infamous "anti-cosmopolitans" (in fact, anti-Semitic) campaign but was allowed to continue the seminar. Gelfand also worked at the Steklov Institute and for many years at the Institute for Applied Mathematics. There he took part in the secret program related to the Soviet version of the Manhattan Project and its extensions. Andrei Sakharov mentioned his work with Gelfand in [1].

In 1953 Gelfand was elected a Corresponding Member of the Academy of Science (an important title in the Soviet hierarchy). This happened right after Stalin's death and at the end of the anti-cosmopolitans campaign. According to Gelfand, the political uncertainty of the times made his election possible. Later the situation in the USSR stabilized, anti-Semitism became part of the Soviet system, and Gelfand became a full member of the Soviet Academy only in 1984 after being elected to leading foreign academies.

In 1989 Gelfand moved to the United States. After spending some time at Harvard and MIT, he became a professor at Rutgers University, where he worked until his death.

Gelfand's parents.

Gelfand at age 3, 1916.

The articles in the *Notices* present various descriptions of Gelfand's multifaceted research, his way of doing mathematics, and interaction with people. A leading American expert once told me that after reading all definitions in Gelfand's papers, he could easily prove all his theorems. Well, this easiness was based on long computations and a thorough consideration of a variety of carefully selected examples. Gelfand himself liked to repeat a statement by a Moscow mathematician: "Gelfand cannot prove hard theorems. He just turns any theorem into an easy one."

Gelfand always was surrounded by numerous collaborators attracted by his legendary intuition and the permanent flow of new ideas: in every decade he was establishing a new area of research. He had completely different approaches to different people, as described by A. Vershik, A. Zelevinsky, and me in our recollections. His students and collaborators represent an unusual variety of styles and interests. The only similarity for members of the Gelfand school was their inherited passion for mathematics.

One cannot write about Gelfand without mentioning his legendary seminar, which started in 1943 and continued until his death. Gelfand considered the seminar one of his most important creations. It is hard to describe the seminar in a few words: it was a "mathematical stock exchange", a breeding ground for young scientists, a demonstration of how to think about mathematics, a one-man show, and much more. It was not about functional analysis or geometry, it was about mathematics. Some would come to the seminar just to hear Gelfand's jokes and paradoxes. For example, after his visit to the U.S. he stated, "[The] Mathematical

Gelfand in 1934.

Gelfand, circa 1950.

world is not a metric space: The distance from Harvard to MIT is greater than the sum of distances from Harvard to Moscow and from Moscow to MIT."

One should add that sometimes Gelfand's jokes were rather sharp, but for a young person to become a subject of Gelfand's joke meant to be noticed, to be knighted. The seminar was also the right place to find out about fresh preprints coming from the inaccessible West. But most participants were attracted by the power and originality of Gelfand's approach to mathematics. He used the simplest grass-roots examples, but he could turn them around in a totally unexpected way.

Gelfand always paid special attention to students, who formed the majority of the seminar audience. From time to time he would repeat: "My seminar is for high school students, decent undergraduates, bright graduates, and outstanding professors." One of the best descriptions of the seminar was given by S. Gindikin [3]. You may also see the notes by Zelevinsky and me in this collection (my notes will appear in the next issue). Dusa McDuff described the seminar impressions of a young foreigner.

The seminar also served as a constant supply of Gelfand's collaborators, who were already familiar with Gelfand's style and his way of thinking. Their roles were quite different. Sometimes they would discuss specific examples, sometimes very vague ideas; sometimes they would bring their own suggestions that would be ridiculed, torn apart, turned upside down, and then transformed into something exquisite. Only a few could bear the task, but the pool of mathematicians in Moscow was enormous.

The seminar was a reflection of Gelfand's passion to teach, as he tried to teach everyone and everywhere. Among his former students are F. Berezin, J. Bernstein, E. Dynkin, A. Goncharov, D. Kazhdan, A. Kirillov, M. Kontsevich, and A. Zelevinsky. The number of his informal students is hard to estimate.

From the early period of his life also comes Gelfand's interest in education, especially in education for school students living far from research centers. He was among the founders of the Moscow Mathematical Olympiads and later organized his famous Mathematics School by Correspondence for middle and high school students (see the recollections by Sergei Tabachnikov in the second part of this article). Gelfand founded, ran, and wrote several textbooks for the school. His university textbooks *Linear Algebra* and *Calculus of Variations* (written with S. Fomin) also bear the imprint of his style and personality.

It is hard to describe all of Gelfand's achievements in mathematics. He left his unique and powerful imprint everywhere (excluding, probably, mathematical logic). Some (but far from all) of his breakthroughs are described here by Simon Gindikin, David Kazhdan, Bertram Kostant, Peter Lax, Isadore Singer, and Anatoly Vershik. But Gelfand's interests spread far beyond pure and applied mathematics. He left a number of papers in biology, physiology, medicine, and other fields.

Gelfand was the first to obtain the Wolf Prize, in 1978 (together with C. L. Siegel), and had many other awards, including the Kyoto Prize (1989) and the MacArthur Fellowship (1994). He was elected to all leading academies and had honorary degrees from many universities.

This collection of articles about Gelfand also contains an impression of the Gelfand seminar by a foreigner (Dusa McDuff), a look at the Gelfand school from inside (Andrei Zelevinsky and Vladimir Retakh), a description of Gelfand's School by Correspondence (Sergei Tabachnikov), and an essay, "Gelfand at 92" by Mark Saul.

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I. M. Singer

I. M. Gelfand

Israel Gelfand was one of the most influential mathematicians of the twentieth century—I dare say, the most outstanding in the last sixty years.

Unfortunately, our society neither understands nor appreciates mathematics. Despite its many applications, despite its intellectual power which has changed the way we do science, mathematicians are undervalued and ignored. Its practitioners, its leaders go unrecognized. They have neither power nor influence. Watching the negative effects

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popularity causes in other fields and looking at the few superficial articles about mathematics, I think it is just as well.

Faced constantly with problems we can't solve, most mathematicians tend to be modest about themselves and their accomplishments. Perhaps that is why we have failed to recognize a giant in our midst. I won't compare Gelfand with other outstanding mathematicians or scientists of the twentieth century; if I did, you would start checking for yourselves whether you agree with me. But focus on my point—we had a giant in our midst. I turn to other fields to find comparable achievements: Balanchine in dance or Thomas Mann in literature or Stravinsky, better still, Mozart in music, but for me, a better comparison is with artists like Cezanne and Matisse. I commend to you the great poet Paul Rilke's letter on Cezanne. He said, "Paul Cezanne has been my supreme example, because he has remained in the innermost center of his work for forty years…which explains something beyond the freshness and purity of his paintings" (of course, much longer for Gelfand).

Evoking Matisse is perhaps more apt. A Matisse is breathtaking. No matter what his personal circumstance, he turns to new frontiers with joy and energy. Particularly outstanding is his later work: Jazz and the remarkable "papier-decouples", efforts done in the early 1880s.

Gelfand always dazzled us with new and profound ideas. One of his latest works, the book with Kapranov and Zelevinsky, is a major effort that maps out new directions for decades to come.

In preparing this article, I asked many people for topics I should emphasize. You will be interested in what happened. First, there was little intersection in the subjects my correspondents chose. Second, everyone gave me a five- to twenty-minute enthusiastic lecture on the essence of Gelfand's contribution—simple and profound.

Reviewing Gelfand's contributions to mathematics is an education. Let me remind you of some of his main work.

1. Normed rings
2. C*-Algebras (with Raikov)—the GNS Construction
3. Representations of complex and real semisimple groups (with Naimark and Graev)
4. Integral Geometry—Generalizations of the Radon Transform
5. Inverse scattering of Sturm-Liouville systems (with Levitan)
6. Gelfand-Dickey on Lax operators and KdV
7. The treatises on generalized functions
8. Elliptic equations
9. The cohomology of infinite dimensional Lie algebras (with Fuchs)

After receiving honorary degree at Oxford in 1973. Left with Gian-Carlo Rota.

10. Combinatorial characteristic classes (beginning with MacPherson)
11. Dilogarithms, discriminants, hypergeometric functions
12. The Gelfand seminar

It is impossible to review his enormous contributions in a short note. I will just comment on a few results that affected me.

As a graduate student, one of the first strong influences on me was Gelfand's normed ring paper. Marshall Stone had already taught us that points could be recaptured in Boolean algebras as maximal ideals. But Gelfand combined analysis with algebra in a simple and beautiful way. Using the maximal ideal in a complex commutative Banach algebra, he represented such algebras as algebras of functions. Thus began the theory of commutative Banach algebras. The spectral theorem and the Wiener's Tauberian theorem were elementary consequences. I was greatly influenced by the revolutionary view begun there.

A natural next step for Gelfand was the study of noncommutative C*-algebras. He represented such algebras as operator algebras using the famous GNS construction. It seemed inevitable to find unitary representations of locally compact groups using their convolution algebras. The representation theory of complex and real semisimple Lie groups followed quickly. What struck me most was the geometric approach Gelfand and his coworkers took. Only recently it appears this subject has become geometric again.

In 1963 twenty American experts in PDE were on their way to Novosibirsk for the first visit of foreign scientists to the academic city there. It was in the midst of Khrushchev's thaw. When I learned about it, I asked whether I could be added to the list of visitors, citing the index theorem Atiyah and I had just proved. After reading his early papers, I wanted to meet Gelfand. Each day of my two-week stay in Novosibirsk I asked Gelfand's

With M. Atiyah, 1973.

With I. M. Singer, 1973.

students when he was coming. The response was always "tomorrow". Gelfand never came. I sadly returned to Moscow. When I got to my room at the famous Hotel Ukraine, the telephone rang and someone said Gelfand wanted to meet me, could I come downstairs. There was Gelfand. He invited Peter Lax and me for a walk. During this walk, Peter tried to tell Gelfand about his work on $SL(2,R)$ with Ralph Phillips. Gelfand tried to explain his own view of $SL(2,R)$ to Peter, but his English was inadequate. (He was rusty; within two days his English was fluent.) I interrupted and explained Gelfand's program to Peter. At the corner Gelfand stopped, turned to me and said, "But you are my student." I replied, "Indeed, I am your student." (By the way, Gelfand told me he didn't come to Novosibirsk, because he hated long conferences.)

Although it was an honor to be Gelfand's student, it was also a burden. We tried to imitate the depth and unity Gelfand brought to mathematics. He made us think harder than we believed possible. Gelfand and I became close friends in a matter of minutes, and we remained so ever since. I was ill in Moscow, and Gelfand took care of me.

I didn't see him again for ten years. He was scheduled to receive an honorary degree at Oxford, where I was visiting. It was unclear that he would be allowed to leave the Soviet Union to visit the West. I decided not to wait and returned home. A week later, I received a telegram from Atiyah: Gelfand was coming—the queen had asked the Russian ambassador to intercede. I flew back to England and accompanied Gelfand during his visit—a glorious time. Many things stand out, but I'll mention only one: our visit to a Parker fountain pen store. Those of you who have ever shopped with Gelfand will smile; it was always an unforgettable experience. Within fifteen minutes, he had every salesperson scrambling for different pens. Within an hour I knew more about the construction of fountain pens than I ever cared to know and had ever believed possible! Gelfand's infinite curiosity and focused energy on details were unbelievable; those, coupled with his profound intuition of essential features, are rare among human beings. He was beyond category.

Talking about Oxford, let me emphasize Gelfand's paper on elliptic equations. In 1962 Atiyah and I had found the Dirac operator on spin manifolds and already had the index formula for geometric operators coupled to any vector bundle, although it took another nine months to prove our theorem. Gelfand's paper was brought to our attention by Smale. It enlarged our view considerably, as Gelfand always did, and we quickly realized, using essentially the Bott periodicity theorem, that we could prove the index theorem for any elliptic operator.

I should also mention the application of Gelfand's work to physics: Gelfand-Fuchs, for example, on vector fields on the circle, the so-called Virasoro algebra, which Virasoro did not in fact define. Although I mentioned Gelfand-Dickey, I have to stress its influence on matrix model theory. And I have to describe how encouraging he was and how far ahead of his time he was in understanding the implication of a paper which seemed obscure at the time.

Claude Itzykson told me that his famous paper with Brezin, Parisi, and Zuber that led to triangulating moduli space at the beginning went unnoticed by scientists. The authors then received one request for a reprint—from Gelfand.

Ray and I were very excited about our definition of determinants for Laplacian-like operators and its use in obtaining manifold invariants-analytic torsion. The early response in the United States was silence; Gelfand sent us a congratulatory telegram.

In conclusion, I want to mention one special quality of Gelfand. He was a magician. It is not very difficult—not very difficult at all—for any of us mere mortals to keep the difference in our ages a constant function of time. But with Gelfand, when I met him at his fiftieth, and in his sixtieth, I thought he was older than I. Ten years later, I felt that we were the same age. Later it became clear to me that Gelfand was, in fact, much younger than most mathematicians.

David Kazhdan

Works of I. Gelfand on the Theory of Representations

The theory of group representations was the center of interest of I. Gelfand. I think this is related to the nature of this domain, which combines analysis, algebra, and topology in a very intricate fashion. But this richness of the representation theory should not be taken as self-evident. To a great extent we owe this understanding to works of I. Gelfand, to his unique way of seeing mathematics as a unity of different points of view.

In the late thirties, when Gelfand started his mathematical career, the theory of representations of compact groups and the general principles of harmonic analysis on compact groups were well understood due to works of Hermann Weyl. Harmonic analysis on locally compact abelian groups was developed by Lev Pontryagin. The general structure of operator algebras was clarified by Murray and von Neumann. But the representation theory of noncompact noncommutative groups was almost nonexistent. The only result I know of is the work of Eugene Wigner on representations of the inhomogeneous Lorentz group. Wigner has shown that the study of physically interesting irreducible representations of this group can be reduced to the study of irreducible representations of its compact subgroups.

It was not at all clear whether the theory of representations of real semisimple noncompact groups is "good", i.e., whether the set of irreducible representations could be parameterized by points of a "reasonable" set, and whether the unitary representations can be uniquely decomposed into irreducible ones. The conventional wisdom was to expect that the beautiful theory of Murray-von Neumann factors is necessary for the description of representations of real semisimple noncompact groups. On the other hand, Gelfand, for whom Gauss and Riemann were the heroes, expected that the theory of representations of such groups should possess classical beauty.

Gelfand's first result in 1942 with D. Raikov in the theory of representations of groups is a proof of the existence of "sufficiently many" unitary representations for any locally compact group G. In other words, any unitary irreducible representation of G is a direct integral of irreducible ones. The proof of this result is based on the very important observation that the representation theory of the group G is identical to the theory of representations of the convolution algebra of measures on G with compact support and on an application of Gelfand's theory of normed rings.

Next, in the late forties, there was a stream of papers (most of them jointly with M. Naimark) that developed the main concepts of the representation theory of noncompact classical groups G. It would be a much simpler task to describe the concepts that appeared later than to describe the richness of this work.

Photograph by Carol Tate.

On Raoul Bott's porch, 1976. Left to right, I. Gelfand, R. MacPherson, R. Bott, D. Kazhdan.

Gelfand believed that the space $\hat{G}$ of irreducible representations of G is a reasonable "classical" space. If I understand correctly, the first indication of the correctness of this intuition came from the theory of spherical functions, developed in the early forties but published only in 1950. Let $K \subset G$ be the maximal compact subgroup and $\hat{G}_0 \subset \hat{G}$ be the subset of irreducible representations of *class* 1 (that is, the representations (π, V) of G such that $V^K \neq 0$). Gelfand observed that the subset $\hat{G}_0$ is equal to the set of irreducible representations of the subalgebra of two-sided K-invariant functions on G, proved the commutativity of this algebra, and identified the space of its maximal ideals with the quotient ${}^LT/W$ where ${}^LT$ is the torus dual to the maximal split torus $T \subset G$ and W is the Weyl group. The generalization of this approach developed in the fifties by Harish-Chandra and Godement led to the proof of the uniqueness of the decomposition of any representation of the group G into irreducibles.

Given such a nice classification of irreducible representations of class 1, it was natural to guess that the total space $\hat{G}$ is also an algebraic variety. But for this purpose one had to find a way to construct irreducible representations of G. Gelfand introduced the notion of parabolic induction (for classical groups) and, in particular, studied representations π_ξ of G induced from a character ξ of a Borel subgroup $B \subset G$. He showed that, for generic character ξ of $T = B/U$, the representation π_ξ is irreducible and that the representations $\pi_\xi, \pi_{\xi'}$ are equivalent if and only if the characters ξ, ξ' of T are conjugate under the action of the Weyl group W. The proof is based on the decomposition $G = \bigcup_{w \in W} BwB$ for classical groups. This decomposition was extended by Harish-Chandra to the case of an arbitrary

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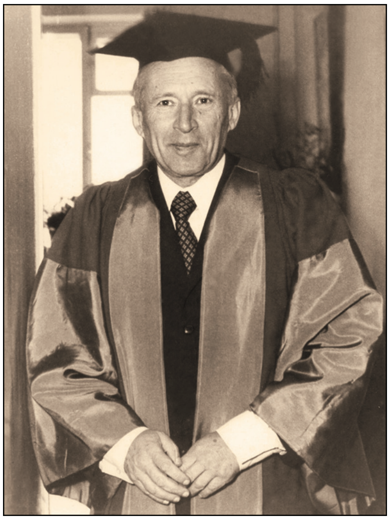

Photograph by Elena Ermakova.

60th birthday.

semisimple group and is known now as the Bruhat decomposition.

This construction gives many irreducible representations. But how to show that not much is missing? In the case of a compact group G it is well known that all the representations of G are constituents of the regular representation. Therefore, to see that a list of representations π_a, $a \in A$, of G is complete, it is sufficient to show that one can write the delta function δ_e on G as a linear combination of the characters $tr(\pi_a)$. But for representations (π, V) of a noncompact group G, which are typically infinite-dimensional, the trace of the operator $\pi(g)$, $g \in G$, is not defined. The ingenious idea of Gelfand was to define characters $tr(\pi)$ as *distributions*. That is, he showed that for any smooth function $f(g)$ with compact support, the operator

$$\pi_\xi(f) := \int f(g)\pi_\xi(g)dg$$

is of trace class and defined the distribution $tr(\pi)$ by $tr(\pi)(f) := tr(\pi(f))$. Now one could look for a W-invariant measure (called the *Plancherel measure*) μ_X on the space X of unitary characters of T such that

$$\delta_e = \int \xi \in X tr(\pi_\xi)\mu_X. \tag{1}$$

It is not difficult to see that such a measure μ_X is unique (if it exists), and the knowledge of μ_X is equivalent to the explicit decomposition of the regular representation $L^2(G)$ into irreducible ones.

In a series of joint works with M. Naimark, Gelfand was able to guess a beautiful algebraic expression for the Plancherel measure μ_X in the case of classical complex groups and to prove equality 1 by very intricate explicit calculations—a great reward for difficult work.

As a continuation of this series of works, Gelfand posed a number of questions (he was able to answer them only in particular cases) which influenced the development of representation theory for many years.

1. In joint work with M. Graev, Gelfand classified generic irreducible representations of the group $SL(n, \mathbb{R})$. They found that $\hat{G}$ is a union of pieces called *series* which correspond to conjugacy classes of maximal tori. Moreover, the series corresponding to nonsplit tori have realizations in spaces of (partially) analytic functions. Gelfand conjectured that the analogous description of the space $\hat{G}$ should be true for all real semisimple groups and that it should be possible to realize discrete series in appropriate spaces of analytic functions. The first part of the conjecture was justified by Harish-Chandra, who constructed discrete series of representations for real semisimple groups and found the Plancherel measure concentrated on unitary representations induced from discrete series of Levi subgroups. The second part was modified by Langlands, who suggested the realization of discrete series (which exist if and only if there is a maximal compact torus $T^c \subset G$) in the space of cohomologies $H^i(G/T^c, \mathcal{F})$ of homogeneous holomorphic vector bundles $\mathcal{F}$ on G/T^c (while Gelfand considered only the realization in sections of such bundles).

2. In joint work with Graev, Gelfand constructed the analog of the Paley-Wiener theorem for groups $SL_2(\mathbb{C})$ and $SL_2(\mathbb{R})$ (that is, a decomposition of the representation of G on the space $\mathbb{C}_c^\infty(G)$ of smooth functions with compact support) and raised the question about the extension of this result to other groups. This generalization was obtained by J. Arthur in 1983.

3. Also in joint work with Graev, Gelfand constructed the decomposition of representations of the group $SL_2(\mathbb{C})$ on the space $L^2(SL_2(\mathbb{C})/SL_2(\mathbb{R}))$ and asked the question about the decomposition of representations of the group G on $L^2(G/H)$ where $H \subset G$ is the set of fixed points of an involution. An extension of this result to arbitrary such pairs (G, H) was achieved only recently (see the talk by P. Delorme at the ICM Congress, 2002).

4. Gelfand showed that many special functions, such as Bessel and Whittaker functions and Jacobi and Legendre polynomials, appear as matrix coefficients of irreducible representations. This interpretation of special functions immediately explains the functional and differential equations for these functions. It is clear now that (almost) all special functions studied in the nineteenth and twentieth centuries can be interpreted as matrix coefficients or traces of representations of groups or their quantum analogs (see, for example, papers of Tsuchiya-Kanie, Koelink, Noumi, Rosengren, Stokman, Sugitani, and others on representation theoretic interpretation of Askey-Wilson, Macdonald, and Koornwinder polynomials).

The next series of Gelfand's work (with M. Tsetlin) was on irreducible finite-dimensional representations of classical groups G. The classification of such representations (π_λ, V_λ) was known, but Gelfand asked a new question, partially influenced by his interest in physics: how to find a "good" realization of these representations. In other words, how to find a basis in V_λ which allows one to compute matrix coefficients of $\pi_\lambda(g)$, $g \in G$, in these bases. Such a basis (Gelfand-Tsetlin basis) was constructed for irreducible representations of groups SL_n and SO_n and became the core of many works in representation theory and combinatorics.

Gelfand and Graev found the expression for the matrix coefficients of representations π_λ in terms of discrete versions of Γ-functions. This realization of finite-dimensional representations has an important analog for infinite-dimensional representations of groups over local fields.

As part of the theory of finite-dimensional representations, Gelfand studied the Clebsch-Gordan coefficients, which give a decomposition of tensor products of irreducible representations into irreducible components. He noticed (at least for $G = SL_2$) that Clebsch-Gordan coefficients of G are discrete analogs of Jacobi polynomials which are matrix coefficients of irreducible representations of G. Possibly an explanation of this can be given by using the theory of quantum groups, where multiplication and comultiplication are almost symmetric to each other.

The next series of Gelfand's work was on representations of groups over finite and local fields F. The basic results here are the proof of the uniqueness of a Whittaker vector, the existence of a Whittaker vector for cuspidal representations of $GL_n(F)$, the construction of an analog of the Gelfand-Tsetlin basis for such representations, and the description of cuspidal representations of $GL_n(F)$ in terms of Γ-functions (joint work with M. Graev and D. Kazhdan). But I think that his most important work in this area is the complete description of irreducible representations of groups $SL_2(F)$ and $GL_2(F)$ for local fields with the residue characteristic different from 2 (joint work with M. Graev and A. Kirillov). They showed that irreducible representations of $GL_2(F)$ are essentially parameterized by conjugacy classes of pairs (T, ξ) where $T \subset GL_2(F)$ is a maximal torus and $\xi : T \to \mathbb{C}$ is a character. Moreover, they found a formula for the characters $tr_{T_\xi}(g)$ of these representations and an explicit expression for the Plancherel measure. A striking and until now unexplained feature of these formulas is that they are essentially algebraic. For example, the Mellin transform $L(g, t)$ in ξ of $tr_{T_\xi}(g)$, which is a function of $GL_2(F) \times T$, is given by

$$L(g,t) = \delta(\det(g), \mathbf{N}m(t))\epsilon_T \times (tr(g) - \mathbf{t}r(t))/|tr(g) - \mathbf{t}r(t)|.$$

Here $T = E^*$ where E is a quadratic extension of F, $\epsilon_T : F^* \to \pm$ is the quadratic character corresponding to E, tr is the matrix trace, and $\mathbf{t}r$ and $\mathbf{N}m$ are the trace and norm maps from E to F.

The understanding of the existence of an intrinsic connection between the structure of irreducible representations of groups over local fields and number theory was greatly clarified by Langlands. On the other hand, a generalization of algebraic formulas for the Mellin transform of characters and for the Plancherel measure was never found.

Talking at 90th birthday conference, 2003.

In other work, Gelfand and Graev found a description of the irreducible representations of the multiplicative group D^* of quaternions over F as induced from 1-dimensional representations of appropriate subgroups. This was of constructions of irreducible representations of D^* that were later generalized by R. Howe to other p-adic groups.

The description of representations of the groups $SL_2(F)$ and $GL_2(F)$ for local fields is presented in the book *Generalized Functions, Volume 6* written with Graev and I. Piatetski-Shapiro. In the same book, Gelfand developed the theory of representations of semisimple adelic groups $G(\mathbb{A}_K)$ for global fields K. He defined the cuspidal part

$$L_0^2(G(\mathbb{A}_K)/G(K)) \subset L^2(G(\mathbb{A}_K)/G(K))$$

of the space of automorphic forms, proved that the representation of $G(\mathbb{A}_K)$ on $L_0^2(G(\mathbb{A}_K)/G(K))$ is a direct sum of irreducible representations, and developed a representation-theoretical interpretation of the theory of modular forms. The work of Langlands is very much influenced by these results of Gelfand.

It became clear that generic representations of any semisimple Lie group or Lie algebra almost do not depend on a choice of a particular group, so Gelfand tried to find a way to express this similarity in an intrinsic way. In a series of papers with A. Kirillov he studied the structure of the skew-field $F(\mathfrak{g})$ of fractions for the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. He found that the skew-fields $F(\mathfrak{G})$ are almost defined by the transcendence degree of the center $Z(\mathfrak{g})$ (equal to the rank $r(\mathfrak{g})$ of $\mathfrak{g}$) and by their *Gelfand-Kirillov dimension* (equal to $(\dim(\mathfrak{g}) - r(\mathfrak{g}))/2$). These results are the foundation of works of A. Joseph on the structure of the category of Harish-Chandra modules.

The last series of work of Gelfand on representation theory was on category $\mathcal{O}$ of representations of a semisimple Lie algebra. This category of representations was defined by Verma, but the basic results are due to J. Bernstein, I. Gelfand, and S. Gelfand. They constructed a resolution of finite-dimensional representations by Verma modules $V_w, w \in W$ (known as the BGG-resolution), discovered the duality between irreducible and projective modules in the category $\mathcal{O}$, and found the relation between the category $\mathcal{O}$ and the category of Harish-Chandra modules. These results form the cornerstone of the theory of representations of semisimple Lie algebras and their affine analogs.

However, their main discovery was the existence of a strong connection between algebraic geometry of the flag space $\mathcal{B}$ of a semisimple group and the structure of the category $\mathcal{O}$. For example, they showed that there is an embedding of V_w into $V_{w'}$ if and only if the Bruhat cell $BwB \subset \mathcal{B}$ belongs to the closure of $Bw'B$. This connection between algebraic geometry and the category of representations is the basis for the recent geometric theory of representations.

There are other important works of Gelfand on representation theory (such as indecomposable representations of semisimple Lie groups, models of representations, and representations of infinite-dimensional groups), but I want to mention two series of works that originated in representation theory but have an independent life. The appearance of such works is very natural, since for Gelfand representation theory was a part of a much broader structure of analysis.

Integral geometry is an offshoot of representation theory. The proof of the Plancherel theorem for complex groups is equivalent to the construction of the inversion formula, which gives the value of a function in terms of its integrals over horocycles. Gelfand (in a series of joint works with Graev, Z. Shapiro, S. Gindikin, and others) found inversion formulas for reconstruction of the value of a function on a manifold in terms of its integrals over an appropriate family of submanifolds. The existence of such inversion formulas found applications in such areas as symplectic geometry, multi-dimensional complex analysis, algebraic analysis, nonlinear differential equations, and Riemannian geometry, as well as in applied mathematics (tomography). (A more detailed description of Gelfand's work on integral geometry is given by S. Gindikin in the current issue of the *Notices*.)

Analogously, the work of Gelfand with V. Ponomarev and, later, with J. Bernstein on quivers was motivated by the problems in representation theory—the description of indecomposable representations for the Lorentz group. But the inner development of this subject led to a beautiful and deep theory which later made a full circle in works of Ringel, Lusztig, and Nakajima and became the foundation for geometric representation theory of Lie algebras and quantum groups.

Anatoly Vershik

Gelfand, My Inspiration

The achievement of Israel Moiseevich Gelfand was an unusual phenomenon of twentieth-century mathematics. His name must be included on any short list of those who formed the mathematics of that century. He pioneered many new ideas and created whole new provinces of knowledge. An exceptional intuition, a depth and breadth of thought, a liveliness in comprehending mathematics—these were his chief qualities.

But the most important quality of this wonderful mathematician was his ability to inspire others. Many other mathematicians, both in Russia and abroad, felt an amazing sense of inspiration in talking to or corresponding with Gelfand. His ideas and his advice stimulated many discoveries and much research done by others.

Gelfand and Leningrad Mathematics

I was a student in Leningrad in the 1950s, when Gelfand's name was well known to all Leningrad mathematicians, including students interested in the subject. This is not surprising. The interests of most Leningrad mathematicians (including L. V. Kantorovich, V. I. Smirnov, G. M. Fikhtengoltz, and others) centered on functional analysis and its applications, and this "Leningrad approach" was one of the branches of functional analysis that developed in the Soviet Union in the years 1930–1950. I. M. Gelfand was an iconic figure in this development.

The Leningrad approach differed from that taken in Moscow (by A. N. Kolmogorov, I. M. Gelfand, L. A. Lyusternik and A. I. Plessner) and in the Ukraine (by M. G. Krein and N. I. Akhiezer). It was centered around the theory of functions, operator theory, and its applications to differential equations and methods of computation. The main difference between the Moscow approach and the other two was the study advocated by Gelfand of infinite-dimensional representations of groups and algebras and an interest not so much in Banach spaces as in general problems of noncommutative Fourier analysis and, more broadly, in a synthesis of algebra, classical analysis, and the theory of functions. These ideas were not present in Leningrad for many years.

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This segment of the article was translated from the Russian by Vladimir Retakh and Mark Saul.

Gelfand's fame and authority went unchallenged at Leningrad seminars. While still young, Gelfand had broken into the elite club of Moscow mathematicians of the highest level, amazing everyone with his theory of maximal ideals of commutative Banach algebras (the so-called "Gelfand transform"), and he remained a leader of this club until his death.

Among Leningrad mathematicians Gelfand especially valued L. V. Kantorovich, D. K. Faddeev, and V. A. Rokhlin, whom he remembered from his student days. Like Kantorovich, Gelfand played an active role in the atomic research project and generally knew the work of Kantorovich on functional analysis. Both these outstanding mathematicians did much work on applied problems.

In 1986, at the funeral of Kantorovich, Gelfand told me that of all of Kantorovich's results, the one that impressed him the most was his pioneering work on linear programming. That was my view as well. On the other hand, when I asked Kantorovich to submit my paper to *Doklady* in 1971, he noted that the paper really represented the work of Gelfand and that it was disgraceful that Gelfand was not still a full member of the Soviet Academy of Science. Gelfand had direct contact with only a few of the younger Leningrad mathematicians in addition to me—in particular, with L. D. Faddeev and M. S. Birman.

My First Acquaintance with Gelfand's Work

The first significant mathematical papers that I studied were the papers of I. M. Gelfand, D. A. Raikov, and G. E. Shilov (Gimdargesh, as I called them) on commutative normed rings and their papers that followed on generalized Fourier analysis. This theory was a mathematical awakening for me. I was enchanted by its beauty and simplicity, its generality and depth.

Before this I had been hesitant. I could join the Department of Algebra, where I attended lectures by D. K. Faddeev. Or I could work in the Department of Analysis, where my first mentor was G. P. Akilov and where I might choose to specialize in complex analysis (V. I. Smirnov, N. A. Lebedev) or real and functional analysis (G. M. Fikhtengoltz, L. V. Kantorovich, G. P. Akilov). Now functional analysis was my only choice. And my interest was mostly in the analysis of the Moscow school—Gelfand's functional analysis, as I described earlier. Since that time the work of Gelfand and his school in various areas have become my mathematical guide.

One could say a lot about the great variety of Gelfand's interests and works. In fact, he was interested in everything: biology, music, and politics, not just mathematics. I would like to write here about the area of his mathematical activity that was closest to my interests and about the problems on which I was fortunate to work with him.

Lab of Function Theory at Moscow State University, circa 1958. Left to right sitting, I. Gelfand, Polyakov, D. E. Menshov, N. K. Bari, G. P. Tolstov; standing, P. L. Ulyanov, A. G. Kostychenko, F. A. Berezin, G. E. Shilov, R. A. Minlos.

I would go so far as to say that the main legacy of the work of Gelfand's first period was his foundation (along with the group of students he led) of the theory of normed commutative rings (commutative Banach algebras, as they are now called) and, most importantly, the theory of unitary infinite-dimensional representations of locally compact groups. These papers became classics of mathematics despite the youth of the authors.

As a byproduct of this theory, Gelfand and Naimark gave a definition of general C^*-algebras that is now in common usage. Once, as early as the 1970s, Gelfand said to me, "If I could only talk to von Neumann, I would explain to him why our C^*-algebras are more important than W^*-algebras (von Neumann algebras)." I can add that Gelfand was always more interested in topological and smooth problems (which are connected with C^*-algebras) than in theoretical metric problems (which are formulated in terms of W^*-algebras). The representation theory that will forever be connected with his name always was a favorite subject of Gelfand's, but his ideas about relationships and details within this theory kept changing. Among his wonderful traits was a fearless irony towards his own earlier results. When I told him that as a student I was impressed by his work on Banach algebras, Gelfand replied, "But why did we consider only maximal and not prime ideals?" This remark is similar to one he made in a talk in the 1970s, when I mentioned to him my interest in his early papers. "I am always praised for my work that was done five or more years ago, and the same people are not satisfied with what I am doing right now," he explained about his not uncommon critics. In fact, he was always ahead of his time, he was always creating a new fashion.

I remember being strongly influenced by Gelfand's talk in 1956 about problems in functional analysis (he was probably following the example of a similar talk by von Neumann at the International Congress of Mathematicians in 1954). In this address he posed several general problems and talked about their relationship to various mathematical phenomena. One of these problems was a conjecture about a possible connection between Wiener measures and von Neumann factors and of its possible utility in future work. "Such beauty must not perish unused," noted Gelfand. I was a student at that time and I was deeply impressed by this strong assertion of the primacy of aesthetic estimation of mathematical theories above other criteria.

A Few Comments on Gelfand's Work of the 1950s and 1960s

Perhaps Gelfand's highest achievement was the theory of infinite-dimensional unitary representations of complex semisimple Lie groups, created together with M. A. Naimark, M. I. Graev, and others.

It is hardly possible to comment on this development completely here, but it is worth noting that Gelfand faced many difficulties common to all pioneers: many details had to be clarified later or proofs rewritten, and so it would happen that attributions of priority or authorship were given to other researchers. But one must remember that Gelfand was the first to initiate and develop this huge area of mathematics, which is so useful in physics. There was no theory of infinite-dimensional representations before Gelfand's. From reading the recently published diaries of A. N. Kolmogorov, one can understand how impressed the world of mathematics was by these works in the 1940s. It is remarkable how confident Gelfand was in starting up completely new areas.

Gelfand's next great passion in the late 1950s and 1960s—an interest common to many centers of mathematics—was the theory of distributions of L. Schwartz (called in Russia "generalized functions"). Several Russian mathematicians had a keen interest in the subject but with a touch of bitterness and perhaps chagrin. The reason was simple: Gelfand, and before him N. M. Gyunter (a famous Petersburg mathematician) and L. V. Kantorovich and certainly S. L. Sobolev in fact not just made use of but created the theory of generalized functions. In one of his lectures in Leningrad in the early1960s, Gelfand stated directly that he and Naimark used distributions in their work on representation theory. However, the merit of L. Schwartz's work is that he understood the importance of the general theory and substantiated it with many examples. It is important to add that, as often happens in the theory of functions, this reformulation actually gave a new language and new understanding to many concepts.

However, the hope that arose at that time that this reformulation would give essentially new results did not exactly come true, and we cannot talk about a serious change from the theory of Banach spaces to the theory of locally convex spaces, despite the expectations of that time.

The celebrated six-volume series of books *Generalized Functions* by Gelfand and his coauthors demonstrated their unparalleled depth of understanding of the subject. The breadth of the area covered, from partial differential equations to representations and number theory, is characteristic of Gelfand's style.

In particular, Gelfand was among the first to understand the importance of turning from Banach's methods in functional analysis to more general linear topological (nuclear) spaces and their value for the spectral theory of operators (the Gelfand-Kostyuchenko theorem) and the theory of measure in linear spaces (the Minlos theorem). Gelfand also immediately understood the usefulness of generalized stochastic processes (Gelfand-Ito processes). This stimulated the construction of the theory of measures in linear spaces started by A. N. Kolmogorov in the 1930s. The famous "holy trinity" (triples of Hilbert spaces), generalized spectral decompositions, an original approach to Levi processes and to Gelfand-Segal constructions, quasi-invariant measures, and so on were studied and developed by hundreds of mathematicians.

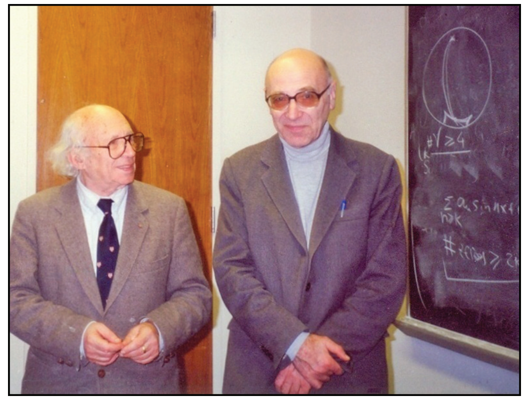
My First Meeting with Gelfand

Not counting a few brief discussions with Gelfand during the Mathematical Congress in 1966 or in the late 1960s or on my rare visits to his seminar when I happened to be in Moscow, our first close meeting took place in the spring of 1972 when I came to Moscow for two months. After the seminar I walked to his home with several participants. I started talking about the work I had recently started on asymptotic statistics of the lengths of cycles of random permutations. This paper initiated a long series written by me and my students on what I later called the asymptotic theory of group representations. Gelfand was interested and invited me to his home the next day to continue the conversation. I remember that Dima Kazhdan, who was also there, understood me sooner than Gelfand, who often asked me to repeat what I had said. His active and sometimes aggressive questioning was very helpful to the development of the theme and the improvement of the talk, if you could ignore the form of the criticism. But I had long ago heard about Gelfand's manner in such things. In talking about mathematics he would openly express his displeasure to his listener. One of his expressions, which is useful in the upbringing of young mathematicians, was "Keep your work and your self-esteem separate." In other words, do not take even harsh intellectual criticism personally.

I told Gelfand about my plans to study the symmetric group and its representations. My motivation for this research lay not only in the study of the subject by itself but also in its applications to optimization theory, linear programming, and combinatorics. (At that time I worked in the Department of Computational Mathematics and Operational Research.) There was also a way of connecting asymptotic theory with ergodic theory. In our discussions, Gelfand remarked that everything seemed to be clear with regard to representations of finite groups, then started talking enthusiastically about symmetric functions. He recommended that I look at a mysterious paper of E. Thoma about characteristics of the infinite symmetric group, which were of special interest to me.

Many things about the symmetric group were unclear to me, and I was not satisfied with the classical exposition of the theory of its representations despite the fact that it had been developed by Frobenius, Schur, and Young, and also by such giants as von Neumann and H. Weyl. It seemed to me then that the foundations of this theory and its relations with combinatorics were not yet clear. Gelfand gave his final conclusion many years later, after he had gotten to know and appreciate my papers with S. Kerov and later with A. Okounkov. He said something like, "It is all clear now." But that was all years later. In fact, the main idea of my approach was, roughly speaking, an application of the Gelfand-Tsetlin method, created in the early 1950s for representation of compact Lie groups, to the theory of symmetric groups. It is worth noting that a dissatisfaction with established theory often begins with those studying the theory in their mature years. That is what happened in my case. I started to study representations theory later than usual and got a certain advantage from this.

In Moscow, during our first conversation at his home, Gelfand once again surprised me. He repeated part of our discussion to Dima Kazhdan, who had come in later, and said, as if it had already been done, that I was studying the asymptotics of Young diagrams. However, I had not mentioned this and was just planning to do it. When I corrected him, Gelfand said, "Yeah, yeah, but you will do it later." This unique ability of Gelfand to see "three meters under the ground where his interlocutor was standing" was one of his most remarkable qualities. It instilled trust in his judgment by many who got to know him, but also drove off many who feared his ability to see through them. He immediately and (usually) correctly guessed what his companions were about to say or what they had in mind but did not express, and generally knew how the discussion would turn out. There are many examples of this, the most striking was his mathematical intuition, which allowed him to guess the result (often without calculations) and to foresee what to expect from a given method or even from the further work of a mathematician or a whole team. During that discussion Gelfand approved of my ideas (later called asymptotic representations theory). Many of these plans were then acted on by S. Kerov, G. Olshanski, A. Okounkov, A. Borodin, and a number of Western mathematicians.



With M. I. Graev, 1999.

At Gelfand's seminar in 1977 I talked with S. Kerov about my results on the limiting form of Young diagrams and about our approximation approach to representations of the infinite symmetric group. At that time Gelfand, Graev, and I had already started our work on the representations of current groups. I. M. always commented on every talk at his seminar. During my talk he said (and repeated this many times) that combinatorics is becoming a central part of mathematics of the future. As a permanent seminar participant said to me, "He was excited."

Our Collaboration

Let me return to 1972 and to the history of our collaboration. After our conversation I said goodbye and intended to leave for Leningrad. In Moscow I always stayed with my old friend, the virologist N. V. Kaverin. He once attended a Gelfand seminar in biology. Gelfand remembered him, but they had no other contact. I mentioned to Gelfand that I was staying with Kaverin. Suddenly, on the day of my departure, Gelfand called me at Kaverin's (having found the phone number with some difficulty[1]) and asked me to come over immediately. He also invited M. I. Graev, and during our long walk together started to talk about a construction of noncommutative integrals of representations of semisimple groups, primarily for $SL(2,R)$. He said that he had been incubating this problem for a long time and had suggested it to his other students, but he was confident that the problem was perfect for me.

[1] *There was no "official" way to get someone's telephone number in Moscow at the time.*

At the time of the ICM, 1966. Second from left, I. Gelfand; third from left, A. Kolmogorov; fifth from left, S. V. Fomin; far right, O. A. Oleinik.

I was a bit surprised that Gelfand was aware of my knowledge of the representation theory of Lie groups, in particular of $SL(2,R)$. We had not talked about this. But Gelfand was right: the problem was posed at just the right time for me. In the early 1970s, independent of my other work, I was lecturing on representations of groups, C^*-algebras, and factors. Perhaps Gelfand had heard about this, but not from me. More likely, this was the result of his ability to foresee things that I mentioned earlier. Gelfand said that in order to construct the multiplicative integral of representations we have to study "infinitesimal" representations, i.e., a neighborhood of the identity representation. In the summer of 1972 Gelfand came to Leningrad for the thesis defense of M. Gromov, and I told him about my preliminary experiments with the Heisenberg group, where a construction of the integral by other methods was already known. In December of 1972 we found a solution, a spherical function of the required representation of $SL(2,R)$, or the "canonical state", as Gelfand suggested we call it. I came to Moscow and we drafted the text. The problem was solved in a few months. The next spring Gelfand came to Leningrad with M. I. Graev and stayed with us. My domestic helpers prepared food as instructed by Zorya Yakovlevna.[2] While we were working on the paper with Graev, I. M. talked with my wife, Rita, about his favorite music and paintings. Of course, we went to the Hermitage Museum, where Gelfand talked a lot about a painting of El Greco (*Saints Peter and Paul*) and where we also discussed how to continue our work.

I recall next Gelfand's last visit to Leningrad in 1984. I invited him to talk about his own life at a meeting of the Leningrad Mathematical Society (he had turned seventy-six months earlier). It was an interesting talk with many details about his first steps in mathematics, including his discovery at a young age of the Euler-Maclaurin formula (recall that he taught himself almost entirely and never attended a high school).

[2] *Zorya Yakovlevna Shapiro was then Gelfand's wife, with whom he wrote several important papers.*

Our first paper was published in *Uspekhi* in 1973 in an issue honoring the seventieth birthday of Kolmogorov, and this was the beginning of my collaboration with Gelfand and Graev, which continued with some breaks over ten years. (At some point I will write more about this.) The first paper in this series (in Gelfand's opinion and mine, the best one) touched on many subjects that were important at that time. In particular, we discussed cohomology with coefficients in irreducible representations. In the next paper we described the cohomology of all semisimple groups without the Kazhdan property and gave explicit formulas for semisimple groups of rank 1. We also constructed irreducible nonlocal representations of the corresponding current groups. The next paper was devoted to representations of groups of diffeomorphisms and the start of the development of the geometry of configurations and its application to representation theory. We had no doubts (and it was later confirmed and reconfirmed) that this series of papers would have various applications and that the work would continue. In recent years M. I. Graev and I have found new constructions and new representations of functional groups, and this idea certainly has a future.

I would like to recall here one particular story. Our first paper had a natural continuation related to representations of groups of smooth functions on a manifold with values in a simple compact Lie group. The properties of these so-called "energy" representations depend on the dimension of the manifold.

This is not the place for details, but we were able to prove that the representations are reducible when the dimension is three or higher and when the dimension is two under some conditions on the length of a simple root (see our papers and also papers by R. Ismagilov, R. Hoeg-Krohn, S. Albaverio, and N. Wallach). Dimension one is special, and even at the beginning of our work, Gelfand said that we would probably have to deal with a von Neumann factor-representation. There were no obvious reasons or even preliminary estimates for this at that time; we had just started to work on the subject. After some years Gelfand's prediction was confirmed: in dimension one the energy representation is a factor-representation of type III1, and it turned out to be connected with a Wiener measure (not with the scalar one, but with a measure generated by Brownian motion on a compact group). Do you remember the mystic prediction by Gelfand fifty years ago that factors should connect with Wiener measures? Today the subject is actively being developed.

The Seminar

Gelfand left us an enormous number of statements on different mathematical subjects. Someone with patience should collect them. Similarly, someone should have kept a diary of the Gelfand Seminar. Unfortunately, no one took the responsibility. Now we can pick up only bits and pieces. The seminars were like one-man shows, sometimes successful, sometimes rough, with salty humor, often relevant and instructive. I gave several talks at Gelfand Seminars in Moscow and later at Rutgers. I have already described my first very successful talk. The second was shorter and less successful but also distinctive. When I mentioned a not very interesting but very new fact, Gelfand called a graduate student to the blackboard and asked him to give a proof right away. Should I take this as an offense? Certainly not.

I attended the seminar for the first time when I was a graduate student. A French mathematician was talking about a paper by von Neumann. The talk was not very clear. Then Gelfand said, "Next time so-and-so will give a talk on this paper" and mentioned several names of professors and graduate students and immediately started to evaluate the future talks. He said that one professor's talk would give an illusion of understanding, but in reality neither the speaker nor the listeners would have any idea of what was going on. Another one would speak about something else, including his own results, and so on. All this added some drama to the seminar. One could take it seriously or with a smile. Much later Gelfand told me that L. D. Landau ran a seminar in the style of Pauli (which bore a surface resemblance to Gelfand's but was somewhat more rude). Gelfand himself attended Landau's seminar and once repeated to me a joke made by M. Migdal: "Gelfand goes to the physicists as an intellectual to the peasantry."[3]

At the same time, we must note that the Gelfand style was not for everyone. It drove some away and even complicated their lives. For example, one of his first, and most beloved, students was F. A. Berezin. At one point they parted ways completely. In the 1970s I tried unsuccessfully to get them to meet and talk. Nowadays, Berezin's work, especially in supermathematics, has gotten worldwide recognition, but he did not live to see this. Because I was not nearby and did not talk to Gelfand very often, I was able to avoid the frustration of some of his coauthors who worked with him more closely. On the other hand, with me he was always extremely polite and friendly. His telegrams on my birthday were always very complimentary.

Life in the USSR

It is more and more difficult nowadays to explain to my Western colleagues, as well as young people in Russia, what academic life was like in the USSR, or even what other parts of life were like. For example, why did Gelfand, with all his achievements in science and contributions to classified state projects (which were considered exceptionally important), become a full member of the Soviet Academy of Science only after a shameful delay? Of course, one of the reasons is an official academic and even governmental anti-Semitism. But, for example, this feeling was not so strong among physicists. There were other reasons as well.

Once Gelfand told me, "The situation was quite simple for me in the beginning of the 1950s [the time of the infamous "fight with cosmopolitism",[4] when Gelfand lost his position in Moscow State University]: only those who were really interested in mathematics became my students." It was true at that time (and also later) that it was better to have another advisor for one's career. But he still had quite a number of students. And here we face one of the most important and, in some ways, transcendental qualities of Gelfand: he could attract very different kinds of people. He surrounded himself with a whirlpool of established mathematicians, newcomers to the field, biologists, and others. His seminar was a place for meetings, corridor discussions, exchanges of news, opinions, and so on. It was not by chance that the famous "letter from 99 mathematicians" defending A. Esenin-Volpin[5] was openly distributed for signatures at his seminar. It was met by authorities with resentment and fear at the highest levels. In a totalitarian state such as the USSR, only those who were trusted and tested by the state could be a magnet for people. For this reason, officials, including official mathematicians, did not like Gelfand and his circle, not just out of consideration of their origins but also by the principle of "either with us or against us".

"Of course, we live in a prison," Gelfand once told me during a conference outside Moscow in the late 1970s during lively discussions between our scientists and others visiting from the West. Still, he always refused to take any Samizdat books from me. People should write more about this.

Before any of his few visits abroad, Gelfand had to pass a number of unpleasant procedures, as did almost all Soviet people. My friends and I,

[3] *A reference to the political movement in Russia in the 1870s known as "Going to the People"* (Хождение к народу)*, during which intellectuals went to the countryside in expectations of "enlightening" the peasants.*

[4] *A euphemism for an anti-Semitic government program.*

[5] *In 1969 Esenin-Volpin was thrown into a psychiatric "hospital" for political reasons.*

who were openly denied the right to travel abroad for any purpose, were for that reason spared the examinations by party commissions of our behavior and of our knowledge of party politics. These seemed to us unfortunate, but at the same time as rather a prestigious distinction.

At the end of the 1980s and the beginning of the 1990s the desire of many scientists to leave Russia (quite understandably, especially at that time) touched Gelfand as well. His Moscow seminar died soon after that. It was impossible to replace its leader. A relatively modest version of the seminar was reestablished at Rutgers. One can only guess what might have happened had Gelfand stayed in Moscow.

I met Gelfand in the U.S. several times. While mathematical life was different in the U.S., Gelfand found his place. The celebration of his ninetieth birthday at Harvard was perfectly organized and became a real scientific event with his brilliant talk.

I have never had any doubts that the name of I. M. Gelfand will be one of the icons of mathematics in the twentieth century, because the century was not just a century of outstanding achievements but also of new conceptions. I. M. Gelfand valued and created exactly this sort of mathematics.

Bertram Kostant

I. M. Gelfand

I first heard of I. M. Gelfand when I was a graduate student in Chicago in the early 1950s. At that time, Gelfand's paper "Normed rings" played a major role in the area of modern harmonic analysis, which was then very popular with students. My thesis advisor, Irving Segal, had arranged for me to spend the years 1953–1955 at the Institute for Advanced Study in Princeton. I already knew Chern and Weil from my years in Chicago, and at Princeton I became friendly with Lefschetz, Hermann Weyl, von Neumann, and Einstein. Certainly I considered Gelfand to be in the same class as these twentieth-century math and physics luminaries, and I looked forward to meeting him as well.

My chance for this meeting came about when I was invited to a 1971 summer school in Budapest. The activities of the school were organized by Gelfand himself, and I believe this was the first time he had been given permission to attend a conference outside the Soviet Union. A large number of his students and colleagues came with him. It was an unforgettable experience for me to be a participant in one of his famous multihour seminars. Gelfand is a coauthor in a vast number of papers. No doubt many of these papers were outgrowths of these seminars. He seemed to have a very distinctive style of inspiring research by posing probing questions to potential collaborators and insisting on not letting go until there was some sort of resolution. I went to Budapest carrying a recently written paper on the spherical principal series. Gelfand requested that I submit the paper to the proceedings (Lie Groups and Their Representations) of the Budapest conference. Even though this paper was later (1990) to win the Steele Prize, it was probably a mistake for me to publish the paper in the proceedings of the Budapest conference. The initial publisher, Halsted Press, went out of business, and the book did not appear until 1974. When it finally did appear, many people reported that it was very hard to find.

In June 1972, responding to an invitation, I went to Moscow, accompanied by my wife, Ann, for a three-week stay. I have to say that I was deeply touched and happy to be the recipient of Gelfand's warmth, friendship, and respect. In his 1970 ICM report he had included me in a very small (5) group of people whom he said had made outstanding contributions to representation theory. In Moscow, Gelfand made arrangements for me to address a meeting of the Moscow Mathematics Society, chaired by Shafarevich, with whom I had a pleasant lunch. Gelfand also made arrangements for me to meet on a regular basis with a number of his students, including Kazhdan, Bernstein, Kirillov, Gindikin, and his son, Sergei. I was also introduced to his colleagues Manin, Novikov, and Graev, and I spent an afternoon with Berezin in Gorky Park talking about (from my perspective, geometric) quantization.

One of the mathematical topics that came up during my conversations with Gelfand was the results in the first of the BGG, [BGG-71] papers. Somewhat earlier I had solved a problem of Bott which asked for a determination of the polynomial functions on the dual of a Cartan subalgebra which mapped, via Borel transgression, onto the dual basis of the Schubert classes of a general flag manifold. Acknowledging my precedence in solving this problem, [BGG-71] made a penetrating further development in the solution of this problem by introducing the very important BGG operators.

What was apparent to me during my stay was that Gelfand seemed to have created an environment where he was involved with both the personal as well as the professional lives of the many people around him. Gelfand's apartment was like Grand Central Station, with any number of people going in and out. He seemed to be carrying on n conversations. I can't speak for the people orbiting around him, but as an outsider looking in I was charmed. Doing mathematics for me has

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had many lonely moments, but in this exciting environment I sensed that mathematics (like much of physics) could be done in a communal, socially satisfying way.

There are any number of unforgettable incidents which occurred during my Moscow visit. The following was very embarrassing for me (but not without its comic aspects). One day Gelfand invited Ann and me to have some borscht with him at the restaurant of the Soviet Academy. All the tables in the room were filled with people, except for the table opposite us. That table was occupied by only one man. Suddenly Gelfand started whispering to me, but I couldn't make out what he was saying. Then he repeated it somewhat louder. Finally, when I heard what he was saying, I foolishly blurted out loud the name Lysenko. The room became still, and then I realized what I had done. "Oh my God," I thought. "I have gotten Gelfand into trouble with the authorities." For a moment visions of him being dragged off to the Gulag went through my head. But all turned out well. By the early seventies Lysenko had apparently been defanged.

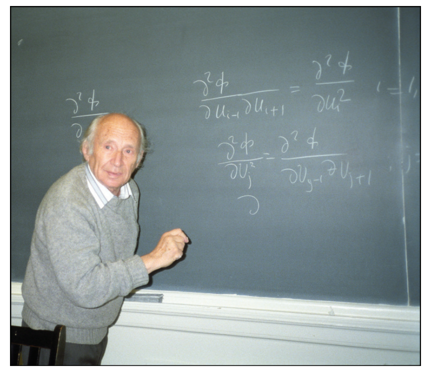
Much later Ann and I saw a great deal of Gelfand when he was invited to Cambridge, Massachusetts, in the United States to receive an honorary degree from Harvard. A number of scenes during that visit still stick in my head. One was at my house during a party as I watched him on the floor with a bunch of young kids, somehow managing to keep their interest by telling them some mathematical gems. Another was the scene at the breakfast table after Gelfand spent the night in our house. Gelfand was complaining about the absence of good bread in the U.S. and the difficulty of finding healthy food. I countered by pointing to the Swiss muesli in my breakfast bowl. Exhibiting a sample of his delicious (no pun intended) sense of humor, he then proceeded to eat the fruits and raisins in my bowl, all the while cautioning me not to have that for breakfast because it wasn't good for me.

Sometime during his visit I met him in New York City and introduced him to my son, Steven, then engaged in filmmaking. Gelfand gave a long discourse to Steven and his friends on Stanislavski. Apparently method acting was one of Gelfand's many artistic interests. I also took him to see Jean Renoir's famous film *Le Grande Illusion*, whose main theme was the disintegration of European aristocracy in the wake of World War I. I was totally surprised by his reaction to the film. Since it was made in the late thirties, he thought it was unconscionable that a prominent film was produced on such a topic at a time when Hitler was planning his march across Europe.

Another scene that sticks in my head was when, while driving Gelfand to the airport for his flight back to Russia, he started to recite poems of Ossip Mandelstam, Anna Akhmatova, and others. I remember recording him on tape, but unfortunately I seem to have misplaced the tape.

With Serre and MacPherson at Monday night seminar, Moscow State University, 1984.

Several years later my mathematical interests intersected with those of Gelfand in the area of completely integrable systems. In 1979 I had written a paper showing that the complete integrability of the open Toda lattice arises from a consideration of a certain coadjoint orbit of the Borel subgroup. Gelfand brilliantly put this result in an infinite-dimensional context and applied it to the theory of pseudodifferential operators. I believe a similar observation was made by Mark Adler.



Lecture at MIT, 1990.

I would like to end here by citing a mathematically philosophical statement of Gelfand which I think deserves considerable attention. It also opens a little window, presenting us with a view of the way Gelfand's mind sometimes worked. One of my first papers gave a formula for the multiplicity of a weight in finite-dimensional (Cartan-Weyl) representation theory. A key ingredient of the formula was the introduction of a partition function on the positive part of the root lattice. The partition function was very easy to define combinatorially, but giving an expression for its value at a particular lattice point was altogether a different matter. Gelfand was very interested in this partition function and mentioned it on many occasions. He finally convinced himself that no algebraic formula existed which would give its values everywhere. He dealt with this realization as follows. One day he said to me that in any good mathematical theory there should be at least one "transcendental" element and this transcendental element should account for many of the subtleties of the theory. In the Cartan-Weyl theory, he said that my partition function was the transcendental element.

Simon Gindikin

50 Years of Gelfand's Integral Geometry

As I begin this paper, the entire fifty-year history of integral geometry in Gelfand's life unfolds before my eyes. I was fortunate enough to collaborate with him during some key moments. I would like to discuss here those points that appear to me to be the most important in this wonderful endeavor.

First Presentation

I remember well a meeting of Dynkin's seminar on Lie groups, most likely in the spring of 1959. The seminar, which had been only for undergraduate students at first, at this point combined both undergraduate students (Kirillov, Vinberg, me) and well-known mathematicians (Karpelevich, Berezin, Piatetski-Shapiro). Suddenly, Gelfand appeared (late as usual and accompanied by Graev), and with enormous enthusiasm he began talking about their new work [1]. This was the first time their work on integral geometry was presented.

It may appear strange that Gelfand did not present this in his own seminar (in fact, he almost never presented his new results there) and that he selected what was mostly a student seminar. Without a doubt, this was no accident, since Gelfand was always very precise in selecting venues for his talks, and this fit well into traditions of Moscow's mathematical life. My mind's eye does not see any specifics of the presentation but rather recalls Gelfand's excitement, obvious to the listeners, and his certainty that something significant had opened before him—a new direction of geometric analysis, which he proposed to call "integral geometry", as it was equaled in importance only by differential geometry. He commented that Blaschke used this term for a certain class of problems connected with calculations of geometric measures but that this narrow area did not deserve such an ambitious title, and so he felt justified in appropriating it for this new field. The contentious nature of such a position is obvious. Such things worried Gelfand only a little, and I will not venture an opinion on the subject. I think this is important for understanding Gelfand's emotional state. He reminded us of his frequent saying that "representation theory is all of mathematics." (I heard this many times, and Manin once recalled that he also heard from Gelfand that "all mathematics is representation theory," noting the delicate difference between these aphorisms.) From now on, Gelfand said, he considered that "integral geometry is all of mathematics."

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Now a word about Gelfand's mathematical motivation. He considered that the main problem of the theory of representations was the decomposition of certain reducible representations—most importantly, regular representations on homogeneous spaces—into a sum (maybe continuous) of irreducible representations (Plancherel formulas). The class of groups and homogeneous spaces was not specified in the talk. It was only clear that complex semisimple Lie groups, together with their homogeneous spaces with maximally compact and Cartan isotropy subgroups, were in this class. The first two spaces are symmetric, but the third is not. This is of principal importance—spaces that are not symmetric are considered. From the very beginning the decomposition into irreducible representations is interpreted as a noncommutative analog of the Fourier integral: under the conditions of a continuous simple spectrum, we can talk about the "projections" of a function on a homogeneous space onto irreducible representations. The usual Fourier transform on $\mathbb{R}^n$ has a geometric twin in the Radon transform: integration on hyperplanes. They are connected with each other via the one-dimensional Fourier transform. The marvelous discovery of Gelfand-Graev was that, in the case of semisimple Lie groups, there is an analogous geometric twin: on homogeneous spaces we consider horospheres—the set of orbits of all maximally unipotent subgroups—and the horospherical transform (the operator of integration on horospheres). The generalized Fourier transform and the horospherical transform are connected by the (commutative) Mellin transform, equivalently, with the commutative Fourier transform. For this reason, problems about the Plancherel formula and inversion of the horospherical transform on a homogeneous space are trivially reduced to each other. Horospheres in hyperbolic geometry were well known (and highly valued) already by their creators: they are spheres of infinite radius with centers at infinity and different from hyperbolic hyperplanes. For other homogeneous spaces (including symmetric spaces), they had remained unnoticed before the work of Gelfand-Graev.

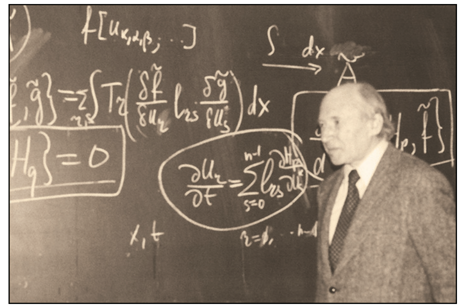
A posteriori, the idea of the horospherical transform seems almost obvious. Let G be a complex semisimple Lie group and let $B = MAN$ be a Borel subgroup; here N is the nilpotent radical, MA is a Levi subgroup (which is commutative in this case) with M and A its compact and vector subgroups, respectively. Then the manifold of horospheres is $\Xi = G/N$. Gelfand and Graev called it the "principal affine space". On this space there are two commuting actions: that of the group G acting by left shifts and that of the group MA acting by right shifts; the latter action is well defined because MA normalizes N.

The decomposition of the regular representation on Ξ relative to the action of MA gives precisely the irreducible representations of group G realized in the spaces of sections of line bundles over the flag manifold $F = G/B$ (the "principal projective space" in Gelfand-Graev's terminology). This is simply a rephrasing of the conventional realization of the principal series representations. Therefore, if the horospherical transform (which is an intertwining operator) is injective, the decomposition of the regular representation is reduced to the decomposition relative to the action of MA on the space of horospheres and the above-mentioned Mellin transform along the abelian subgroup A.

Prehistory

Needless to say, the real road to the discovery just outlined was entirely different. I was lucky to hear about it from Gelfand, and I want to share it here. Three papers appeared in 1947—by Bargmann, Gelfand-Naimark, and Harish-Chandra—in which unitary representations of the Lorentz group were found. I believe that I once heard Gelfand's observation that important things in mathematics spring forward independently from three places at once (the history of mathematics indeed delivers striking examples of this, beginning with non-Euclidean geometry). The Lorentz group is locally isomorphic to group $SL(2;\mathbb{C})$. Not surprisingly, the initiative in this problem belonged to the physicists, and the authors were targeting physical applications. (Bargmann was solving Pauli's problem and discussed it with Wigner and von Neumann; the problem was suggested to Harish-Chandra by Dirac; the first publication of Gelfand-Naimark was in a physics journal.) Because representations of the rotation group play a key role in nonrelativistic quantum mechanics, it was natural to expect representations of the Lorentz group to play an analogous role in the relativistic case. The difficulty lay in the fact that, while the rotation group is compact and all its irreducible unitary representations are finite dimensional, the Lorentz group is not compact and its only finite-dimensional unitary representation is the trivial one-dimensional representation. (It is precisely the unitary representations that have a physical interpretation.)

In the 1930s Dirac and Wigner considered certain infinite-dimensional unitary representations of the Lorentz group. From the other direction, in the early 1940s Gelfand-Raikov constructed an analog to the theory of Peter-Weyl for general locally compact groups using infinite-dimensional unitary representations. It seems likely that there was a consensus that the description of all unitary representations of the Lorentz group could not be done in an explicit form and had to include difficult considerations of von Neumann factors. Doubtless the biggest surprise for the authors was how explicitly and simply all irreducible unitary representations of the Lorentz group can be described.



Lecture at Collège de France, 1978.

Bargmann also considered representations of the "real" Lorentz group $SL(2;\mathbb{R})$ and discovered representations of the discrete series, realized in holomorphic functions on the disk. The central result is the completeness of the constructed unitary representations, although Bargmann and Harish-Chandra proved only an infinitesimal version of the completeness. Bargmann also proved a certain statement of the completeness of matrix coefficients.

At the same time, Gelfand and Naimark [2] went much further, establishing essentially all the principal concepts of the theory of unitary representations. They introduced characters of irreducible unitary representations as distributions, showed that characters define representations up to equivalence, and computed explicitly the character values on regular elements, thus obtaining an exact analog of the classical Weyl character formula. The paper contains a complicated analytical proof of the fact that their list of irreducible unitary representations is complete; however, the focus of the work is an analog of the Plancherel formula. This provides the (continuous) decomposition into irreducibles of the two-sided regular representation on the L^2 space of the group relative to the invariant measure.

It turned out that in this decomposition only some unitary representations appear; they were called representatives of the *principal series* (the rest of the unitary representations make up the *complementary series*). It is remarkable that the Plancherel measure, which appears in the expression of a norm of a function on a group through the norms of its projections on irreducible components, was computed explicitly. However, this remarkable formula, reminiscent of the classical Weyl formula for finite-dimensional representations, was a result of almost ten pages of dense computations without any visible attempts to uncover the conceptual structure. Later this was pointed out by Mautner while refereeing a more conceptual proof found by Harish-Chandra. This fitted the style of Gelfand, who at the beginning was more concerned with the beauty and clarity of the final result than with a simplification of the proof.


Photograph by Carol Tate.

Talking with Renée Thom at IHÉS, 1974.

However, it is typical of Gelfand that years afterwards he persistently tried to clarify the situation. Already at the time of the publication of [2] several notes had appeared announcing generalizations to the case of the group $SL(n;\mathbb{C})$. In 1950 a big book of Gelfand and Naimark appeared dedicated to unitary representations of the classical groups (Gelfand liked to call the book "Blue" because of the color of the cover, and the paper about $n = 2$ "Red"). This book is a result of an incredible amount of work, which to me appears to be nothing short of heroic. The style is clear: derivation of explicit formulas through intense analytic attacks, overcoming many obstacles. The proof of an analog of the Plancherel formula, the peak of the theory, especially stands out. The authors were unable to generalize an already difficult proof from the paper about the Lorentz group; the proof given in the book is exceptionally difficult and contains many brilliant inventions. Gelfand especially prized the calculations in a certain generalization of elliptic coordinates.

Around 1950 significant changes took place in Gelfand's mathematical life. First of all, he concluded his extraordinarily fruitful collaboration with Naimark. More than that, he decisively changed the organization of his work. If before, he concentrated on some one direction (Banach spaces, Banach algebras, theory of representations) at every moment, now he worked simultaneously with several coauthors on problems from very different areas. He did not abandon the theory of representations, and for many years Graev became his primary collaborator. Soon Harish-Chandra gave a conceptual proof of the Plancherel formula for all complex semisimple Lie groups and also for $SL(2;\mathbb{R})$. However, something kept Gelfand from being satisfied, and from time to time he continued returning to the Plancherel formula. In 1953 he and Graev offered an exceptionally elegant way to derive it through application of the result of M. Riesz about regularization of powers of a quadratic form.

Gelfand kept coming back to his proof of the Plancherel formula for the Lorentz group, which had not been generalized to other groups and in which he felt something important was left not understood. In 1958 suddenly he saw in this proof something that had remained hidden for almost ten years, namely, that a large part of the proof was dedicated to the solution of the following elementary-sounding problem of geometrical analysis: Consider a function $f(\alpha, \beta, \delta)$ of three complex variables. Integrate this function (in the real sense) on all complex lines intersecting the hyperbola $\alpha = \lambda$, $\delta = \lambda^{-1}$, $\beta = 0$, where $\lambda \in \mathbb{C}$. Then reconstruct f through all of these integrals. What connection do these lines have to the Lorentz group $SL(2;\mathbb{C})$? Consider the set of horocycles in this group: the two-sided shifts of the unipotent subgroup N of unit upper triangular matrices $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ with $u \in \mathbb{C}$. This set turns out to be exactly the three-parameter family (on $\mathbb{C}$) of all (complex) lines in the group, considered as a hyperboloid $\alpha\delta - \beta\gamma = 1$ in $\mathbb{C}^4$. If we project the hyperboloid onto the plane with coordinates (α, β, δ), then almost all the horocycles (excluding those that project into points) transform into lines intersecting the hyperbola. The solution to the inversion problem in [1] is a very simple formula, reminiscent of the Radon inversion formula. For the inverse of the generalized Fourier transform on the group (Plancherel formula), it remains only to invert the usual Mellin transform. After this interpretation, it was straightforward for Gelfand and Graev to define horospheres and the horospherical transform in the general case. The authors of [2] apparently did not know of this geometrical interpretation, but looking at the paper, one can't stop marveling at how closely they followed it in their calculations (the corresponding text about this problem of integral geometry in the fifth volume of *Generalized Functions* [4] differs from the text of the 1947 paper only in the addition of certain words without any substantial changes to any of the formulas).

However, at this point the romantic side of the story concludes and prosaic everyday mathematical problems begin. There is no doubt about the beauty of the horospherical transform and its significance, but it is fair to raise the question: what does it contribute to the theory of representations? From the very beginning, Gelfand had the idea that there have to be direct methods to invert certain generalized Radon transforms, including the horospherical transforms, and there had to be a more natural way to derive Plancherel formulas. However, in 1959 such a way was not uncovered, and in [1] the situation is essentially the opposite: the formula for the inversion of the horospherical transform for complex semisimple Lie groups and certain homogeneous spaces was derived from the already-known Plancherel formula. A remarkable formula was produced, which resembled the the Radon inversion formula in odd-dimensional spaces but which also showed which modifications are required for spaces of rank greater than 1: one needed to average (over the set of horospheres passing through a point) the result of applying an explicit differential operator (of the order equal to the rank) that acts along parallel horospheres. It remained to develop direct methods of integral geometry and to understand what advantages this approach had to harmonic analysis. This was accomplished, partially, only ten years later [5].

Even so, some important supporting observations slowly accumulated. Gelfand noticed in the very beginning that the symmetric Riemannian space for the group $SL(2;\mathbb{C})$ is the three-dimensional hyperbolic space, with horospheres in the classical sense. It turned out that the inversion formula for this horospherical transform, with the appropriate definitions, exactly matches the formula for the inversion of the Radon transform in three-dimensional Euclidean space. By contrast, the Plancherel formulas for the Fourier transform in Euclidean and hyperbolic spaces differ dramatically. In a sense, the horospherical transform is independent of curvature! There was no doubt of the importance of this observation, but for a surprisingly long time there were no attempts to incorporate this into some general result. Today it is clear that if one considers for a Riemannian symmetric space of negative curvature its "flat" twin (sending curvature to zero), then the inversion formulas for the horospherical transforms will be identical for these two spaces. It seems to me that this explains the nature of the simple explicit formulas for the representations of semisimple Lie groups (which surprised their creators): from the point of view of the horospherical approach, the problems turn out to be equivalent to flat ones. There exists another important circumstance which doubtlessly was tacitly understood, even though I've never encountered a discussion of it. Namely, in the standard approach to the Plancherel formula, integrals on conjugacy classes are regularized so as to be meaningful as "integrals" on singular conjugacy classes, eventually, on the unit class. Horospheres, in a certain sense, are generators of conjugacy classes; bringing them into the analysis illuminates the consideration of singular classes.

Integral Geometry of Lines and Curves

The biggest success in the following period was connected with the interpretation of horocycles on the Lorentz group as lines intersecting a hyperbola. Everything began with a natural question: What happens if the hyperbola is replaced by another curve? Kirillov proved that under such a replacement the inversion formula remains unchanged. This meant something very important: with a replacement of the hyperbola by another curve, the group disappeared, and it became clear that a significant part of the theory of representations of the Lorentz group is connected not with the group structure, but rather with some sort of more general geometrical structure. This was the first supporting evidence for Gelfand's project to embed the theory of representations into a wider area of geometrical analysis.

The next step was again evident: understand the nature of the condition on a family of lines "to intersect a fixed curve". The family of all lines in $\mathbb{C}^3$ depends on four (complex) parameters. For integral geometry it is natural to consider families with three parameters (complexes of lines in classical terminology), because then integration transforms a function of three variables into a function of three variables. For which complexes of lines are there inversion formulas of Radon's type? Gelfand and Graev [4] showed that this is possible not only for complexes of lines intersecting a fixed curve but also for complexes of lines tangent to a fixed surface and not for any others. They called such complexes *admissible*. It is worthwhile to say a few words about the proof of this astounding result.

F. John discovered that in the real case the image of the operator of integration along lines in a three-dimensional space is described by the ultrahyperbolic differential equation. In the complex case one has to consider the holomorphic and antiholomorphic versions of this differential equation. It turns out that existence of the required inversion formula is equivalent to solvability of the Goursat problem and therefore coincides with the characteristic condition for this operator. This is a nonlinear equation that can be integrated using the method of Hamilton-Jacobi. The consideration of bicharacteristics gives the description of admissible complexes. These complexes had already appeared in classical differential geometry. They are, in a natural sense, maximally degenerate: the lines of the complex that intersect one fixed line of the complex lie (infinitesimally) in one plane.

This result is generalized to complexes of lines in general position in spaces of any dimension. The natural next step is to consider the possibility of the replacement of lines by curves. The final result in this direction was obtained by Bernstein and me [12] (after Gelfand, Shapiro, and I previously found a universal structure of inversion formulas for complexes of curves [6]). Admissible complexes of curves turned out to be exactly infinitesimally full families of rational curves (infinitesimally, they are spaces of all sections of a certain vector bundle on the projected line). Here the most significant condition is for the curves to be simultaneously rational. This is a quite effective condition, allowing the construction of many explicit examples of families of curves with inversion formulas of Radon type, for example, curves of the second order. In these constructions it is natural to give up the requirement that the family of curves be a complex (the number of parameters of the family equals the dimension of the manifold). In this case, the possibility exists to consider a certain generalization of a problem of integral geometry, but more instructive are the possibilities of applications beyond the theory of representations. There exists, for example, a connection with the twistor theory of Penrose: the four-parameter admissible family of curves corresponds to conformal right-flat four-metrics (the self-dual part of Weyl curvature equals zero). This allowed us to develop a method of construction of explicit solutions of the Einstein self-dual equation. It seems to me that, in the case of curves, integral geometry has been successfully fully developed. It appears to be some part of nonlinear analysis, with a focus on explicit integrable problems, and it is connected with the method of the inverse problem for the case of one spectral parameter.

Complexes of Planes

The situation with submanifolds of dimension higher than one is significantly more complicated, and there may be no opportunity in this case for work to advance as far. It appears that it is similar to the possibility of the integrability in inverse problems with several spectral parameters. However, a few essential results have been achieved. Let us recall that beginning with the first work on integral geometry [1], the grand challenge was to find a proof of the Plancherel formula through inversion of the horospherical transform. Ten years passed before Gelfand, Graev, and Shapiro did this [6], [5] for the group $SL(n;\mathbb{C})$. We have already discussed the fact that when $n = 2$, horospheres can be considered as lines. For arbitrary n, horospheres for this group can also be considered as planes of dimension $k = n(n-1)/2$ in $\mathbb{C}^N$, where $N = n^2 - 1$. The dimension of this family of planes coincides with N, so we have a complex of k-planes. The key idea is to study the problem of reconstructing a function through its integrals on planes from an arbitrary complex, not limiting ourselves to a complex of horospheres. Let us note that the problem of reconstructing a function through its integrals on all planes is extremely overdetermined and the transition to complexes is a natural way to make the problem well defined. It turns out that for reconstruction at a point it is always possible to write a formula reminiscent of the Radon inversion formula. This formula involves averaging a certain explicit differential operator (of order $2k$) along the set of planes of the complex which pass through a point. This operator is defined on the set of all k-planes; however, we can compute it using its restriction to the set of planes of the complex only under very strong conditions. Let us call complexes satisfying these conditions *admissible*. There again arises a certain condition involving characteristics. We can describe the situation slightly differently. The image of the operator of integration over k-planes is described via a certain system of differential equations of the second order, generalizing the John equation, and admissible complexes must be characteristic for this system. In some sense, the admissible complexes are maximally degenerate.

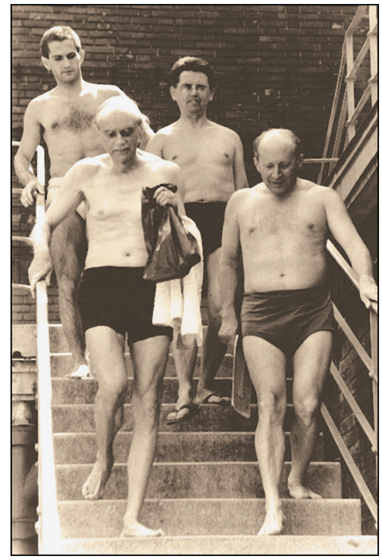
The nonlinear problem of the description of admissible complexes is unlikely to be integrable for $k > 1$. However, Gelfand and Graev [5] checked directly that the complex of horospheres in $SL(n;\mathbb{C})$ is admissible. As a result, we get an inversion of the horospherical transform and as a consequence the Plancherel formula. I am certain that there is something very significant in the degeneracy of the manifold of horospheres. I think that the geometric background of the theory of representations of semisimple Lie groups lies in this phenomenon. Clearly, this result makes substantial use of the possibility of considering horospheres on $SL(n;\mathbb{C})$ as planes. For other semisimple Lie groups this is impossible, but I showed that the construction described above may be generalized for these groups by considering admissible complexes of arbitrary submanifolds [8].

Nonlocal Problems and Integral Geometry for Discrete Series

In the case of complex groups the structure of inversion formulas with averaging of a differential operator means that, in particular, these formulas are local: for reconstruction of the function at a point, it is sufficient to know the integrals on the horospheres close to this point. In the case of the Radon transform, this is the case only in odd-dimensional spaces. In even-dimensional spaces the inversion formula is nonlocal: in this situation a pseudodifferential operator is averaged. If we were to go from complex groups to real ones and their homogeneous spaces, the first new difficulty would be connected with the inevitable appearance of nonlocal formulas. It is natural to begin with Riemannian symmetric spaces of noncompact type $X = G/K$, where G is a real semisimple Lie group and K is a maximally compact subgroup of G. Then the horospherical transform is always injective, and the inversion formula is local if and only if the multiplicities of all roots are even. In this case the inversion formula can be derived using the method described above for complex groups. However, the general case must involve nonlocal inversion formulas and requires new ideas. After Karpelevich and I computed the Plancherel measure on these spaces through the product formula for the Harish-Chandra c-function, we found [13] in this case the analog of the approach of Gelfand-Graev to complex groups in their first paper [1]: we computed the kernel of the inverse horospherical transform as the inverse Mellin transform of the Plancherel density. We made this calculation for classical groups. Later, Beerends made it in the general case. The next step should be a direct derivation of these inversion formulas using methods of geometric analysis (as in the case of even multiplicities), but so far this has not been done. We tried with Gelfand [9] to move in this direction, developing a certain symmetric analog of differential forms on real Grassmannians, which doubtless has its own interest. However, we did not achieve significant progress in this problem. It seems to me that today many points look clearer, and this problem looks realistically solvable.

Finally, we can begin discussing the main obstacle to the development of the theory of representations on the basis of the horospherical transform, which was clear from the very beginning. For real semisimple Lie groups the horospherical transform as a rule has a kernel, which consists of all of its representation series except for the maximally continuous ones. This is already true for the group $SL(2;\mathbb{R})$: if we decompose its regular representation into the sum of subspaces L_c, L_d corresponding to continuous and discrete series respectively, then the kernel of the horospherical transform coincides with L_d and the image is isomorphic to L_c. There does not appear to be a way to invert the horospherical transform or to derive in this way the Plancherel formula. It is easy to interpret this as a dramatic limitation of the power of integral geometry in the case of real groups, where discrete series play a central role. This appears to be the reason that Gelfand's idea of applying integral geometry to the theory of representations has not evoked much enthusiasm amongst experts on representation theory.

I can be a witness to the fact that Gelfand never believed that the area of applications of horospheres is bounded by continuous series. In his opinion it was necessary to understand what corresponds to horospheres in the case of discrete series, and this is perfectly possible. This resonated with his faith in the aesthetic harmony of mathematics. In his first talk, which it befell me to hear (in 1955), he said about von Neumann factors of the second type (which did not have any known applications at that time): "Such beauty must not vanish!" Only in one case—an imaginary three-dimensional hyperbolic space—did Gelfand and Graev manage [7] to find an appropriate expansion: the discrete series there was connected with certain degenerate horospheres. However, this was connected with certain very special circumstances, and there was no chance of direct generalization.


Budapest swimming pool, 1966. First row, P. Dirac, I. Gelfand; second row, B. Bollobas, M. Arato.

The case of $SL(2;\mathbb{R})$ remained the first call to action. In 1977 Gelfand and I attempted to understand it [10]. The philosophy of our approach was the fact that the Plancherel formula has to be derived in two stages. First, one has to find projections into subspaces corresponding to the representation series. Series of representations correspond to equivalence classes of Cartan subgroups. The decomposition of each series into irreducible subspaces is reduced to a commutative Fourier transform (continuous or discrete) corresponding to the associated Cartan subgroup. So the first stage is the principal one. Two of the main connected problems are the finding of projections into the series and the internal analytic characterization of subspaces for the series. We solved these problems for $SL(2;\mathbb{R})$. Subspaces corresponding to holomorphic and antiholomorphic discrete series can be characterized as boundary values on $SL(2;\mathbb{R})$ of holomorphic functions in certain tube domains in the complex group $SL(2;\mathbb{C})$. Projections may be interpreted as certain analogs of the Cauchy integral formula. I remember how happy Gelfand was with this result. He often said that new, significant things in representations have to be already nontrivial for $SL(2)$. We expected that in the general case as well the series would be connected with certain tube domains, which may not be Stein manifolds, and then it would be required to consider $\bar{\partial}$ cohomology on these domains. Later this was called the Gelfand-Gindikin program. Significant progress was achieved in this program, but only for holomorphic discrete series.

It could have been expected that the projections into the series had to be somehow connected with integral geometry, but at the time it proved impossible to find an appropriate generalization of the horospheres. Much later I discovered a certain natural construction [14]. On $G = SL(2;\mathbb{C})$ there are two classes of horospheres, each of which allows the possibility of constructing a horospherical transform. We have discussed the one-dimensional horocycles. However, it is also possible to consider two-dimensional horospheres—orbits of maximal unipotent groups in $SL(2;\mathbb{C}) \times SL(2;\mathbb{C})$ under the two-sided action. Their geometrical characterization consists of sections of the group, considered as a hyperboloid, by isotropic planes. Horocycles are linear generators of two-dimensional horospheres; for this reason, both versions of the horospherical transform are equivalent. For $SL(2;\mathbb{R})$ both transforms have kernels. The idea is that since there are not enough real horospheres, we must consider certain complex ones.

Let us consider those complex horospheres which do not intersect the real subgroup $SL(2;\mathbb{R})$. There are three types of such horospheres. Instead of integration on real horospheres, we consider the convolution (on the real group) of Cauchy kernels with singularities on the complex horospheres without real points. As a result, the horospherical transform of their three components is defined. It already does not have a kernel, and images of the various components can be decomposed using representations of various series. The inversion of the horospherical representation gives projectors onto series and then also the Plancherel formula. At the same time, the continuation of functions from the continuous series is obtained as a one-dimensional $\bar{\partial}$-cohomology in a certain tube domain, in agreement with our hypothesis. It is my hope that the correct development of the understanding of complex horospheres for real semisimple Lie groups is sufficient for the construction of the integral-geometrical equivalent of the theory of representations. Interestingly, this can be done even for compact Lie groups, for which there are no real horospheres at all.

Gelfand (in collaboration with Graev and me) worked on many other aspects of integral geometry. I have not discussed these results because I wanted to concentrate here on integral geometry connecting with the theory of representations, which seems to me to be the most important part of this project. It is intriguing that Gelfand was interested in the Radon transform long before its connection with the theory of representations was discovered. He liked to offer problems concerning the Radon transform to students, though he never worked in this direction himself. This was almost a premonition of future connections, an ability often shown by exceptional mathematicians and typical for him. Integral geometry was not one of his most successful projects. It did not gain him broad recognition or a multitude of followers. Yet we find in it, in a very pronounced way, his inimitable approach to mathematics. Like every other mathematician lucky enough to have worked with Gelfand, I have gotten used to trusting his extraordinary intuition, and I am certain that the story is not over. Perhaps I will see the realization of his fantastic project yet.

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Peter Lax

I. M. Gelfand

Like everyone else, I first heard Gelfand's name as the author of the famous theorem on maximal ideals in commutative Banach algebras and its application to prove Wiener's theorem about functions with absolutely convergent Fourier series. The following story describes the reception of this result in the U.S.

Shortly after its publication, Ralph Phillips presented this result at Harvard. It created such a

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stir that he was asked to repeat the lecture, this time to the whole faculty. And then he was asked to present it a third time to G. D. Birkhoff alone.

Other early basic results were the Gelfand formula for the spectral radius and the Gelfand-Levitan inverse spectral theory for ordinary differential operators. If you have a completely integrable system, you are supposed to be able to integrate it completely. For the KdV equation, the landmark result of Gelfand and Levitan on the inverse spectral problem for second-order ODEs turned out to be the key to perform the integration.

It is a measure of Gelfand's lifetime achievements that these spectacular early results are viewed today as merely a small part of his total work.

Here is a story that sheds light on Gelfand's modesty. I was one of three members of a committee to award the first Wolf Prize. We all agreed that the prize should be given to the greatest living mathematician. That person, in my opinion, was Carl Ludwig Siegel, but another member of the committee insisted that it was Israel Moiseevich Gelfand. So we argued back and forth fruitlessly until we decided to divide the prize. Some time later Gelfand remarked in a chance conversation, "It was a great honor for me to share a prize with Siegel."

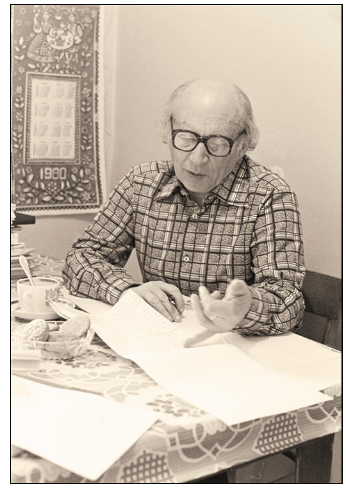
The world shall not see the like of Israel M. Gelfand for a long, long time.

Andrei Zelevinsky

Remembering I. M. Gelfand

On December 6, 2009, two months after I. M. Gelfand passed away, Rutgers University, his last place of work, held a Gelfand Memorial. This event brought together the crème de la crème of several generations of Moscow mathematicians (now scattered all around the world) whose life in mathematics was to a large extent shaped by I. M.'s influence. Despite the sad occasion, it was a pleasure to see many old friends and to share memories of our student years when we all attended the famous Gelfand Seminar at Moscow State University. And of course to share stories about I. M. He was such a huge presence in so many lives, and his passing left a gap which will be impossible to fill.

I first met I. M. in the early fall of 1970. The meeting was arranged by Victor Gutenmacher, who at the time worked at the School by Correspondence organized by I. M. (with the purpose of bringing mathematics into the lives of schoolchildren all around the Soviet Union). I. M., who as usual was simultaneously involved in a myriad of various projects both scientific and pedagogical, had decided to organize and run a special mathematical class of seventh-graders within the famous Moscow School No. 2, which already had a decade-long tradition of such classes. To help him run this program, I. M. asked Victor to find him several young assistants who had themselves passed through such a class during their school years. I was very fortunate to become one of four such assistants. (Another was my old friend and classmate since the seventh grade Borya Feigin, who is now a distinguished mathematician.) All four of us had graduated a year earlier from the same School No. 2, all were math undergraduates beginning our second year at Moscow State University, and all felt a little lost in our mathematical studies.


Photograph by Carol Tate.

Math in the kitchen, 1982.

The organizational matters could have been resolved within a few minutes, but our first meeting with I. M. lasted for several hours. The four of us (plus poor Victor) walked with I. M. for hours, and he talked to us about all kinds of things, mathematical and not. He asked us what we most loved about mathematics and what seminars and elective courses we had attended during our freshman year. Of course, he declared that we had done everything wrong and were almost lost for mathematics, but there was still some little hope for us if we started to attend his seminar at once. He explained to us how to study a new mathematical subject: focus on the most basic things at the foundation and dwell upon them until you reach full understanding; then the technicalities of the subject would be understood very quickly and effortlessly. I remember vividly how he illustrated this by explaining to us the foundations of linear algebra: a subspace in a vector space is characterized by one integer, its dimension; a pair of subspaces is characterized by three integers (dimensions of the two subspaces and of their intersection). What about a triple of subspaces? and what about a quadruple?[6]

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Zelevinsky is grateful to his daughter, Katya, for helpful editorial suggestions.

[6]*Triples of subspaces demonstrate the limitations of a fruitful analogy between subspaces of a finite-dimensional vector space and subsets of a finite set: the subspace lattice is only modular but not distributive. Thinking about this analogy led me to my first two published notes, which appeared within the next couple of years. As for the quadruples, as I*

This meeting was definitely one of my most life-changing experiences. I have never met a person with such personal magnetism and such an ability to ignite enthusiasm about mathematics as I. M. I remember returning home late in the evening, totally exhausted but happy, with the definite feeling that my mathematical fate had been sealed that day. I could not sleep and spent most of the night thinking about mysteries of triples of subspaces! I attended the Gelfand Seminar for almost twenty years. A lot has been said about its unique character, including horror stories about I. M.'s rough treatment of speakers and participants. I had my share of humbling experiences there, both as a speaker and as a "control listener" who was sent by I. M. to the blackboard in the middle of a talk to explain what the speaker was trying to say. Many people, including some excellent mathematicians, could not stand this style and stopped attending the seminar. Those who stayed (myself included) decided that such a great learning experience was worth a little suffering. Equally important, or maybe even more important than the talks themselves, was Gelfand's choice of topics; his comments and monologues, which often deviated a lot from the original topic; and of course his famous "pedagogical" jokes and stories.

The official starting time of the seminar was 7 p.m. (or was it 6:30?) on Mondays, but it almost always started with much delay, sometimes after up to two hours! I am sure I. M. did this on purpose, because these weekly get-togethers before the seminar with numerous friends coming to the university from all around the city were also a big attraction. Sometimes even after his arrival, I. M. did not immediately start the seminar but stayed in the corridor for some time and chatted with people just like everybody else.

I think of myself as I. M.'s "mathematical grandson": my first real teacher and de facto Ph.D. advisor was Joseph Bernstein, one of Gelfand's best students.[7] However, at some point after Joseph's emigration (in 1980, I believe), I. M. approached me and suggested that we start working together. Our close mathematical collaboration lasted about a decade (from the first joint note in 1984 to our 1994 book with Misha Kapranov). It would take volumes to give a comprehensive account of I. M.'s mathematical contributions, so let me just share my personal impressions of some of his unique features as a mathematician and as a teacher.[8]

Working with him could be a very frustrating experience. Just looking at his incredibly prolific scientific output, one would imagine him as a model of efficiency, never wasting a moment of his time. But most of the time during our daily meetings was filled with numerous distractions, jumps from one topic to another, his long phone conversations with an amazing variety of people on an amazing variety of topics, etc. Quite often after long hours spent like this I felt totally exhausted and had a depressing feeling that we had just wasted a perfectly good working day. But almost without exception there came a moment (sometimes when I was already saying good-bye at the door) of his total concentration on our project that led to extremely rapid progress, completely justifying all the torturous hours leading to this moment. It seemed as if his subconscious mind never stopped working on our project (and probably on a multitude of other things at the same time), and it just took I. M. a long time to become ready to spell out the results of this work.

This "nonlinearity" of I. M.'s thinking process was also one of the many features that made his seminar so unique. He would spend an inordinate amount of time asking everybody to explain to him some basic definitions and facts, and just when most of the participants (starting with the speaker, of course) would get totally frustrated, I. M. would suddenly switch gears and say something very illuminating, making it all worthwhile.[9]

I have never met any other mathematician with such an ability to see the "big picture" and always go to the heart of the matter, ignoring unnecessary technicalities. He had an uncanny ability to ask the "right" questions and to find unexpected connections between different mathematical fields. I. M. was fully aware of this gift and liked to illustrate it by one of his numerous "pedagogical" stories: "An old plumber comes to repair a heater. He goes around it, thinks for a moment, and hits it once with a hammer. The heater immediately starts working. The plumber charges 200 rubles for his service. The owner says, 'But you just spent two minutes and didn't do anything.' The plumber replies, 'I am charging you 3 rubles for hitting your

realized much later, I. M. had just finished his remarkable paper with V. A. Ponomarev, "Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space", so he was talking to us about his own cutting-edge research!

[7] *Joseph could not be my official advisor, since he was never affiliated with the mathematics department at Moscow State, which was also the case for many other first-rate mathematicians from the "Gelfand circle". A. A. Kirillov Sr., also a student of Gelfand, kindly agreed to serve as my official advisor. Of course, I also learned a lot from him and from attending his seminar.*

[8] *For I. M. these two occupations were inseparable. His way of doing mathematics always involved close personal interaction with his innumerable students and collaborators.*

[9] *I. M. often interrupted speakers with the words "May I ask a stupid question?" The best reply to this was given by Y. I. Manin during one of his rare appearances as a speaker: "No, I. M., I don't think you are capable of such a thing!"*

heater with a hammer and the rest for knowing where to hit, which took me forty years to learn!'[10]

I. M. had such an enormous wealth of ideas and plans that he needed many collaborators to help him realize even a small part of them. I was always very impressed with his great intuitive grasp of people. A few minutes of penetrating questioning of a new acquaintance allowed him to take the full measure of a person, understand his or her scientific potential, strengths and weaknesses, even how much pressure the person could withstand. In my own case, it seems that I. M. detected my soft spot for algebraic combinatorics (quite rare in Moscow at that time, I must say) even before I realized it myself. Very soon after I joined his seminar, he asked me to study (and then give a talk on) the old book *The Theory of Group Characters and Matrix Representations of Groups* by D. E. Littlewood, where the representation theory was treated with a strong combinatorial flavor. This was truly a sniper's shot: the facts and ideas that I learned from this book continue to serve me to this day. Another truly inspired suggestion by I. M. was to bring together Borya Feigin and Dmitry Borisovich Fuchs, which led to many years of very fruitful collaboration. Like so many of my friends and colleagues, I feel very fortunate for having known Israel Moiseevich and for being given a chance to be close to this gigantic, complex, wise, inspiring, and infinitely fascinating personality.

[10] *I. M.'s amazing record of initiating new fruitful directions of mathematical research provides plenty of examples of "knowing where to hit." Let me just mention one example: the theory of general hypergeometric functions initiated by I. M. (where a significant part of my collaboration with him took place) grew from his insight that hypergeometric functions should live on the Grassmannians.*

Closed Forms: What They Are and Why We Care

Jonathan M. Borwein and Richard E. Crandall

Closed Forms: What They Are

Mathematics abounds in terms that are in frequent use yet are rarely made precise. Two such are *rigorous proof* and *closed form* (absent the technical use within differential algebra). If a rigorous proof is "that which 'convinces' the appropriate audience," then a closed form is "that which looks 'fundamental' to the requisite consumer." In both cases, this is a community-varying and epoch-dependent notion. What was a compelling proof in 1810 may well not be now; what is a fine closed form in 2010 may have been anathema a century ago. In this article we are intentionally informal as befits a topic that intrinsically has no one "right" answer.

Let us begin by sampling the Web for various approaches to informal definitions of "closed form".

Definitions

First Approach to a Definition of Closed Form. The first comes from MathWorld [56] and so may well be the first and last definition a student or other seeker-after-easy-truth finds.

> An equation is said to be a closed-form solution if it solves a given problem in terms of functions and mathematical operations from a given generally accepted set. For example, an infinite sum would generally not be considered closed-form. However, the choice of what to call closed-form and what not is rather arbitrary since a new "closed-form" function could simply be defined in terms of the infinite sum.
>
> —Eric Weisstein

There is not much to disagree with in this, but it is far from rigorous.

Second Approach. The next attempt follows a September 16, 1997, question to the long-operating "Dr. Math" site[1] and is a good model of what interested students are likely to be told.

> Subject: Closed form solutions
>
> Dear Dr. Math: What is the exact mathematical definition of a closed form solution? Is a solution in "closed form" simply if an expression relating all of the variables can be derived for a problem solution, as opposed to some higher-level problems where there is either no solution, or the problem can only be solved incrementally or numerically?
>
> Sincerely, ….

The answer followed on September 22:

> This is a very good question! This matter has been debated by mathematicians for some time, but without a good resolution.
>
> Some formulas are agreed by all to be "in closed form". Those are the ones which contain only a finite number of symbols, and include only the operators $+, -, *, /,$

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[1] *Available at* `http://mathforum.org/dr/math/`.

and a small list of commonly occurring functions such as nth roots, exponentials, logarithms, trigonometric functions, inverse trigonometric functions, greatest integer functions, factorials, and the like.

More controversial would be formulas that include infinite summations or products, or more exotic functions, such as the Riemann zeta function, functions expressed as integrals of other functions that cannot be performed symbolically, functions that are solutions of differential equations (such as Bessel functions or hypergeometric functions), or some functions defined recursively. Some functions whose values are impossible to compute at some specific points would probably be agreed not to be in closed form (example: $f(x) = 0$ if x is an algebraic number, but $f(x) = 1$ if x is transcendental. For most numbers, we do not know if they are transcendental or not). I hope this is what you wanted.

No more formal but representative of many dictionary definitions is:

Third Approach. A coauthor of the current article is at least in part responsible for the following brief definition from a recent mathematics dictionary [17]:

> **closed form** n. an expression for a given function or quantity, especially an integral, in terms of known and well understood quantities, such as the evaluation of
> $$\int_{-\infty}^{\infty} \exp(-x^2)\,\mathrm{d}x$$
> as $\sqrt{\pi}$.
>
> —Collins Dictionary

And of course one cares more for a closed form when the object under study is important, such as when it engages the normal distribution as above.

With that selection recorded, let us turn to some more formal proposals.

Fourth Approach. Various notions of elementary numbers have been proposed.

> **Definition** [31]**.** A subfield F of $\mathbb{C}$ is *closed under* exp and log if (1) $\exp(x) \in F$ for all $x \in F$ and (2) $\log(x) \in F$ for all nonzero $x \in F$, where log is the branch of the natural logarithm function such that $-\pi < \mathrm{Im}(\log x) \leq \pi$ for all x. The field $\mathbb{E}$ of EL numbers is the intersection of all subfields of $\mathbb{C}$ that are closed under exp and log.
>
> —Tim Chow

Tim Chow explains nicely why he eschews capturing all algebraic numbers in his definition, why he wishes only elementary quantities to have closed forms, whence he prefers $\mathbb{E}$ to Ritt's 1948 definition of *elementary numbers* as the smallest algebraically closed subfield $\mathbb{L}$ of $\mathbb{C}$ that is closed under exp and log. His reasons include that:

> Intuitively, "closed-form" implies "explicit", and most algebraic functions have no simple explicit expression.

Assuming *Shanuel's conjecture* that *given n complex numbers $z_1, \ldots, z_n$ which are linearly independent over the rational numbers, the extension field*

$$\mathbb{Q}\,(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$$

has transcendence degree of at least n over the rationals, then the algebraic members of $\mathbb{E}$ are exactly those solvable in radicals [31]. We may thence think of Chow's class as the smallest plausible class of closed forms. Only a mad version of Markov would want to further circumscribe the class.

Special Functions

In an increasingly computational world, an explicit/implicit dichotomy is occasionally useful, but not very frequently. Often we will prefer computationally the numerical implicit value of an algebraic number to its explicit tower of radicals, and it seems increasingly perverse to distinguish the root of $2x^5 - 10x + 5$ from that of $2x^4 - 10x + 5$ or to prefer $\arctan(\pi/7)$ to $\arctan(1)$. We illustrate these issues further in Examples 7, 9, and 14.

We would prefer to view all values of classical *special functions* of mathematical physics [54] at algebraic arguments as being closed forms. Again there is no generally accepted class of special functions, but most practitioners would agree that the solutions to the classical second-order algebraic differential equations (linear or, say, Painlevé) are included. But so are various *hypertranscendental functions*, such as Γ, B, and ζ, which do not arise in that way.[2]

Hence, we do not wish to accept any definition of special function which relies on the underlying functions satisfying natural differential equations. The class must be extensible; new special functions are always being discovered.

A recent *American Mathematical Monthly* review[3] of [47] says:

---

[2]*Of course, a value of a hypertranscendental function at algebraic argument may be very well behaved; see Example 4.*

[3]*Available at* `http://www.maa.org/maa%20reviews/4221.html`.

There's no rigorous definition of special functions, but the following definition is in line with the general consensus: functions that are commonly used in applications have many nice properties and are not typically available on a calculator. Obviously this is a sloppy definition, and yet it works fairly well in practice. Most people would agree, for example, that the Gamma function is included in the list of special functions and that trigonometric functions are not.

Once again, there is much to agree with and much to quibble about in this reprise. That said, most serious books on the topic are little more specific. For us, special functions are nonelementary functions about which a significant literature has developed because of their importance in either mathematical theory or in practice. We certainly prefer that this literature include the existence of excellent algorithms for their computation. This is all consonant with—if somewhat more ecumenical than—Temme's description in the preface of his lovely book [54, Preface, p. xi]:

> [W]e call a function "special" when the function, just like the logarithm, the exponential and trigonometric functions (the elementary transcendental functions), belongs to the toolbox of the applied mathematician, the physicist or engineer.

Even more economically, Andrews, Askey, and Roy start the preface to their important book *Special Functions* [1] by writing:

> Paul Turan once remarked that special functions would be more appropriately labeled "useful functions".

With little more ado, they then start to work on the Gamma and Beta functions; indeed, the term "special function" is not in their index. Near the end of their preface, they also write:

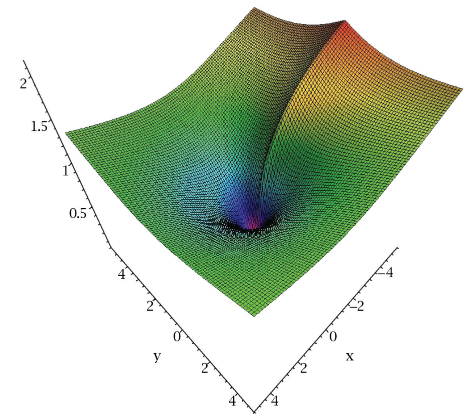
> [W]e suggest that the day of formulas may be experiencing a new dawn.

Example 1 (Lambert's W)**.** The *Lambert W* function, $W(x)$, is defined by appropriate solution of $y \cdot e^y = x$ [21, pp. 277–279]. This function has been implemented in computer algebra systems (CAS) and has many uses despite being unknown to most scientists and only relatively recently named [41]. It is now embedded as a primitive in Maple and Mathematica with the same status as any other well-studied special or elementary function. (See, for example, the tome [26].) The CAS know its power series and much more. For instance, in Maple entering:

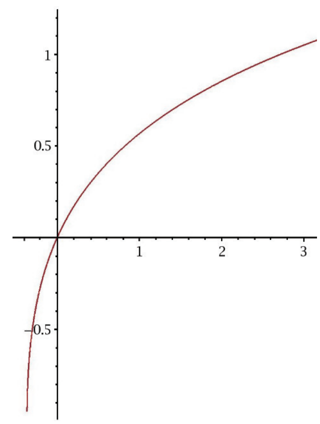
```
> fsolve(exp(x)*x=1);identify(%);
```

returns

```
0.5671432904, LambertW(1)
```



(a) modulus of ***W***



(b) ***W*** **on real line**

Figure 1. The Lambert ***W*** **function.**

We consider this to be a splendid closed form even though, again assuming Shanuel's conjecture, $W(1) \notin \mathbb{E}$ [31]. Additionally, it has only recently rigorously been proven that W is not an elementary function in Liouville's precise sense [26]. We also note that successful simplification in a modern CAS [29] requires a great deal of knowledge of special functions. □

Further Approaches

Fifth Approach. PlanetMath's offering, as of February 15, 2010,[4] is certainly in the elementary number corner.

> **expressible in closed form** (Definition) An expression is expressible in a closed form if it can be converted (simplified) into an expression containing only elementary functions, combined by a finite amount of rational operations and compositions.
>
> —PlanetMath

[4] *Available at* `http://planetmath.org/encyclopedia/ClosedForm4.html`.

This reflects both much of what is best and what is worst about "the mathematical wisdom of crowds". For the reasons adduced above, we wish to distinguish—but admit both—those closed forms that give analytic insight from those which are sufficient and prerequisite for effective computation. Our own current preferred class [7] is described next.

Sixth Approach. We wish to establish a set $\mathbb{X}$ of *generalized hypergeometric* evaluations; see [7] for an initial, rudimentary definition which we shall refine presently. First, we want any convergent sum

$$x = \sum_{n\geq 0} c_n z^n \tag{1}$$

to be an element of our set $\mathbb{X}$, where z is algebraic, c_0 is rational, and for $n > 0$,

$$c_n = \frac{A(n)}{B(n)} c_{n-1}$$

for integer polynomials A, B with $\deg A \leq \deg B$. Under these conditions the expansion for x converges absolutely on the open disk $|z| < 1$. However, we also allow x to be any finite analytic-continuation value of such a series. Moreover, when z lies on a branch cut, we presume both branch limits to be elements of $\mathbb{X}$. (See ensuing examples for some clarification.) It is important to note that our set $\mathbb{X}$ is closed under rational multiplication due to freedom of choice for c_0.

Example 2 (First members of $\mathbb{X}$)**.** The *generalized hypergeometric function* evaluation

$${}_{p+1}F_p\left(\begin{matrix} a_1, \ldots a_{p+1} \\ b_1, \ldots, b_p \end{matrix} \middle| z\right)$$

for rational a_i, b_j with all b_j positive has branch cut $z \in (0, \infty)$, and the evaluation is an element of $\mathbb{X}$ for complex z not on the cut (and the evaluation on each side of said cut is also in $\mathbb{X}$).

The *trilogarithm* $\mathrm{Li}_3(z) := \sum_{n\geq 1} z^n/n^3$ offers a canonical instance. Formally,

$$\frac{1}{z}\mathrm{Li}_3(z) = {}_4F_3\left(\begin{matrix} 1,1,1,1 \\ 2,2,2 \end{matrix} \middle| z\right),$$

and for $z = 1/2$ the hypergeometric series converges absolutely, with

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{1}{6}\log^3 2 - \frac{\pi^2}{12}\log 2.$$

Continuation values at $z = 2$ on the branch cut can be inferred as

$$\lim_{\epsilon\to 0^+} \mathrm{Li}_3(2 \pm i\epsilon) = \frac{7}{16}\zeta(3) + \frac{\pi^2}{8}\log 2 \pm i\frac{\pi}{4}\log 2,$$

so both complex numbers on the right here are elements of $\mathbb{X}$. The *quadralogarithmic* value $\mathrm{Li}_4\left(\frac{1}{2}\right)$ is thought not to be similarly decomposable but likewise belongs to $\mathbb{X}$. $\square$

Now we are prepared to posit

> **Definition** [7]**.** The *ring of hyperclosure* $\mathbb{H}$ is the smallest subring of $\mathbb{C}$ containing the set $\mathbb{X}$. Elements of $\mathbb{H}$ are deemed *hyperclosed*.

In other words, the ring $\mathbb{H}$ is generated by all general hypergeometric evaluations under the $\cdot, +$ operators, all symbolized by

$$\mathbb{H} = \langle \mathbb{X} \rangle_{\cdot,+}.$$

$\mathbb{H}$ will contain a great many interesting closed forms from modern research. Note that $\mathbb{H}$ contains all closed forms in the sense of Wilf and Zeilberger [48, Ch. 8], wherein only finite linear combinations of hypergeometric evaluations are allowed.

So what numbers are in the ring $\mathbb{H}$? First off, *almost no* complex numbers belong to this ring! This is easily seen by noting that the set of general hypergeometric evaluations is countable, so the generated ring must also be countable. Still, a great many fundamental numbers are provably hyperclosed. Examples follow, in which we let ω denote an arbitrary algebraic number and n any positive integer:

$$\omega, \log\omega, e^\omega, \pi;$$

the dilogarithmic combination

$$\mathrm{Li}_2\left(\frac{1}{\sqrt{5}}\right) + (\log 2)(\log 3);$$

the elliptic integral $\mathrm{K}(\omega)$;

the zeta function values $\zeta(n)$;

special functions such as

the Bessel evaluations $J_n(\omega)$.

Incidentally, it occurs in some modern experimental developments that the real or imaginary part of a hypergeometric evaluation is under scrutiny. Generally, $\Re, \Im$ operations preserve hyperclosure simply because the series (or continuations) at z and z^* can be linearly combined in the ring $\mathbb{H}$. Referring to Example 2, we see that for algebraic z, the number $\Re(\mathrm{Li}_3(z))$ is hyperclosed, and even on the cut, $\Re(\mathrm{Li}_3(2)) = \frac{7}{16}\zeta(3) + \frac{\pi^2}{8}\log 2$ is hyperclosed. In general, $\Re\left({}_{p+1}F_p\left(\begin{matrix}\cdots\\ \cdots\end{matrix}\middle| z\right)\right)$ is hyperclosed.

We are not claiming that hyperclosure is any kind of final definition for "closed forms", but we do believe that any defining paradigm for closed forms must include this ring of hyperclosure $\mathbb{H}$. One way to reach further is to define a *ring of superclosure* as the closure

$$\mathbb{S} := \langle \mathbb{H}^{\mathbb{H}} \rangle_{\cdot,+}.$$

This ring contains numbers such as

$$e^\pi + \pi^e, \ \frac{1}{\zeta(3)^{\zeta(5)}},$$

and of course a vast collection of numbers that may not belong to $\mathbb{H}$ itself. If we say that an element of

$\mathbb{S}$ is *superclosed*, we still preserve the countability of all superclosed numbers. Again, any good definition of "closed form" should incorporate whatever is in the ring $\mathbb{S}$.

Seventh Approach. In a more algebraic topological setting, it might make sense to define closed forms to be those arising as *periods*, that is, as integrals of rational functions (with integer parameters) in n variables over domains defined by algebraic equations. These ideas originate in the theory of elliptic and abelian integrals and are deeply studied [42]. Periods form an algebra and certainly capture many constants. They are especially well suited to the study of L-series, multizeta values, polylogarithms, and the like but again will not capture all that we wish. For example, e is conjectured not to be a period, as is Euler's constant γ (see the section "Profound Curiosities"). Moreover, while many periods have nice series, it is not clear that all do.

As this takes us well outside our domain of expertise, we content ourselves with two examples originating in the study of Mahler measures. We refer to a fundamental paper by Deninger [39] and a very recent paper of Rogers [50] for details.

Example 3 (Periods and Mahler measures [39])**.** The logarithmic *Mahler measure* of a polynomial P in n-variables can be defined as

$$\mu(P) := \int_0^1 \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\right) \right| \times d\theta_1 \cdots d\theta_n.$$

Then $\mu(P)$ turns out to be an example of a period, and its exponential, $M(P) := \exp(\mu(P))$, is a mean of the values of P on the unit n-torus. When $n = 1$ and P has integer coefficients, $M(P)$ is always an algebraic integer. An excellent online synopsis can be found in Dave Boyd's article, `http://eom.springer.de/m/m120070.htm`. Indeed, Boyd has been one of the driving forces in the field. A brief introduction to the univariate case is also given in [22, pp. 358–359].

There is a remarkable series of recent results—many more discovered experimentally than proven—expressing various multidimensional $\mu(P)$ as arithmetic quantities. Boyd observes that there appears to be a tight connection to K-theory. An early result due to Smyth (see [51], also [53]) is that $\mu(1 + x + y) = L_3'(-1)$. Here L_3 is the Dirichlet L-series modulo three. A partner result of Smyth's is that $\mu(1 + x + y + z) = 7\zeta(3)/\pi^2$, a number that is certainly hyperclosed since both $\zeta(3)$ and $1/\pi$ are. A conjecture of Deninger [39], confirmed to over fifty places, is that

$$\mu(1 + x + y + 1/x + 1/y) \overset{?}{=} \frac{15}{\pi^2} L_E(2) \tag{2}$$

is an L-series value over an elliptic curve E with conductor 15. Rogers [50] recasts (2) as

$$F(3,5) \overset{?}{=} \frac{15}{\pi^2} \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{(1/16)^{2n+1}}{2n+1}, \tag{3}$$

where

$$F(b,c) := (1+b)(1+c) \times \sum_{n,m,j,k} \frac{(-1)^{n+m+j+k}}{\left((6n+1)^2 + b(6m+1)^2 + c(6j+1)^2 + bc(6k+1)^2\right)^2}$$

is a four-dimensional lattice sum.

While (3) remains a conjecture,[5] Rogers is able to evaluate many values of $F(b,c)$ in terms of Meijer-G or hypergeometric functions. We shall consider the most famous crystal sum, the Madelung constant, in Example 15. □

It is striking how beautiful combinatorial games can be when played under the rubric of hyper- or superclosure.

Example 4 (Superclosure of Γ at rational arguments)**.** Let us begin with the *Beta function*

$$B(r,s) := \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)},$$

with $\Gamma(s)$ defined, if one wishes, as $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt$. It turns out that for any rationals r, s the Beta function is hyperclosed. This is immediate from the hypergeometric identities

$$\frac{1}{B(r,s)} = \frac{rs}{r+s}\,{}_2F_1\left(\left.\begin{matrix} -r, -s \\ 1 \end{matrix}\right| 1\right),$$

$$B(r,s) = \frac{\pi \sin \pi(r+s)}{\sin \pi r \, \sin \pi s} \frac{(1-r)_M (1-s)_M}{M!(1-r-s)_M} \times {}_2F_1\left(\left.\begin{matrix} r, s \\ M+1 \end{matrix}\right| 1\right),$$

where M is any integer chosen such that the hypergeometric series converges, say $M = \lceil 1 + r + s \rceil$. (Each of these Beta relations is a variant of the celebrated Gauss evaluation of ${}_2F_1$ at 1 [1], [54] and is also the reason B is a period.)

We did not seize upon the Beta function arbitrarily, for, remarkably, the hyperclosure of $B(r,s)^{\pm 1}$ leads to compelling results on the Gamma function itself. Indeed, consider for example this product of four Beta-function evaluations:

$$\frac{\Gamma(1/5)\Gamma(1/5)}{\Gamma(2/5)} \cdot \frac{\Gamma(2/5)\Gamma(1/5)}{\Gamma(3/5)} \cdot \frac{\Gamma(3/5)\Gamma(1/5)}{\Gamma(4/5)} \cdot \frac{\Gamma(4/5)\Gamma(1/5)}{\Gamma(5/5)}.$$

We know this product is hyperclosed. But upon inspection we see that the product is just $\Gamma^5(1/5)$.

[5] *Equation (2) is now proven, see M. Rogers and W. Zudilin, "On the Mahler measure of* $1 + X + 1/X + Y + 1/Y$*", preprint (2011),* `http://arxiv.org/abs/1102.1153`.

Along such lines one can prove that for any positive rational a/b (in lowest terms), we have hyperclosure of powers of the Gamma function in the form

$$\Gamma^{\pm b}(a/b) \in \mathbb{H}.$$

Perforce, we therefore have a superclosure result for any Γ(rational) and its reciprocal:

$$\Gamma^{\pm 1}(a/b) \in \mathbb{S}.$$

Again, like calculations show that $\Gamma^b(a/b)$ is a period [42]. One fundamental consequence is thus $\Gamma^{-2}\left(\frac{1}{2}\right) = \frac{1}{\pi}$ is hyperclosed; thus every integer power of π is hyperclosed.

Incidentally, deeper combinatorial analysis shows that in spite of our $\Gamma^5\left(\frac{1}{5}\right)$ Beta-chain above, it really takes only *logarithmically many* (i.e., $O(\log b)$) hypergeometric evaluations to write Gamma-powers. For example,

$$\Gamma^{-7}\left(\frac{1}{7}\right) = \frac{1}{2^3 7^6}\, {}_2F_1\left(\begin{matrix}-\frac{1}{7}, -\frac{1}{7}\\ 1\end{matrix}\,\middle|\, 1\right)^4 \times {}_2F_1\left(\begin{matrix}-\frac{2}{7}, -\frac{2}{7}\\ 1\end{matrix}\,\middle|\, 1\right)^2 {}_2F_1\left(\begin{matrix}-\frac{4}{7}, -\frac{4}{7}\\ 1\end{matrix}\,\middle|\, 1\right).$$

We note also that for $\Gamma(n/24)$ with n integer, elliptic integral algorithms are known that converge as fast as those for π [27], [22]. □

The above remarks on superclosure of $\Gamma(a/b)$ lead to the property of superclosure for special functions such as $J_\nu(\omega)$ for algebraic ω and rational ν and for many of the mighty Meijer-G functions, as the latter can frequently be written by Slater's theorem [15] as superpositions of hypergeometric evaluations with composite-gamma products as coefficients. (See Example 8 for instances of Meijer-G in current research.)

There is an interesting alternative way to envision hyperclosure or at least something very close to our above definition. This is an idea of J. Carette [28] to the effect that solutions at algebraic end-points and algebraic initial points for *holonomic* ODEs—i.e., differential-equation systems with integer-polynomial coefficients—could be considered closed. One might say *diffeoclosed*. An example of a diffeoclosed number is $J_1(1)$, i.e., from the Bessel differential equation for $J_1(z)$ with $z \in [0,1]$; it suffices without loss of generality to consider topologically clean trajectories of the variable over $[0,1]$. There is a formal ring of diffeoclosure, which ring is very similar to our $\mathbb{H}$; however, there is the caution that trajectory solutions can sometimes have nontrivial topology, so precise ring definitions would need to be effected carefully.

It is natural to ask, "what is the complexity of hypergeometric evaluations?" Certainly for the converging forms with variable z on the open unit disk, convergence is geometric, requiring $O(D^{1+\epsilon})$ operations to achieve D good digits. However, in very many cases this can be genuinely enhanced to $O(D^{1/2+\epsilon})$ [22].

Closed Forms: Why We Care

> In many optimization problems, *simple, approximate* solutions are more useful than complex exact solutions.
>
> —Steve Wright

As Steve Wright observed in a recent lecture on *sparse optimization*, it may well be that a complicated analytic solution is practically intractable, but a simplifying assumption leads to a very practical closed form approximation (e.g., in compressed sensing). In addition to appealing to Occam's razor, Wright instances that:

(a) the data quality may not justify exactness,
(b) the simple solution may be more robust,
(c) it may be easier to *explain/actuate/implement/store*,
(d) and it may conform better to prior knowledge.

As mathematical discovery more and more involves extensive computation, the premium on having a closed form increases. The insight provided by discovering a closed form ideally comes at the top of the list, but efficiency of computation will run a good second.

Example 5 (The amplitude of a pendulum)**.** Wikipedia,[6] after giving the classical small angle (simple harmonic) approximation

$$p \approx 2\pi\sqrt{\frac{L}{g}}$$

for the period p of a pendulum of length L and amplitude α, develops the exact solution in a form equivalent to

$$p = 4\sqrt{\frac{L}{g}}\,\mathrm{K}\left(\sin\frac{\alpha}{2}\right)$$

and then says:

> This integral cannot be evaluated in terms of elementary functions. It can be rewritten in the form of the elliptic function of the first kind (also see Jacobi's elliptic functions), which gives little advantage since that form is also insoluble.

True, an elliptic-integral solution is not elementary, yet the notion of insolubility is misleading for two reasons: First, it is known that for some special angles α the pendulum period can be given a closed form. As discussed in [33], one exact solution is, for

[6] *Available at* http://en.wikipedia.org/wiki/Pendulum_(mathematics).

$\alpha = \pi/2$ (so pendulum is released from horizontal-rod position),

$$p = \left(2\pi\sqrt{\frac{L}{g}}\right)\frac{\sqrt{\pi}}{\Gamma^2(3/4)}.$$

It is readily measurable in even a rudimentary laboratory that the excess factor here, $\sqrt{\pi}\Gamma^{-2}(3/4) \approx 1.18034$, looks just right; i.e., a horizontal-release pendulum takes 18 percent longer to fall. Moreover, there is an *exact* dynamical solution for the time-dependent angle $\alpha(t)$, namely, for a pendulum with $\alpha(\pm\infty) = \pm\pi$ and $\alpha(0) = 0$; i.e., the bob crosses angle zero (hanging straight down) at time zero, but in the limits of time $\to \pm\infty$ the bob ends up straight vertical. We have period $p = \infty$, yet the *exact* angle $\alpha(t)$ for given t can be written in terms of elementary functions!

The second misleading aspect is this: K is, for any α, remarkably tractable in a computational sense. Indeed K admits a quadratic transformation

$$\text{(4)} \quad \mathrm{K}(k) = (1 + k_1)\,\mathrm{K}(k_1), \quad k_1 := \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}},$$

as was known already to Landen, Legendre, and Gauss.

In fact all elementary functions to very high precision are well computed via K [22]. So the comment was roughly accurate in the world of slide rules or pocket calculators; it is misleading today if one has access to any computer package. Nevertheless, both deserve to be called closed forms: one exact and the other an elegant approximate closed form (excellent in its domain of applicability, much as with Newtonian mechanics) that is equivalent to

$$\mathrm{K}\left(\sin\frac{\alpha}{2}\right) \approx \frac{\pi}{2}$$

for small initial amplitude α. To compute $K(\pi/6) = 1.699075885\ldots$ to five places requires using (4) only twice and then estimating the resultant integral by $\pi/2$. A third step gives the ten-digit precision shown. □

It is now the case that much mathematical computation is *hybrid*: mixing numeric and symbolic computation. Indeed, which is which may not be clear to the user if, say, numeric techniques have been used to return a symbolic answer or if a symbolic closed form has been used to make possible a numerical integration. Moving from classical to modern physics, both understanding and effectiveness frequently demand hybrid computation.

Example 6 (Scattering amplitudes [2])**.** An international team of physicists, in preparation for the Large Hadron Collider (LHC), is computing scattering amplitudes involving quarks, gluons, and gauge vector bosons in order to predict what results could be expected on the LHC. By default, these computations are performed using conventional double precision (64-bit IEEE) arithmetic. Then if a particular phase space point is deemed numerically unstable, it is recomputed with double-double precision. These researchers expect that further optimization of the procedure for identifying unstable points may be required to arrive at an optimal compromise between numerical accuracy and speed of the code. Thus they plan to incorporate arbitrary precision arithmetic into these calculations. Their objective is to design a procedure where instead of using fixed double or quadruple precision for unstable points, the number of digits in the higher precision calculation is dynamically set according to the instability of the point. Any subroutine which uses a closed form symbolic solution (exact or approximate) is likely to prove much more robust and efficient. □

Detailed Examples

We start with three examples originating in [16].

In the January 2002 issue of *SIAM News*, Nick Trefethen presented ten diverse problems used in teaching *modern* graduate numerical analysis students at Oxford University, the answer to each being a certain real number. Readers were challenged to compute ten digits of each answer, with a $100 prize to the best entrant. Trefethen wrote,

> If anyone gets 50 digits in total, I will be impressed.

To his surprise, a total of ninety-four teams, representing twenty-five different nations, submitted results. Twenty of these teams received a full one hundred points (ten correct digits for each problem). The problems and solutions are dissected most entertainingly in [16]. One of the current authors wrote the following in a review [19] of [16]:

> Success in solving these problems required a broad knowledge of mathematics and numerical analysis, together with significant computational effort, to obtain solutions and ensure correctness of the results. As described in [16] the strengths and limitations of *Maple*, *Mathematica*, MATLAB (The *3Ms*), and other software tools such as PARI or GAP, were strikingly revealed in these ventures. Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided.

Example 7 (Trefethen problem #2 [16], [19])**.**

A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x, y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius $1/3$ has been erected. How far from the origin is the photon at $t = 10$?

Using *interval arithmetic* with starting intervals of size smaller than 10^{-5000}, one can actually find the position of the particle at time 2000, not just at time ten. This makes a fine exercise in very high-precision interval computation, but in the absence of any closed form, one is driven to such numerical gymnastics to deal with error propagation. □

Example 8 (Trefethen's problem #9 [16], [19]).

The integral $I(a) = \int_0^2 [2 + \sin(10\alpha)]x^\alpha \times \sin(\alpha/(2-x))\,dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

The maximum parameter is expressible in terms of a *Meijer-G function*, which is a special function with a solid history. While knowledge of this function was not common among the contestants, Mathematica and Maple both will figure this out [15], and then the help files or a Web search will quickly inform the scientist.

This is another measure of the changing environment. It is usually a good idea—and not at all immoral—to data-mine. These Meijer-G functions, first introduced in 1936, also occur in quantum field theory and many other places [8]. For example, the moments of an n-step random walk in the plane are given for $s > 0$ by

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s dx. \tag{5}$$

It transpires [24], [36] that for all complex s

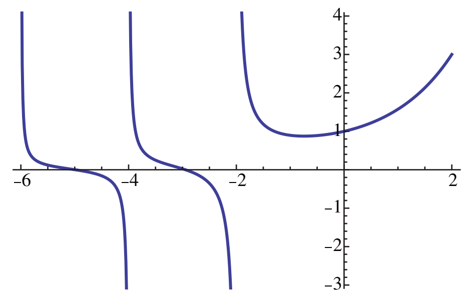
$$W_3(s) = \frac{\Gamma(1+s/2)}{\Gamma(1/2)\Gamma(-s/2)}\, G^{2,1}_{3,3}\left(\begin{matrix} 1,1,1 \\ \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2} \end{matrix} \middle| \frac{1}{4} \right). \tag{6}$$

Moreover, for s not an odd integer, we have

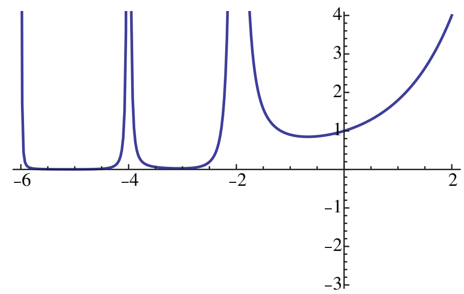
$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| \frac{1}{4} \right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, -\frac{s-1}{2} \end{matrix} \middle| \frac{1}{4} \right).$$

We have not given the somewhat technical definition of `MeijerG`, but Maple, Mathematica, Google searches, Wikipedia, the DLMF, or many other tools will.

There are two corresponding formulae for W_4. We thus know from our "Sixth Approach" section, in regard to superclosure of Γ-evaluations, that *both* $W_3(q), W_4(q)$ *are superclosed for rational argument* q for q not an odd integer. We illustrate by showing graphs of W_3, W_4 on the real line in Figure 2 and in the complex plane in Figure 3. The latter highlights the utility of the Meijer-G representations. Note the poles and removable singularities.



(a) W_3



(b) W_4

Figure 2. Moments of n-step walks in the plane. W_3, W_4 analytically continued to the real line.

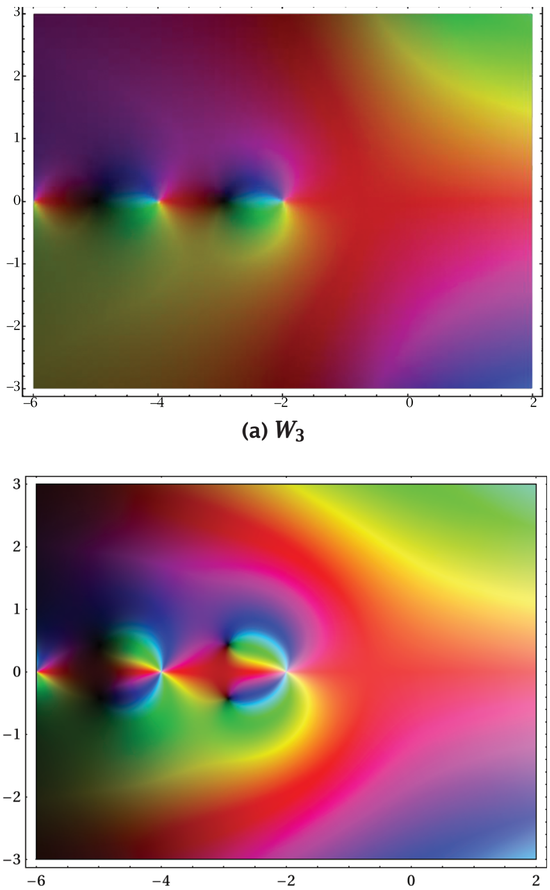
The Meijer-G functions are now described in the newly completed *Digital Library of Mathematical Functions*[7] and as such are now full, indeed central, members of the family of special functions. □

Example 9 (Trefethen's problem #10 [16], [19]).

A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Hitting the Ends. Bornemann [16] starts his remarkable solution by exploring *Monte-Carlo methods*, which are shown to be impracticable. He then reformulates the problem *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate* harmonic measure [57] *of the ends*, arising from a five-point discretization of *Laplace's equation* with Dirichlet boundary conditions. This is then solved by a well-chosen *sparse Cholesky*

[7] *A massive revision of Abramowitz and Stegun, with the now redundant tables removed, is available at* `http://dlmf.nist.gov`. *The hard copy version is also now out* [45]. *It is not entirely a substitute for the original version, as coverage has changed.*



(a) W_3

(b) W_4

Figure 3. W_3 via (6) and W_4 in the complex plane.

solver. At this point a reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained, and the posed problem is solved *numerically* to the requisite ten places.

This is the warm-up. We may proceed to develop two analytic solutions, the first using *separation of variables* on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that

$$(7)\quad p(a,b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right),$$

where $\rho := a/b$. A second method using *conformal mappings* yields

$$(8)\qquad \operatorname{arccot}\rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right),$$

where K is again the *complete elliptic integral* of the first kind. It will not be apparent to a reader unfamiliar with inversion of elliptic integrals that (7) and (8) encode the same solution—though they must, as the solution is unique in $(0,1)$—and each can now be used to solve for $\rho = 10$ to arbitrary precision. Bornemann ultimately shows that the answer is

$$(9)\qquad p = \frac{2}{\pi} \arcsin\left(k_{100}\right),$$

where

$$k_{100} := \left(\left(3 - 2\sqrt{2}\right)\left(2 + \sqrt{5}\right)\left(-3 + \sqrt{10}\right) \times \left(-\sqrt{2} + \sqrt[4]{5}\right)^2\right)^2.$$

No one (except harmonic analysts perhaps) anticipated a closed form, let alone one like this.

Where Does This Come From? In fact, [22, (3.2.29)] shows that

$$(10)\quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) = \frac{1}{2}\arcsin k$$

exactly when k_{ρ^2} is parameterized by *theta functions* in terms of the so-called *nome*, $q = \exp(-\pi\rho)$, as Jacobi discovered. We have

$$(11)\quad k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}, \quad q := e^{-\pi\rho}.$$

Comparing (10) and (7) we see that the solution is

$$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7},$$

as asserted in (9).

The explicit form now follows from classical nineteenth-century theory, as discussed, say, in [16], [22]. In fact, k_{210} is the singular value sent by Ramanujan to Hardy in his famous letter of introduction [21], [22]. If Trefethen had asked for a $\sqrt{210} \times 1$ box or, even better, a $\sqrt{15} \times \sqrt{14}$ one, this would have shown up in the answer, since in general

$$(12)\qquad p(a,b) = \frac{2}{\pi} \arcsin\left(k_{a^2/b^2}\right).$$

Alternatively, armed only with the knowledge that the singular values of rational parameters are always algebraic, we may finish entirely computationally as described in [19]. ☐

We finish this section with two attractive applied examples from optics and astrophysics respectively.

Example 10 (Mirages [46])**.** In [46] the authors, using geometric methods, develop an exact but implicit formula for the path followed by a light ray propagating over the earth with radial variations in the refractive index. By suitably simplifying, they are able to provide an explicit integral closed form. They then expand it asymptotically. This is done with the knowledge that the approximation is good to six or seven places, more than enough to use it on optically realistic scales. Moreover, in the case of quadratic or linear refractive indices, these steps may be done *analytically*.

In other words, as advanced by Wright, a tractable and elegant approximate closed form is

(a) A superior mirage

(b) An inferior mirage (Photo © Ctein.)

Figure 4. Two impressive mirages.

obtained to replace a problematic exact solution. From these forms interesting qualitative consequences follow. With a quadratic index, images are uniformly magnified in the vertical direction; only with higher-order indices can nonuniform vertical distortion occur. This sort of knowledge allows one, for example, to correct distortions of photographic images as in Figure 4 with confidence and efficiency. □

Example 11 (Structure of stars)**.** The celebrated *Lane-Emden equation*, presumed to describe the pressure χ at radius r within a star, can be put in the form

$$r^{n-1}\frac{d^2\chi}{dr^2} = -\chi^n, \tag{13}$$

with boundary conditions $\chi(0) = 0, \chi'(0) = 1$, and positive real constant n, all of this giving rise to a unique trajectory $\chi_n(r)$ on $r \in [0, \infty)$. (Some authors invoke the substitution $\chi(r) := r\theta(r)$ to get an equivalent ODE for temperature θ; see [30].) The beautiful thing is, where this pressure trajectory crosses zero for positive radius r is supposed to be the *star radius*; call that zero z_n.

Amazingly, the Lane-Emden equation has known exact solutions for $n = 0, 1, 5$, the pressure trajectories for which indices n being respectively

$$\chi_0(r) = -\frac{1}{6}r^3 + r, \tag{14}$$

$$\chi_1(r) = \sin r, \tag{15}$$

$$\chi_5(r) = \frac{r}{\sqrt{1 + r^2/3}}. \tag{16}$$

The respective star radii are thus closed forms $z_0 = \sqrt{6}$ and $z_1 = \pi$, while for (16), with index $n = 5$ we have infinite star radius (no positive zero for the pressure χ_5).

In the spirit of our previous optics example, the Lane-Emden equation is a simplification of a complicated underlying theory—in this astrophysics case, hydrodynamics—and one is rewarded by some closed-form star radii. But what about, say, index $n = 2$? We do not know a closed-form function for the χ trajectory in any convenient sense. What the present authors *have* calculated (in 2005) is the $n = 2$ star radius, as a high-precision number

$$z_2 = 4.35287459594612467697357006152614262811236536321300883530215\,1\ldots.$$

If only we could gain a closed form for this special radius, we might be able to guess the nature of the whole trajectory! □

Recent Examples Relating to Our Own Work

Example 12 (Ising integrals [5], [8])**.** We recently studied the following classes of integrals [5]. The D_n integrals arise in the Ising model of mathematical physics (showing ferromagnetic temperature-driven phase shifts; see Figure 5 and [32]), and the C_n have tight connections to quantum field theory [8]:

$$C_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2}\frac{\mathrm{d}u_1}{u_1}\cdots\frac{\mathrm{d}u_n}{u_n},$$

$$D_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i - u_j}{u_i + u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2}\frac{\mathrm{d}u_1}{u_1}\cdots\frac{\mathrm{d}u_n}{u_n},$$

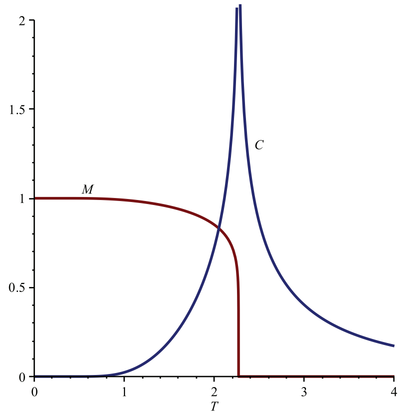
$$E_n = 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n}\frac{u_k - u_j}{u_k + u_j}\right)^2 \mathrm{d}t_2\,\mathrm{d}t_3\cdots\mathrm{d}t_n,$$

where (in the last line) $u_k = \prod_{i=1}^k t_i$.

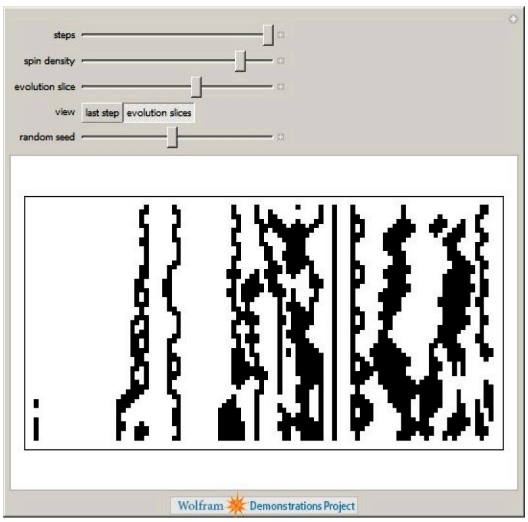
Needless to say, evaluating these multidimensional integrals to high precision presents a daunting computational challenge. Fortunately, in the first case, the C_n integrals can be written as one-dimensional integrals:

$$C_n = \frac{2^n}{n!}\int_0^\infty pK_0^n(p)\,\mathrm{d}p,$$

where K_0 is the *modified Bessel function*. After computing C_n to 1000-digit accuracy for various n, we were able to identify the first few instances of C_n



(a) Critical temperature



(b) *Wolfram Player* Demonstration

Figure 5. The 2-dimensional Ising Model of Ferromagnetism (a) plotting magnetization C (blue, with peak) and specific heat M (red, decaying) per site against absolute temperature T (image provided by Jacques Perk) [44, pp. 91–93, 245].

in terms of well-known constants, e.g.,

$$C_3 = \mathrm{L}_{-3}(2) \;:=\; \sum_{n\geq 0}\left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right),$$

$$C_4 = \frac{7}{12}\,\zeta(3),$$

where ζ denotes the Riemann zeta function. When we computed C_n for fairly large n, for instance,

$$C_{1024} = 0.6304735033743867961220401927108789043545870787127323 4\ldots,$$

we found that these values rather quickly approached a limit. By using the new edition of the Inverse Symbolic Calculator,[8] we identified this numerical value as

$$\lim_{n\to\infty} C_n = 2e^{-2\gamma},$$

where γ is the *Euler constant*; see the section "Profound Curiosities". We later were able to prove this fact—this is merely the first term of an asymptotic expansion—and thus showed that the C_n integrals are fundamental in this context [5].

The integrals D_n and E_n are much more difficult to evaluate, since they are not reducible to one-dimensional integrals (as far as we can tell); but with certain symmetry transformations and symbolic integration we were able to symbolically reduce the dimension in each case by one or two.

In the case of D_5 and E_5, the resulting 3-D integrands are extremely complicated (see Figure 6), but we were nonetheless able to numerically evaluate these to at least 240-digit precision on a highly parallel computer system. This would have been impossible without the symbolic reduction. We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations, all of which, except the last, we subsequently were able to prove:

$$\begin{aligned}
D_2 &= 1/3,\\
D_3 &= 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2),\\
D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2,\\
E_2 &= 6 - 8\log 2,\\
E_3 &= 10 - 2\pi^2 - 8\log 2 + 32\log^2 2,\\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2,\\
&\quad - 256(\log^3 2)/3 + 16\pi^2\log 2 - 22\pi^2/3,
\end{aligned}$$

and

(17)

$$\begin{aligned}
E_5 \stackrel{?}{=} 42 &- 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3)\\
&- 1272\zeta(3)\log 2 + 40\pi^2\log^2 2\\
&- 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2\\
&+ 464\log^2 2 - 40\log 2,
\end{aligned}$$

where Li denotes the polylogarithm function.

In the case of D_2, D_3, and D_4, these are confirmations of known results. We tried but failed to recognize D_5 in terms of similar constants (the 500-digit numerical value is accessible[9] if anyone wishes to try to find a closed form or, in the manner of the hard sciences, to confirm our data values). The conjectured identity shown here for E_5 was confirmed to 240-digit accuracy, which is 180 digits beyond the level that could reasonably be ascribed to numerical round-off error. Thus we are quite

[8] *Available at* `http://isc2.carma.newcastle.edu.au/`.

[9] *Available at* `http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf`.

$$
\begin{aligned}
E_5 = \int_0^1 \int_0^1 \int_0^1 & \Big[2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \\
& \Big(-\Big[4(x+1)(xy+1)\log(2)\big(y^5z^3x^7-y^4z^2(4(y+1)z+3)x^6-y^3z\big(\big(y^2+1\big)z^2 \\
& +4(y+1)z+5\big)x^5+y^2\big(4y(y+1)z^3+3\big(y^2+1\big)z^2+4(y+1)z-1\big)x^4 \\
& +y\big(z\big(z^2+4z+5\big)y^2+4\big(z^2+1\big)y+5z+4\big)x^3+\big(\big(-3z^2-4z+1\big)y^2 \\
& -4zy+1\big)x^2-(y(5z+4)+4)x-1\big)\Big]/\Big[(x-1)^3(xy-1)^3(xyz-1)^3\Big] \\
& +\Big[3(y-1)^2y^4(z-1)^2z^2(yz-1)^2x^6+2y^3z\big(3(z-1)^2z^3y^5+z^2\big(5z^3+3z^2 \\
& +3z+5\big)y^4+(z-1)^2z\big(5z^2+16z+5\big)y^3+\big(3z^5+3z^4-22z^3-22z^2 \\
& +3z+3\big)y^2+3\big(-2z^4+z^3+2z^2+z-2\big)y+3z^3+5z^2+5z+3\big)x^5 \\
& +y^2\big(7(z-1)^2z^4y^6-2z^3\big(z^3+15z^2+15z+1\big)y^5+2z^2\big(-21z^4+6z^3 \\
& +14z^2+6z-21\big)y^4-2z\big(z^5-6z^4-27z^3-27z^2-6z+1\big)y^3+\big(7z^6 \\
& -30z^5+28z^4+54z^3+28z^2-30z+7\big)y^2-2\big(7z^5+15z^4-6z^3-6z^2+15z \\
& +7\big)y+7z^4-2z^3-42z^2-2z+7\big)x^4-2y\big(z^3\big(z^3-9z^2-9z+1\big)y^6 \\
& +z^2\big(7z^4-14z^3-18z^2-14z+7\big)y^5+z\big(7z^5+14z^4+3z^3+3z^2+14z \\
& +7\big)y^4+\big(z^6-14z^5+3z^4+84z^3+3z^2-14z+1\big)y^3-3\big(3z^5+6z^4-z^3 \\
& -z^2+6z+3\big)y^2-\big(9z^4+14z^3-14z^2+14z+9\big)y+z^3+7z^2+7z+1\big)x^3 \\
& +\big(z^2\big(11z^4+6z^3-66z^2+6z+11\big)y^6+2z\big(5z^5+13z^4-2z^3-2z^2+13z \\
& +5\big)y^5+\big(11z^6+26z^5+44z^4-66z^3+44z^2+26z+11\big)y^4+\big(6z^5-4z^4 \\
& -66z^3-66z^2-4z+6\big)y^3-2\big(33z^4+2z^3-22z^2+2z+33\big)y^2+\big(6z^3 \\
& +26z+6\big)y+11z^2+10z+11\big)x^2-2\big(z^2\big(5z^3+3z^2+3z+5\big)y^5 \\
& +z\big(22z^4+5z^3-22z^2+5z+22\big)y^4+\big(5z^5+5z^4-26z^3-26z^2+5z+5\big)y^3 \\
& +\big(3z^4-22z^3-26z^2-22z+3\big)y^2+\big(3z^3+5z^2+5z+3\big)y+5z^2+22z+5\big)x \\
& +15z^2+2z+2y(z-1)^2(z+1)+2y^3(z-1)^2z(z+1)+y^4z^2\big(15z^2+2z+15\big) \\
& +y^2\big(15z^4-2z^3-90z^2-2z+15\big)+15\Big]/\Big[(x-1)^2(y-1)^2(xy-1)^2(z-1)^2 \\
& (yz-1)^2(xyz-1)^2\Big]-\Big[4(x+1)(y+1)(yz+1)\big(-z^2y^4+4z(z+1)y^3 \\
& +\big(z^2+1\big)y^2-4(z+1)y+4x\big(y^2-1\big)\big(y^2z^2-1\big)+x^2\big(z^2y^4-4z(z+1)y^3 \\
& -\big(z^2+1\big)y^2+4(z+1)y+1\big)-1\big)\log(x+1)\Big]/\Big[(x-1)^3x(y-1)^3(yz-1)^3\Big] \\
& -\Big[4(y+1)(xy+1)(z+1)\big(x^2\big(z^2-4z-1\big)y^4+4x(x+1)\big(z^2-1\big)y^3 \\
& -\big(x^2+1\big)\big(z^2-4z-1\big)y^2-4(x+1)\big(z^2-1\big)y+z^2-4z-1\big)\log(xy+1)\Big]/ \\
& \Big[x(y-1)^3y(xy-1)^3(z-1)^3\Big]-\Big[4(z+1)(yz+1)\big(x^3y^5z^7+x^2y^4(4x(y+1) \\
& +5)z^6-xy^3\big(\big(y^2+1\big)x^2-4(y+1)x-3\big)z^5-y^2\big(4y(y+1)x^3+5\big(y^2+1\big)x^2 \\
& +4(y+1)x+1\big)z^4+y\big(y^2x^3-4y(y+1)x^2-3\big(y^2+1\big)x-4(y+1)\big)z^3 \\
& +\big(5x^2y^2+y^2+4x(y+1)y+1\big)z^2+((3x+4)y+4)z-1\big)\log(xyz+1)\Big]/ \\
& \Big[xy(z-1)^3z(yz-1)^3(xyz-1)^3\Big]\Big)\Big] \\
& /\Big[(x+1)^2(y+1)^2(xy+1)^2(z+1)^2(yz+1)^2(xyz+1)^2\Big]\,dx\,dy\,dz.
\end{aligned}
$$

Figure 6. The reduced multidimensional integral for E_5, which integral has led via extreme-precision numerical quadrature and PSLQ to the conjectured closed form given in (17).

confident in this result, even though we do not have a formal proof [5].

Note that every one of the D, E forms above, including the conjectured last one, is hyperclosed in the sense of our "Sixth Approach" section. $\square$

Example 13 (Weakly coupling oscillators [49], [6])**.** In an important analysis of coupled *Winfree oscillators*, Quinn, Rand, and Strogatz [49] developed a certain N-oscillator scenario whose bifurcation phase offset ϕ is implicitly defined, with a conjectured asymptotic behavior: $\sin\phi \sim 1 - c_1/N$, with experimental estimate $c_1 = 0.605443657\ldots$. In [6] we were able to derive the exact theoretical value of this "QRS constant" c_1 as the unique zero of the Hurwitz zeta $\zeta(1/2, z/2)$ on $z \in (0,2)$. In so doing, we were able to prove the conjectured behavior. Moreover, we were able to sketch the higher-order asymptotic behavior, something that would have been impossible without discovery of an analytic formula.

Does this deserve to be called a closed form? In our opinion, resoundingly yes unless all inverse functions such as that in Bornemann's (12) are to be eschewed. Such constants are especially interesting in light of even more recent work by Steve Strogatz and his collaborators on *chimera*, coupled systems which can self-organize in parts of their domain and remain disorganized elsewhere; see Figure 7 taken from [43]. In this case, observed numerical limits still need to be put in closed form. $\square$

It is a frequent experience of ours that, as in Example 13, the need for high-accuracy computation drives the development of effective analytic expressions (closed forms?), which in turn typically shed substantial light on the subject being studied.

Example 14 (Box integrals [3], [7], [23])**.** There has been recent research on the calculation of the expected distance of points inside a hypercube to the hypercube. Such expectations are also called "box integrals" [23]. So, for example, the expectation $\langle|\vec{r}|\rangle$ for random $\vec{r} \in [0,1]^3$ has the closed form

$$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right).$$

Incidentally, box integrals are not just a mathematician's curiosity; the integrals have been used recently to assess the randomness of brain synapses positioned within a parallelepiped [38]. Indeed, we had cognate results for

$$\Delta_d(s) := \int_{[0,1]^d}\int_{[0,1]^d} \|x-y\|_2^s \, \mathrm{d}x\mathrm{d}y,$$

which gives the moments of the distance between two points in the hypercube.

In a lovely recent paper [52] Stephan Steinerberger has shown that in the limit, as the dimension goes to infinity,

$$\lim_{d\to\infty}\left(\frac{1}{d}\right)^{s/p}\int_{[0,1]^d}\int_{[0,1]^d}\|x-y\|_p^s\,\mathrm{d}x\mathrm{d}y = \left(\frac{2}{(p+1)(p+2)}\right)^{s/p} \tag{18}$$

for any $s, p > 0$. In particular, with $p = 2$, this gives a first-order answer to our earlier published request for the asymptotic behavior of $\Delta_d(s)$.

A quite recent result is that all box integrals $\langle|\vec{r}|^n\rangle$ for integer n and dimensions $1, 2, 3, 4, 5$ are *hyperclosed*, in the sense of our "Sixth Approach"

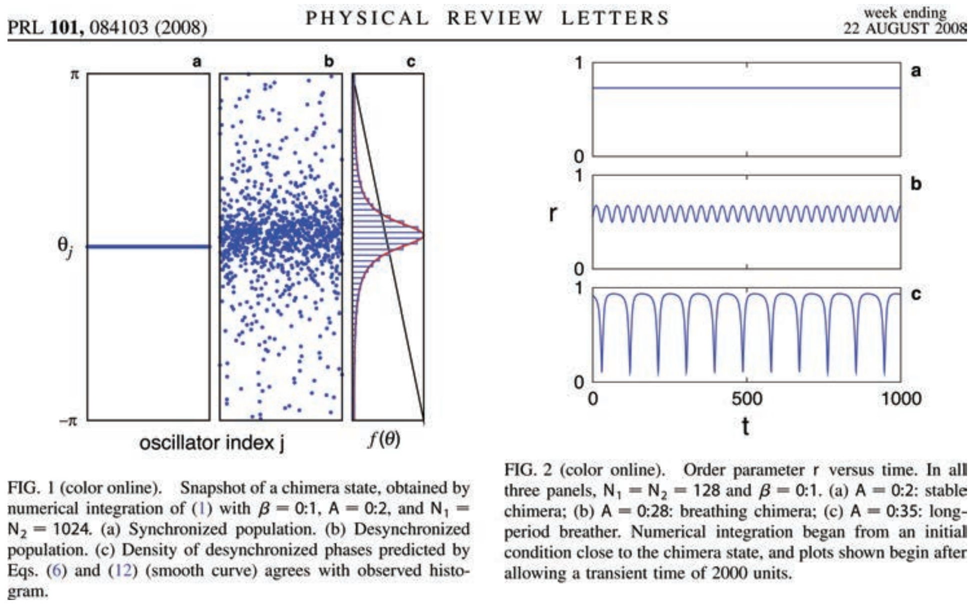

PRL **101,** 084103 (2008) PHYSICAL REVIEW LETTERS week ending 22 AUGUST 2008

FIG. 1 (color online). Snapshot of a chimera state, obtained by numerical integration of (1) with $\beta = 0.1$, $A = 0.2$, and $N_1 = N_2 = 1024$. (a) Synchronized population. (b) Desynchronized population. (c) Density of desynchronized phases predicted by Eqs. (6) and (12) (smooth curve) agrees with observed histogram.

FIG. 2 (color online). Order parameter r versus time. In all three panels, $N_1 = N_2 = 128$ and $\beta = 0.1$. (a) $A = 0.2$: stable chimera; (b) $A = 0.28$: breathing chimera; (c) $A = 0.35$: long-period breather. Numerical integration began from an initial condition close to the chimera state, and plots shown begin after allowing a transient time of 2000 units.

Figure 7. Simulated chimera (figures and parameters from [43]**).**

section. It turns out that five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)}\, dx\, dy. \tag{19}$$

A proof of hyperclosure of $J(t)$ for algebraic $t \ge 0$ is established in [23, Thm. 5.1]. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0,1]^5$ can be written in explicit hyperclosed form involving a 10^5-character symbolic $J(3)$; the authors of [23] were able to reduce the 5-dimensional box integral down to "only" 10^4 characters. A companion integral $J(2)$ also starts out with about 10^5 characters but reduces stunningly to only a few dozen characters, namely,

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right), \tag{20}$$

where Cl_2 is the *Clausen function* $\mathrm{Cl}_2(\theta) := \sum_{n\ge 1} \sin(n\theta)/n^2$ (Cl_2 is the simplest nonelementary Fourier series).

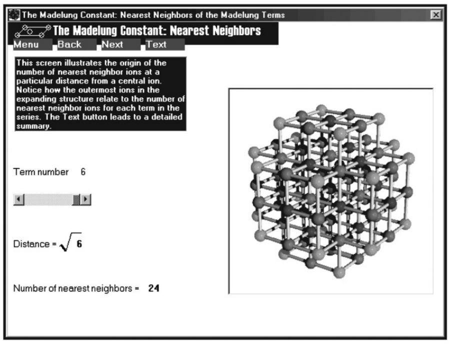
Automating such reductions will require a sophisticated simplification scheme with a very large and extensible knowledge base. With a current research assistant, Alex Kaiser at Berkeley, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper (see [9]). This semi-automated integrity checking becomes pressing when, as above, verifiable output from a symbolic manipulation can be the length of a Salinger novella. □

Profound Curiosities

In our treatment of numbers enjoying hyperclosure or superclosure, we admitted that such numbers are countable, and so almost all complex numbers cannot be given a closed form along such lines. What is stultifying is: *How do we identify an explicit number lying outside such countable sets?* The situation is tantamount to the modern bind in regard to *normal numbers*, numbers which to some base have statistically random digit structure in a certain technical sense. The bind is, though almost all numbers are *absolutely normal* (i.e., normal to every base $2, 3, \ldots$), we do not know a single fundamental constant that is provably absolutely normal. (We do know some "artificial" normal numbers; see [14].)

Here is one possible way out of the dilemma. In the theory of computability, the existence of *noncomputable* real numbers, such as an encoded list of halting Turing machines, is well established. The celebrated *Chaitin constant* Ω is a well-known noncomputable. So a "folk" argument goes: *Since every element of the ring of hyperclosure $\mathbb{H}$ can be computed via converging series, it should be that* $\Omega \notin \mathbb{H}$. A good research problem would be to make this heuristic rigorous.

Let us focus on some constants that *might* not be hyperclosed (nor superclosed). One such constant is the celebrated *Euler constant* $\gamma := \lim_{n\to\infty} \sum_{k=1}^{n} 1/k - \log n$. We know of no hypergeometric form for γ; said constant may well lie outside $\mathbb{H}$ (or even $\mathbb{S}$). There *are* expansions for



A screen from the Madelung Constant section of Lattice Energetics.

(a) NaCl nearest neighbors

(b) CMS Prize Sculpture

Figure 8. Two representations of salt.

the Euler constant, such as

$$\gamma = \log\pi - 4\log\Gamma\left(\frac{3}{4}\right) + \frac{4}{\pi}\sum_{k\geq 1}(-1)^{k+1}\frac{\log(2k+1)}{2k+1},$$

and even more exotic series (see [13]). But in the spirit of the present treatment, we do not want to call the infinite series closed, because it is not hypergeometric per se. Relatedly, the classical Bessel expansion is

$$K_0(z) = -\left(\ln\left(\frac{z}{2}\right)+\gamma\right)I_0(z) + \sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n-1}\frac{1}{k}}{(n!)^2}\left(\frac{z^2}{4}\right)^n.$$

Now $K_0(z)$ has a (degenerate) Meijer-G representation—so potentially is superclosed for algebraic z—and $I_0(z)$ is accordingly hyperclosed, but the nested harmonic series on the right is again problematic. Again, γ is conjectured not to be a period [42].

Example 15 (Madelung constant [22], [37], [58]). Another fascinating number is the *Madelung constant*, $\mathcal{M}$, of chemistry and physics [22, Section 9.3]. This is the potential energy at the origin of an oscillating-charge crystal structure (most often said crystal is NaCl (salt), as illustrated in Figure 8). Image (b) is of a Helaman Ferguson sculpture based on $\mathcal{M}$ which is awarded biannually by the Canadian Mathematical Society as part of the David Borwein Career Award) and is given by the formal (conditionally convergent [18]) sum

$$\mathcal{M} := \sum_{(x,y,z)\neq(0,0,0)}\frac{(-1)^{x+y+z}}{\sqrt{x^2+y^2+z^2}} = -1.747564594633\ldots \tag{21}$$

and has never been given what a reasonable observer would call a closed form. Nature plays an interesting trick here: There are other crystal structures that *are* tractable, yet somehow this exquisitely symmetrical salt structure remains elusive. In general, even-dimensional crystal sums are more tractable than odd for the same modular function reasons that the number of representations of a number as the sum of an even number of squares is. But this does not make them easy, as illustrated by Example 2.

Here we have another example of a constant having no known closed form yet is rapidly calculable. A classical rapid expansion for the Madelung constant is due to Benson:

$$\mathcal{M} = -12\pi\sum_{m,n\geq 0}\operatorname{sech}^2\left(\frac{\pi}{2}\sqrt{(2m+1)^2+(2n+1)^2}\right), \tag{22}$$

in which convergence is exponential. Summing for $m,n \leq 3$ produces $-1.747564594\ldots$, correct to eight digits. There are a great many other such formulae for $\mathcal{M}$ (see [22], [35]).

Through the analytic methods of Buhler, Crandall, Tyagi, and Zucker since 1999 (see [35], [37], [55], [58]), we now know approximations such as

$$\mathcal{M} \approx -\frac{1}{8} - \frac{\log 2}{4\pi} + \frac{8\pi}{3} + \frac{1}{\sqrt{8}} + \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\pi^{3/2}\sqrt{2}} + \log\frac{k_4^2}{16k_4'},$$

where $k_4 := ((2^{1/4}-1)/(2^{1/4}+1))^2$. Two remarkable things: First, this approximation is good to the same thirteen decimals we give in (21); the missing $O(10^{-14})$ error here is a rapidly, exponentially converging—but alas infinite—sum in this modern approximation theory. Second, this six-term approximation itself is indeed hyperclosed, the only problematic term being the Γ-function part, but we did establish in our "Sixth Approach" section that $B(1/8, 3/8)$ and also $1/\pi$ are hyperclosed, which is enough. Moreover, the work of Borwein and Zucker [27] also settles hyperclosure for that term. $\square$

Certainly we have nothing like a proof, or even the beginnings of one, that $\mathcal{M}$ (or γ) lies outside $\mathbb{H}$ (or even $\mathbb{S}$), but we ask on an intuitive basis, Is a constant such as the mighty $\mathcal{M}$ telling us that it is not hyperclosed, in that our toil only seems to bring more "closed-form" terms into play, with no exact resolution in sight?

Concluding Remarks and Open Problems

• We have posited several approaches to the elusive notion of "closed form", but what are the intersections and interrelations of said approaches? For example, can our "Fourth Approach" be precisely absorbed into the evidently more general "Sixth Approach" (hyperclosure and superclosure)?

• How do we find a *single* number that is provably not in the ring of hyperclosure $\mathbb{H}$? (Though no such number is yet known, *almost all* numbers are, as noted, not in said ring!) The same question persists for the ring of superclosure, $\mathbb{S}$. Furthermore, how precisely can one create a *field* out of $\mathbb{H}^{\mathbb{H}}$ via appropriate operator extension?

• Though $\mathbb{H}$ is a subset of $\mathbb{S}$, how might one prove that $\mathbb{H} \neq \mathbb{S}$? (Is the inequality even true?) Likewise, is the set of closed forms in the sense of [48, Ch. 8] (only finite linear combinations of hypergeometric evaluations) properly contained in our $\mathbb{H}$? And what about a construct such as $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$? Should such an entity be anything really new? Lest one remark on the folly of such constructions, we observe that most everyone would say $\pi^{\pi^{\pi}}$ is a closed form!

• Having established the property of hyperclosure for $\Gamma^b(a/b)$, are there any cases where the power b may be brought down? For example, $1/\pi$ is hyperclosed, but what about $1/\sqrt{\pi}$?

• What is a precise connection between the ring of hyperclosure (or superclosure) and the set of periods or of Mahler measures (as in Example 3)?

• There is expounded in reference [23] a theory of "expression entropy", whereby some fundamental entropy estimate gives the true complexity of an expression. So, for example, an expression having one thousand instances of the `polylog` token Li_3 might really involve only about 1,000 characters, with that polylogarithm token encoded as a single character, say. (In fact, during the research for [23] it was noted that the entropy of Maple and Mathematica expressions of the *same* entity often had widely varying text-character counts but similar entropy assessments.)

On the other hand, one basic notion of "closed form" is that explicitly infinite sums not be allowed. Can these two concepts be reconciled? Meaning, can we develop a theory of expression entropy by which an explicit, infinite sum is given infinite entropy? This might be difficult, as, for example, a sum $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ takes only a few characters to symbolize, as we just did! If one can succeed, though, in thus resolving the entropy business for expressions, "closed form" might be rephrased as "finite entropy".

In any event, we feel strongly that the value of closed forms increases as the complexity of the objects we manipulate computationally and inspect mathematically grows, and we hope we have illustrated this. Moreover, we belong to the subset of mathematicians that finds fun in finding unanticipated closed forms.

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Wavelets

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The subject called "wavelets" is made up of several areas of pure and applied mathematics. It has contributed to the understanding of many problems in various sciences, engineering and other disciplines, and it includes, among its notable successes, the wavelet-based digital fingerprint image compression standard adopted by the FBI in 1993 and JPEG2000, the current standard for image compression.

We will begin by describing what wavelets are in one dimension and then pass to more general settings, trying to keep the presentation at a non-technical level as much as possible. We assume that the reader knows a bit of harmonic analysis. In particular, we assume knowledge of the basic properties of Fourier series and Fourier transforms. We start by establishing the basic definitions and notation which will be used in the following.

The space $L^2(\mathbb{R})$ is the Hilbert space of all square (Lebesgue) integrable functions endowed with the inner product $\langle f,g\rangle = \int_{\mathbb{R}} f\,\overline{g}$. The *Fourier transform* $\mathcal{F}$ is the unitary operator that maps $f \in L^2(\mathbb{R})$ into the function $\mathcal{F}f = \hat{f}$ defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x}\, dx$$

when $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and by the "appropriate" limit for the general $f \in L^2(\mathbb{R})$. We refer to the variable x as the *time* variable and to ξ as the *frequency* variable. Notice that the function $\hat{f}$ is also square integrable. Indeed, $\mathcal{F}$ maps $L^2(\mathbb{R})$ one-to-one onto itself. The inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ is defined by

$$(\mathcal{F}^{-1}g)(x) = \check{g}(x) = \int_{\hat{\mathbb{R}}} g(\xi)\, e^{2\pi i x\xi}\, d\xi.$$

The functions $\{e_k(x) = e^{2\pi i k x} : k \in \mathbb{Z}\}$ are 1-periodic and form an orthonormal basis of $L^2(\mathbb{T})$, where $\mathbb{T}$ is the 1-torus and can be identified with any of the sets $(0,1]$ or $[-\frac{1}{2},\frac{1}{2})$ or $[-1,-\frac{1}{2}) \cup [\frac{1}{2},1)$, ..., (all having measure one). We denote the *Fourier series* of f, 1-periodic and in $L^p(\mathbb{T})$, by

$$\sum_{k\in\mathbb{Z}} \langle f, e_k\rangle_{\mathbb{T}}\, e_k \sim f,$$

where $\langle f, e_k\rangle_{\mathbb{T}} = \int_{\mathbb{T}} f\,\overline{e_k}$, and $k \in \mathbb{Z}$.

The paper is organized as follows. In the next section, "Wavelets in $L^2(\mathbb{R})$", we introduce one-dimensional wavelets; in the section "Wavelets in Higher Dimensions", we discuss wavelets in higher-dimensional Euclidean spaces; in "Continuous Wavelets" we introduce continuous wavelets and some applications; finally, in the last section "Various Other Wavelet Topics, Applications and Conclusions", we discuss other applications and make some concluding remarks.

Wavelets in $L^2(\mathbb{R})$

We consider two sets of unitary operators on $L^2(\mathbb{R})$: the *translations* T_k, $k \in \mathbb{Z}$, defined by $(T_k f)(x) = f(x-k)$ and the (dyadic) *dilations* D_j, $j \in \mathbb{Z}$, defined by $(D_j f)(x) = 2^{j/2} f(2^j x)$. A *wavelet* (more precisely, a *dyadic wavelet*) is a function $\psi \in L^2(\mathbb{R})$ having the property that the system $\mathcal{W}_\psi = \{\psi_{j,k} = (D_j T_k)\psi : j,k \in \mathbb{Z}\}$ is an orthonormal (ON) basis of $L^2(\mathbb{R})$. Notice that the order of applying first translations and then dilations is important: $D_j T_k = T_{2^{-j}k} D_j$.

In this section we will explain why there are many wavelets enjoying a large number of useful

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properties which makes it plausible that various different types of functions (or signals) can be expressed efficiently by appropriate wavelet bases.

It is often stated that in 1910 Haar [19] exhibited a wavelet $\psi = \psi^H$ but that it took about seventy years before a large number of different wavelets appeared in the world of mathematics. This is really not the case. The *Haar wavelet* is defined by $\psi^H = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$, and it is not difficult to show that it is a wavelet (as will be shown below). Another simple example is the *Shannon wavelet*; it appeared in the 1940s, and we will explain in what sense it appeared. It is defined as $\psi^S = \check{\chi}_S$, where $S = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)$. A straightforward calculation shows that, if $\psi \in L^2(\mathbb{R})$, then, for $j, k \in \mathbb{Z}$,

$$\text{(1)} \qquad \hat{\psi}_{j,k}(\xi) = [2^{-j/2} e^{-2\pi i k 2^{-j}\xi}]\, \hat{\psi}(2^{-j}\xi).$$

Let us observe that the sets $2^j S$, $j \in \mathbb{Z}$, form a mutually disjoint covering of $\mathbb{R} \setminus \{0\}$. Moreover, since the system $\{e_{-k}\chi_S : k \in \mathbb{Z}\}$ is an ON basis of $L^2(S)$, the functions within the square bracket in (1), restricted to the set $2^j S$, form an ON basis of $L^2(2^j S)$ for each $j \in \mathbb{Z}$. It follows immediately that the set $\{\psi^S_{j,k} : j, k \in \mathbb{Z}\}$ is an ON basis of $L^2(\mathbb{R})$. This shows that ψ^S is a wavelet.

We mentioned that it is not difficult to show that the Haar function ψ^H is a wavelet. We will do this together with the presentation of a general method for constructing wavelets, the *Multiresolution Analysis* (MRA) method introduced by S. Mallat, with the help of R. Coifman and Y. Meyer [23], [26].

An MRA is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following conditions:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.
(ii) $V_{j+1} = D_1 V_j$ for all $j \in \mathbb{Z}$; that is, $f \in V_j$ iff $f(2\cdot) \in V_{j+1}$.
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(iv) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.
(v) There exists $\phi \in V_0$ such that $\{T_k\phi : k \in \mathbb{Z}\}$ is an ON basis of V_0.

The function ϕ described in (v) is called a *scaling* function of this MRA.

If $\{V_j : j \in \mathbb{Z}\}$ is an MRA, let W_j be the orthogonal complement of V_j within V_{j+1}. An immediate consequence of the above properties is that the spaces W_j, $j \in \mathbb{Z}$, are mutually orthogonal and their orthogonal direct sum $\bigoplus_{j\in\mathbb{Z}} W_j$ satisfies

$$\text{(2)} \qquad \bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}).$$

If there exists a function $\psi \in W_0$ such that $\{T_k\psi : k \in \mathbb{Z}\}$ is an ON basis of W_0, using the observation that $D_j W_0 = W_j$ for each $j \in \mathbb{Z}$ (an easy consequence of the MRA properties), we see that $\{\psi_{j,k} = D_j T_k \psi : k \in \mathbb{Z}\}$ is an ON basis of W_j. It follows from (2) that $\{\psi_{j,k} = D_j T_k \psi : j, k \in \mathbb{Z}\}$ is an ON basis of $L^2(\mathbb{R})$. Thus, ψ is a wavelet. In the case where $\phi = \chi_{[0,1)}$ and V_0 is the span of the ON system $\{T_k\phi : k \in \mathbb{Z}\}$, it is easy to check that $\{V_j = D_j V_0 : j \in \mathbb{Z}\}$ is an MRA. Moreover, it is easy to verify that $\{T_k\psi^H : k \in \mathbb{Z}\}$ is an ON basis of the space W_0 defined by $W_0 = V_0^\perp \subset V_1$. It follows that $\{D_j T_k \psi^H : j, k \in \mathbb{Z}\}$ is an ON basis of $L^2(\mathbb{R})$. This shows that the Haar function ψ^H is indeed a wavelet.

We leave it to the reader to verify that the Shannon wavelet ψ^S is an MRA wavelet as well. In fact, it corresponds to the scaling function $\phi(x) = \mathrm{sinc}(x) = \frac{\sin x\pi}{x\pi}$ (note: $\mathrm{sinc}(0) = 1$). This is a consequence of the fact that $(\mathrm{sinc})^\wedge(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi) = \hat{\phi}(\xi)$.

We point out that there is an important result involving the function sinc, namely, the following elementary theorem.

Theorem 1 (Whittaker-Shannon-Kotelnikov Sampling Theorem). *Let $f \in L^2(\mathbb{R})$ and $\mathrm{supp}\hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]$. Then*

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(x-k),$$

where the symmetric partial sums of this series converge in the L^2-norm, as well as absolutely and uniformly.

We will explain how this result is related to wavelets even though, when it was obtained the notion of wavelets had not yet appeared. The word *sampling* reflects the fact that the functions involved are completely determined if we know their values on the countable set $\mathbb{Z}$. The name *Shannon* is singled out because he is associated with many important aspects and applications of sampling.

Let $\phi \in L^2(\mathbb{R})$, ϕ not the zero function, and $\mathcal{T}_\phi = \{\phi_k = T_k\phi : k \in \mathbb{Z}\}$. Then $\mathcal{T}_\phi$ generates the closed space $V_\phi := \langle\phi\rangle = \overline{\mathrm{span}\{\phi_k : k \in \mathbb{Z}\}}$, that is, the closure of all finite linear combinations of the functions ϕ_k. This space is shift-invariant and is called the *Principal Shift-Invariant Space* (PSIS) generated by ϕ. In case $\phi = \mathrm{sinc}$, then $\mathcal{T}_\phi$ is an orthonormal system (recall that $(\mathrm{sinc})^\wedge(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi)$) and we have that $\sum_{k\in\mathbb{Z}} |f(k)|^2 < \infty$, where f is the function in Theorem 1. If $V_0 = \langle\mathrm{sinc}\rangle$, then the set $\{V_j = D_j V_0 : j \in \mathbb{Z}\}$ is an MRA and $\phi = \mathrm{sinc}$ is a scaling function for this MRA. For a general MRA with a scaling function ϕ, there is a bounded 1-periodic function m_0 known as a *low-pass filter* and an associated *high-pass filter*, $m_1(\xi) = e^{2\pi i\xi}\,\overline{m_0(\xi + \frac{1}{2})}$, which produce the so-called *two-scale equations*:

$$\text{(3)} \quad \hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi), \quad \hat{\psi}(2\xi) = m_1(\xi)\hat{\phi}(\xi).$$

In fact, these equations produce the desired wavelet ψ generated by the scaling function ϕ. In the special case we are considering, where $\phi = \text{sinc}$, the low-pass filter, when restricted to the interval $[-\frac{1}{2}, \frac{1}{2}]$, is the function $\chi_{[-\frac{1}{4},\frac{1}{4}]}$. One can indeed verify that, in this case, $\hat{\psi}(\xi) = e^{-i\pi\xi}\,\hat{\psi}^S(\xi) = e^{-i\pi\xi}\,\chi_S(\xi)$. Thus, the wavelet we obtain is essentially the Shannon wavelet, since the factor $e^{-i\pi\xi}$ is irrelevant to the orthonormality of the system.

Let us point out that the spaces V_0 and W_0, in general, are PSIS's, but they have an important difference with respect to the dilation operators we are considering: $V_j = D_j V_0$ is an increasing sequence of closed spaces as $j \to \infty$, while the spaces $W_j = D_j W_0$ are disjoint and satisfy (2).

The properties of shift-invariant spaces have many consequences in the theory of wavelets. If ϕ is not the zero function in $L^2(\mathbb{R})$, let $p_\phi(\xi) = \sum_{j\in\mathbb{Z}} |\hat{\phi}(\xi + j)|^2$ and consider the space $\mathcal{M}_\phi = L^2([0,1], p_\phi)$ of all 1-periodic functions m satisfying

$$\int_0^1 |m(\xi)|^2\, p_\phi(\xi)\, d\xi := \|m\|^2_{\mathcal{M}_\phi} < \infty.$$

It is easy to check that the mapping $J_\phi : \mathcal{M}_\phi \mapsto \langle\phi\rangle = V_\phi$ defined by $J_\phi m = (m\,\hat{\phi})^\vee$ is an isometry onto $V_\phi \subset L^2(\mathbb{R})$. That is, the two spaces $\mathcal{M}_\phi$ and V_ϕ are "essentially equivalent" via the map J_ϕ. It is natural, therefore, to ask how the properties of the weight p_ϕ correspond to the properties of the generating system $\mathcal{T}_\phi$. For example, the functions $e_{-k}(\xi) = e^{-2\pi ik\xi}$, $k \in \mathbb{Z}$, are mapped by the mapping J_ϕ onto the functions $\phi(\cdot - k)$, $k \in \mathbb{Z}$. Since the set $\{e_{-k}(\xi) : k \in \mathbb{Z}\}$ is algebraically linearly independent, so is the system $\mathcal{T}_\phi$. It follows immediately that

(i) *$\mathcal{T}_\phi$ is an ON system iff $p_\phi(\xi) = 1$ a.e.,*

(ii) *$p_\phi(\xi) > 0$ a.e. iff there exists an ON basis of V_ϕ of the form $\{T_k\psi : k \in \mathbb{Z}\}$ for some $\psi \in V_\phi$.*

In [20], [7] many properties of p_ϕ are shown to be equivalent to properties of V_ϕ or $\mathcal{T}_\phi$. For example, one can show the following:

(iii) *The system $\mathcal{T}_\phi$ is a frame for V_ϕ in the sense that we have constants $0 < A \le B < \infty$ for which*

$$A \sum_{k\in\mathbb{Z}} |\langle f, T_k\phi\rangle|^2 \le \langle f, f\rangle \le B \sum_{k\in\mathbb{Z}} |\langle f, T_k\phi\rangle|^2$$

for each $f \in V_\phi$ iff

$$A\,\chi_{\Omega_\phi}(\xi) \le p_\phi(\xi) \le B\,\chi_{\Omega_\phi}(\xi), \quad a.e.,$$

where $\Omega_\phi = \{\xi \in [0,1] : p_\phi(\xi) > 0\}$.

Note that, in general, when $\tilde{\phi} = (\frac{\chi_{\Omega_\phi}}{p_\phi}\,\hat{\phi})^\vee$, then $\tilde{\phi} \in V_\phi = V_{\tilde{\phi}}$ and $p_{\tilde{\phi}} = \chi_\Omega$ a.e. Moreover, $\mathcal{T}_{\tilde{\phi}}$ is a Parseval frame (PF) for V_ϕ; that is, it is a frame with $A = B = 1$. Slightly more general than ON MRA wavelets are PF MRA wavelets ψ where $\mathcal{T}_\psi$ is a PF for W_0 and there is a scaling function ϕ generating a PF for V_0. Examples include the function ψ given by $\hat{\psi} = \chi_{U\setminus\frac{1}{2}U}$, for $U \subset [-\frac{1}{2}, \frac{1}{2})$, where U has positive measure and $\frac{1}{2}U \subset U$.

One of the most celebrated contributions to the construction of MRA wavelets was made by I. Daubechies [1], [2], who used an ingenious construction to produce MRA wavelets that are compactly supported and can have high regularity and many vanishing moments, where the *kth moment* of ψ is defined as the integral $\int_{\mathbb{R}} x^k\,\psi(x)\,dx$. These wavelets are very useful for applications in numerical analysis and engineering, since the wavelet expansions of a piecewise smooth function converge very rapidly to the function. Specifically, suppose that $f \in C^R(\mathbb{R})$, the space of R times differentiable functions such that $\|f\|_{C^R} = \max\{\|f^{(s)}\|_\infty : s = 0, \dots, R\} < \infty$, and ψ is a compactly supported wavelet having at least R vanishing moments. Choose a bijection $\pi : \mathbb{N} \mapsto \mathbb{Z}\times\mathbb{Z}$ such that $|\langle f, \psi_{\pi(k)}\rangle| \ge |\langle f, \psi_{\pi(k+1)}\rangle|$ for all $k \ge 1$; that is, the wavelet coefficients of f are ordered in nonincreasing order of magnitude. Then one can show [21, Thm. 7.16] that

$$|\langle f, \psi_{\pi(m)}\rangle| \le C\,\|f\|_{C^R}\, m^{-(R+1/2)}, \tag{4}$$

where C is a constant independent of f and m. The implication of this is that relatively few coefficients are needed to get a good approximation of f. In fact, letting f_N be the best N-term nonlinear approximation of f, the *nonlinear approximation error* decays as

$$\|f - f_N\|^2_{L^2} \le C \sum_{m>N} |\langle f, \psi_{\pi(m)}\rangle|^2 \le C\,\|f\|_{C^R}\, N^{-2R}. \tag{5}$$

Remarkably, this result holds also if f is R times continuously differentiable up to finitely many jump discontinuities. That is, the wavelet approximation behaves as if the functions had no discontinuities. This behavior is very different from Fourier approximations, in which case the error rate is of the order $O(N^{-2})$. These results have extensions to higher dimensions (see further discussion in the section "Wavelets in Higher Dimensions").

Another important property of an MRA is that it enables an efficient algorithmic implementation of the wavelet decomposition. To explain this, suppose that ψ is a compactly supported MRA wavelet associated with a compactly supported scaling function ϕ. Because ϕ and ψ are in V_1, it follows that $D_{-1}\phi$ and $D_{-1}\psi$ are in V_0. Let us examine equations (3), which, in the time domain, can be written as

(6)
$$(D_{-1}\phi)(x) = \frac{1}{\sqrt{2}}\,\phi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} a_k\,\phi(x-k),$$
$$(D_{-1}\psi)(x) = \frac{1}{\sqrt{2}}\,\psi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} b_k\,\phi(x-k),$$

where only finitely many coefficients a_k and b_k are not zero. Since D_{-1} is unitary, from (6) we obtain

(7)
$$a_k = \langle D_{-1}\phi, T_k\phi\rangle = \langle D_{j-1}\phi, D_jT_k\phi\rangle,$$
$$b_k = \langle D_{-1}\psi, T_k\phi\rangle = \langle D_{j-1}\psi, D_jT_k\phi\rangle.$$

We see, therefore, that there are two ON bases for $V_j = W_{j-1}\oplus V_{j-1}$; they are $D_j\mathcal{T}_\phi$ and $D_{j-1}\mathcal{T}_\psi \cup D_{j-1}\mathcal{T}_\phi$. From equalities (7) we can calculate the matrices of the change of bases derived from these two ON bases (keeping in mind that the order of dilations and translations is important: $T_kD_1 = D_1T_{2k}$). For a given compactly supported $f_j \in V_j$, we apply the appropriate change of basis matrix to compute the orthogonal projection f_{j-1} of f_0 into V_{j-1} and obtain the "correction term" $e_{j-1} = f_j - f_{j-1} \in W_{j-1}$. Iterating this procedure, we see that arithmetic manipulations with finite sets of coefficients are all that are involved to compute $f_0 \in V_0$ and $e_i \in W_i$, $0 \le i \le j-1$, so that $f_j = f_0 + e_1 + \cdots + e_{j-1}$. Together with the excellent approximation properties of wavelets, this remarkably simple technique is one of the main reasons why engineers adopted wavelets (specifically, the so-called compactly supported biorthogonal wavelets [8]) in the design of JPEG2000, the industrial standard for image compression replacing the older Fourier-based JPEG standard.

Another property of MRA wavelets is that, considered as members of a subset of the unit sphere in $L^2(\mathbb{R})$, they form an arcwise connected set. In particular, it is not hard to show that there are continuous paths of wavelets. Suppose that ψ is an MRA wavelet and $\tilde{\psi}$ is a wavelet in the same MRA. Then one can easily show that $(\tilde{\psi})^\wedge = s\,\hat{\psi}$, where s is a 1-periodic unimodular function. In particular, we can choose a 1-periodic function θ for which $(\tilde{\psi})^\wedge = e^{i\theta}\,\hat{\psi}$ and set $s^t = e^{it\theta}$, for $t\in[0,1]$, to establish a continuous path $t \mapsto (s^t\hat{\psi})^\vee$ in $L^2(\mathbb{R})$ connecting ψ to $\tilde{\psi}$. A more complicated argument shows how ψ is continuously connected to the Haar wavelet [13]. Other related questions arise naturally. For example, are all ON wavelets connected? Are any two frame wavelets connected? The answer to this last question is yes, whereas the previous question is still an open problem.

Before moving to the topic of wavelets in higher dimensions, let us state that there are many other facts about one-dimensional wavelets we have not discussed. In particular, there are wavelets not arising from an MRA. Also, wavelets can be defined by replacing dyadic dilations with dilations by $r > 1$, where r need not be an integer. In the situation of nondyadic dilations, the construction of the orthonormal bases associated with the wavelet may require more than one generator; namely, if $r = \frac{p}{q} > 1$ and p, q are relatively prime, then $p - q$ generators are needed.

Wavelets in Higher Dimensions

Many of the concepts in the previous section extend naturally to n dimensions ($n \in \mathbb{N}$, $n > 1$), with $\mathbb{Z}$-translations replaced by $\mathbb{Z}^n$-translations and the dilation set $\{2^j : j \in \mathbb{Z}\}$ replaced by $\{u^j : j \in \mathbb{Z}\}$, where u is an $n \times n$ real matrix, each of whose eigenvalues has magnitude larger than one. Both for theoretical and practical purposes, however, it is convenient to focus our attention on PF wavelets rather than on ON wavelets. Thus, given the matrix u, we seek functions $\psi \in L^2(\mathbb{R}^n)$ for which the wavelet system

(8)
$$\{\psi_{j,k}(x) = (D_u^j\,T_k\,\psi)(x) = |\det u|^{j/2}\psi(u^jx - k),\ j\in\mathbb{Z}, k\in\mathbb{Z}^n\},$$

is a PF for $L^2(\mathbb{R}^n)$. That is,

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} |\langle f, (D_u^j\,T_k\,\psi\rangle|^2 = \|f\|^2_{L^2(\mathbb{R}^2)},$$

for all $f \in L^2(\mathbb{R}^n)$. For example, let $n = 2$ and $u = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. For $U \subset [-\frac{1}{2}, \frac{1}{2})^2$ where U has positive measure and $\frac{1}{2}U \subset U$, the function ψ defined by $\hat{\psi} = \chi_{U\setminus\frac{1}{2}U}$ is a PF MRA wavelet. On the other hand, to obtain an ON MRA wavelet system, we need to use three wavelet generators.

As in the one-dimensional case, we avoid multiple wavelet generators by restricting our attention to $n \times n$ integer matrices u with $|\det u| = 2$. For example, let u be chosen to be the *quincunx matrix* $q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, representing a counterclockwise rotation by $\pi/4$ multiplied by $\sqrt{2}$. We do encounter an "unexpected" fact if we try to find a Haar-type wavelet. There exists a set $D \subset \mathbb{R}^2$ such that χ_D is a scaling function for an MRA (defined as the obvious two-dimensional analogue of the one-dimensional MRA) that produces a Haar-type wavelet as the difference of two disjoint sets. Yet these sets are rather complicated fractal sets, known as the *twin dragons* (see Figure 1), as was observed in [14].

These observations indicate that the general construction of two-dimensional wavelets is significantly more complicated than the one-dimensional case. In particular, it is not known whether there exist continuous compactly supported ON wavelet analogues of the one-dimensional Daubechies wavelets associated with the dilation matrix q. However, it turns out that a rather simple change in the definition of the dilations in (8) produces much simpler constructions of Haar-type wavelets in dimension two.

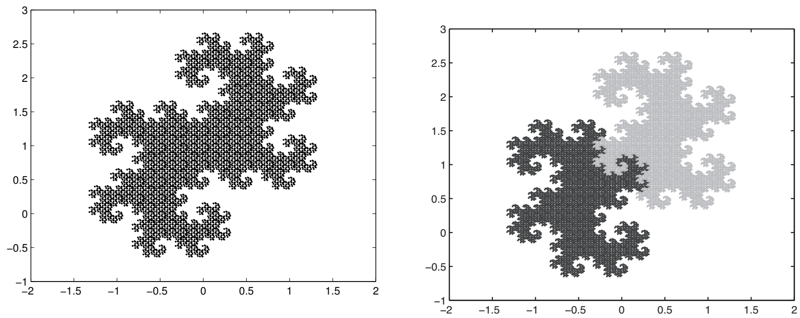

Figure 1. On the left is the fractal set known as the "Twin Dragon", whose characteristic function is the scaling function for the two-dimensional Haar-type wavelet associated with the dilation matrix q. On the right we see the support of the resulting wavelet ψ, whose values are 1 on the darker set, -1 on the lighter set, and 0 elsewhere.

Essentially, the idea consists of adding an additional set of dilations to the ones produced by the integer powers of the quincunx matrix. Specifically, let B be the group of the eight symmetries of the square, given by $B = \{b_j : j = 0, 1, \ldots, 7\}$, where $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $b_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $b_j = -b_{j-4}$, for $j = 4, \ldots, 7$. Let R_0 be the triangle with vertices $(0,0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $R_i = b_i R_0$ for $i = 0, \ldots, 7$ (see Figure 2). Let $\phi = 2^{3/4} \chi_{R_0}$ and V_0 be the closed linear span of $\{D_b T_k \phi : b \in B, k \in \mathbb{Z}^2\}$. Note that $|\det b| = 1$ for all $b \in B$ and V_0 is the subspace of $L^2(\mathbb{R}^2)$ of square integrable functions which are constant on each $\mathbb{Z}^2$-translate of the triangles R_i, $i = 0, \ldots, 7$. It is not difficult to show that there is a structure very similar to the classical MRA consisting of the spaces $V_j = D_q^j V_0$, $j \in \mathbb{Z}$. In fact, let us define the vector-valued function

$$\Phi = \begin{pmatrix} D_{b_0}\phi \\ \vdots \\ D_{b_7}\phi \end{pmatrix} = \begin{pmatrix} \phi^0 \\ \vdots \\ \phi^7 \end{pmatrix}.$$

Then $\{V_j : j \in \mathbb{Z}\}$ is an MRA with a vector-valued scaling function Φ. To derive a Haar-like wavelet, we observe that $R_0 = q^{-1}R_1 \cup q^{-1}(R_6 + \binom{0}{1})$ (see Figure 2). This equality implies that

$$\phi(x) = \phi^0(x) = \phi^1(x) + \phi^6(qx - \tbinom{0}{1}).$$

Applying D_{b_j}, $j = 1, \ldots, 7$, to the expression above, we obtain similar equalities for ϕ^j, $j = 1, \ldots, 7$. Now, let $\psi(x) = \phi^1(qx) - \phi^6(qx - \binom{0}{1})$. This function is the difference of appropriately normalized characteristic functions of disjoint triangles and leads indeed to the desired Haar-like wavelet. In fact, we can define the vector-valued function

$$\Psi = \begin{pmatrix} D_{b_0}\psi \\ \vdots \\ D_{b_7}\psi \end{pmatrix} = \begin{pmatrix} \psi^0 \\ \vdots \\ \psi^7 \end{pmatrix}$$

and observe that the system $\{D_q^j T_k \Psi : j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$ is an ON basis of $L^2(\mathbb{R}^2)$.

The construction above is representative of a much more general situation. For example, let $u = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$, the matrix of counterclockwise rotation by $\pi/3$, normalized to produce $\det u = 2$. Also in this case, there is a fractal Haar wavelet associated with this dilation matrix, but the introduction of an appropriate additional finite group of dilations allows one to derive simpler Haar-like wavelets similar to what we did above. See [18] for details.

In [18] we have shown that all this can be formalized by introducing the notion of *wavelet systems with composite dilations*, which are the systems of the form

$$\{D_a D_b T_k \psi : j \in \mathbb{Z}, a \in A, b \in B, k \in \mathbb{Z}^2\}, \tag{9}$$

where B is a group of matrices with determinant of absolute value 1 (as in the examples above) and A is a group of expanding matrices, in the sense that all eigenvalues have magnitude larger than one (as in the example above, where A is the group of the integer powers of the quincunx matrix). Many different groups of B-dilations have been considered in the literature, such as the crystallographic and shear groups, which are associated with very different properties. As we will see below, one special benefit of this framework is the ability to produce wavelet-like systems with geometric properties going far beyond traditional wavelets. For example, one can construct waveforms whose supports are highly anisotropic and that range not only over various scales and locations but also over various orientations, making these functions particularly useful in image processing applications.

As discussed at the beginning of the previous section, one of the most important properties of wavelets in $L^2(\mathbb{R})$ is their ability to provide

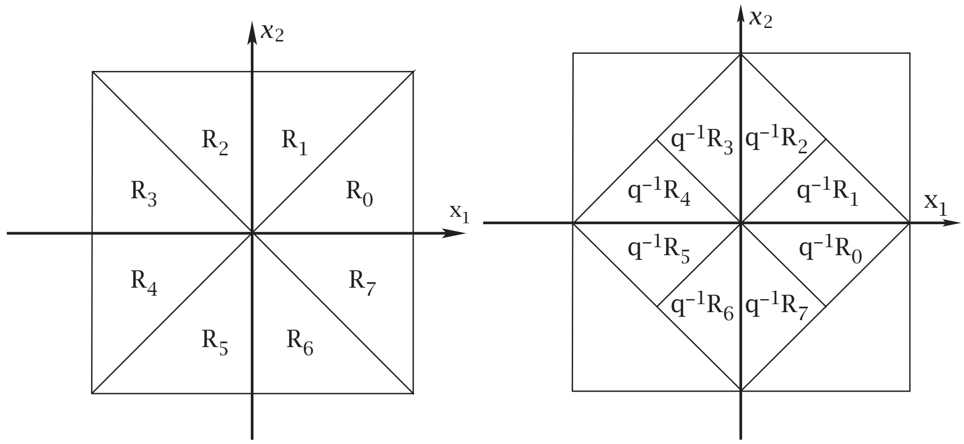

Figure 2. Example of construction of a wavelet system with composite dilations. On the left is illustrated the triangle R_0 and the triangles R_i, $i = 1, \dots, 7$, obtained as $b_i R_0$ (the matrices b_i are described in the text). On the right is illustrated the action of the inverse of the quincunx matrix q on the triangles R_i.

rapidly convergent approximations for piecewise smooth functions. This property implies that wavelet expansion is useful to *compress* functions efficiently since, as described by the nonlinear approximation error estimate (5), most of the L^2-norm of the function can be recovered from a relatively small number of expansion coefficients and not much information is lost by discarding the remaining ones. This idea is the basis for the construction of several wavelet-based data compression algorithms, such as JPEG2000 [27].

There is another important, perhaps less obvious, implication of the wavelet approximation properties which has to do with the classical problem of data denoising. Suppose that we want to recover a function f which is corrupted by zero-mean white Gaussian noise with variance σ^2. Let f_n denote the noisy function. In this case, Donoho and Johnstone [12] have shown that there is a very simple and very effective procedure for estimating f. This consists, essentially, of (i) computing the wavelet expansion of f_n, (ii) setting to zero the wavelet coefficients of f_n whose magnitudes are below a fixed value (which depends on σ), and (iii) computing an estimator of f as a reconstruction from the wavelet coefficients of f_n which have not been set to zero. It turns out that the performance of this procedure depends directly on the decay rate of the nonlinear approximation error estimate (5).

The above observations underline the fundamental importance of the approximation properties of one-dimensional wavelets for applications. Unfortunately, as will be discussed in more detail below, the situation is different in higher dimensions, where the standard multidimensional generalization of dyadic wavelets does not lead to the same type of approximation properties as in the one-dimensional case.

Let us restrict our attention to dimension $n = 2$ (the cases $n > 2$ are similar). Recall that, in dimension $n = 1$, to achieve the desired approximation properties of the wavelet expansions we required wavelets having compact support and sufficiently many vanishing moments. In dimension $n = 2$ we can easily construct a two-dimensional dyadic ON wavelet system starting from a one-dimensional MRA with scaling function ϕ_1 and wavelet ψ_1, as follows. We define three wavelets: $\psi^{(1)}(x_1, x_2) = \phi_1(x_1)\psi_1(x_2)$, $\psi^{(2)}(x_1, x_2) = \psi_1(x_1)\phi_1(x_2)$, $\psi^{(3)}(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$. Then the system

$$\{\psi^{(\ell)}_{j,k_1,k_2}(x_1, x_2) = 2^j \psi^{(\ell)}(2^j(x_1 - k_1), 2^j(x_2 - k_2)) : j, k_1, k_2 \in \mathbb{Z}, \ell = 1, 2, 3\}$$

is an ON basis for $L^2(\mathbb{R}^2)$. This is called a *separable* MRA wavelet basis. Clearly, we can choose ψ_1 and ϕ_1 having compact support and R vanishing moments. In this case, similar to the one-dimensional result, if $f \in C^R(\mathbb{R}^2)$ and $\|f\|_{C^R} < \infty$, then the mth largest wavelet coefficient in magnitude satisfies the estimate

$$|\langle f, \psi_{\pi(m)}\rangle| \le C\, \|f\|_{C^R}\, m^{-(R+1)/2}, \tag{10}$$

where C is a constant independent of f and m. This implies that, letting f_N be the best N-term nonlinear approximation of f, the *nonlinear approximation error* decays as

$$\|f - f_N\|^2_{L^2} \le C \sum_{m>N} |\langle f, \psi_{\pi(m)}\rangle|^2 \le C\, \|f\|^2_{C^R}\, N^{-R}.$$

However, while in the one-dimensional case the approximation properties of the wavelet expansion are not affected if one allows the function f to have a finite number of isolated singularities, the situation now is very different. Let us examine, for example, the wavelet approximation of the function $f = \chi_D$, where $D \subset \mathbb{R}^2$ is a compact set whose boundary has finite length L. Let us consider in particular the wavelet coefficients associated with the boundary of the region D. Since f is bounded, these wavelet coefficients have size $|\langle f, \psi^{(\ell)}_{j,k_1,k_2}\rangle| \sim 2^{-j}$. In addition, since the boundary of D has finite length and the wavelets have compact support, at each scale j there are about $L\,2^j$ wavelets with support overlapping the boundary of D. It follows from these observations that the Nth largest wavelet coefficient in this class is bounded by $C(L)N^{-1}$, and this implies that the nonlinear approximation error rate is only of the order $O(N^{-1})$. Hence, there are a large number of significant wavelet coefficients associated with the edge discontinuity of f, and this is limiting the wavelet approximation rate.

The reason for this limitation of two-dimensional separable MRA wavelet bases is the fact that their supports are isotropic (they are supported on a box of size $\sim 2^{-j} \times 2^{-j}$) so that there are "many" wavelets overlapping the edge singularity and producing significant expansion coefficients. To address this issue and produce better approximations of piecewise smooth multidimensional functions, one has to consider alternative multiscale systems which are more flexible at representing anisotropic features. Several constructions have been introduced, starting with the wedgelets [3] and ridgelets [5]. Among the most successful constructions proposed in the literature, the curvelets [6] and shearlets [15] achieve this additional flexibility by defining a collection of analyzing functions ranging not only over various scales and locations, like traditional wavelets, but also over various orientations and with highly anisotropic supports. As a result, these systems are able to produce (nonlinear) approximations of two-dimensional piecewise C^2 functions for which the nonlinear approximation error decays essentially like N^{-2}, that is, as if the functions had no discontinuities. To give a better insight into this approach and show how the wavelet machinery can be modified to obtain these types of systems, we will briefly describe the shearlet construction in dimension $n = 2$, which is closely related to the framework of wavelets with composite dilations, which we mentioned above. To keep the presentation self-contained, we will sketch only the main ideas and refer the reader to [15] [17] for more details.

For an appropriate function $\psi \in L^2(\mathbb{R}^2)$, a *system of shearlets* is a collection of functions of the form (9), where

$$a = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

Note that a is a dilation matrix whose integer powers produce anisotropic dilations and, more precisely, *parabolic scaling dilations*, since the dilation factors grow quadratically in one coordinate with respect to the other one; the *shear matrix* b is nonexpanding and, as we will see below, its integer powers control the orientations of the elements of the shearlet system. The generator ψ of the system is defined in the frequency domain as

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = w(\xi_1)\, v\left(\frac{\xi_2}{\xi_1}\right),$$

where $w, v \in C^\infty(\mathbb{R})$, $\operatorname{supp} w \subset [-\frac{1}{2}, \frac{1}{2}] \setminus [-\frac{1}{16}, \frac{1}{16}]$ and $\operatorname{supp} v \subset [-1, 1]$. Furthermore, it is possible to choose the functions w, v so that the corresponding system (9) is a PF (Parseval frame) for $L^2(\mathbb{R}^2)$. The geometrical properties of the shearlet system are more evident in the Fourier domain. In fact, a direct calculation gives that

$$\begin{aligned} \hat{\psi}_{j,\ell,k}(\xi) &:= (D_a^j D_b^\ell T_k \psi)^\wedge(\xi) \\ &= 2^{-3j/2} w(2^{-2j}\xi_1) v(2^j \tfrac{\xi_2}{\xi_1} - \ell) e^{2\pi i \xi a^{-j} b^{-\ell} k}, \end{aligned} \tag{12}$$

implying that the functions $\hat{\psi}_{j,\ell,k}$ are supported in the trapezoidal regions

$$\begin{aligned} \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \\ \xi_1 \in [-2^{2j-1}, 2^{2j-1}] \setminus [-2^{2j-4}, 2^{2j-4}], \\ |\frac{\xi_2}{\xi_1} - \ell 2^{-j}| \le 2^{-j}\}. \end{aligned}$$

The last expression shows that the frequency supports of the elements of the shearlet system are increasingly more elongated at fine scales (as $j \to \infty$) and orientable, with orientation controlled by the index ℓ (this is illustrated in Figure 3). These properties show that shearlets are much more flexible than "isotropic" wavelets and explain why shearlets can achieve better approximation properties for functions which are piecewise smooth. Similar properties hold for the curvelets and can be extended to higher dimensions.

As indicated above, shearlets and curvelets are only some of the methods introduced during the last decade to extend or generalize the wavelet approach. We also recall the construction of *bandelets* [25]; these systems achieve improved approximations for functions which are piecewise C^α (α may be larger than 2) by using an adaptive construction. We refer to the volume [24] for the description of several such systems.

Continuous Wavelets

In this section we examine *continuous wavelets* on $\mathbb{R}^n$. The general linear group $GL(n, \mathbb{R})$ of $n \times n$ invertible real matrices acts on $\mathbb{R}$ by linear transformations. The semidirect product G of

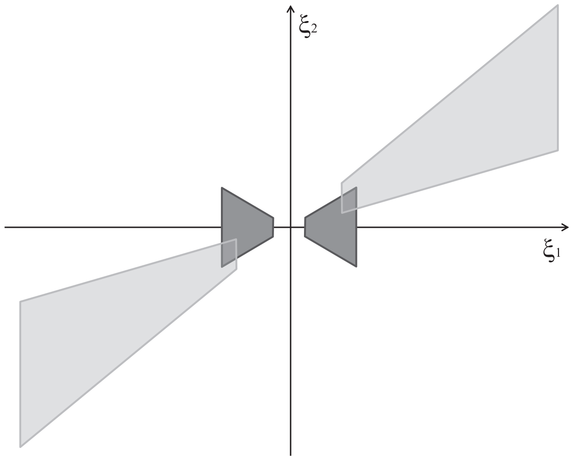

Figure 3. The frequency supports of the elements of the shearlet system are pairs of trapezoidal regions defined at various scales and orientations, dependent on j and ℓ respectively. The figure shows the frequency supports of two representative shearlet elements: the darker region corresponds to $j = 0, \ell = 0$ and the lighter region to $j = 1, \ell = 1$.

$GL(n,\mathbb{R})$ and $\mathbb{R}^n$ is called the *general affine group* on $\mathbb{R}^n$, since each invertible affine map on $\mathbb{R}^n$ has the form $(a,t)\cdot x = a(x+t) = ax+at$ for a unique $(a,t) \in G$. Thus, the group law $(a,t)\cdot(b,s) = (ab, b^{-1}t+s)$ for G corresponds to the composition of the associated affine maps and $(a,t)^{-1} = (a^{-1}, -a^{-1}t)$. We have a unitary representation τ of G acting on $L^2(\mathbb{R}^n)$ defined by

$$\begin{aligned}(\tau_{(a,t)}\psi)(x) &= |\det a|^{-\frac{1}{2}}\psi((a,t)^{-1}\cdot x)\\ &= |\det a|^{-\frac{1}{2}}\psi(a^{-1}x - t) := \psi_{a,t}(x).\end{aligned}$$

We then have that

$$(\tau_{(a,t)}\psi)^\wedge(\xi) = |\det a|^{1/2}\hat{\psi}(a^*\xi)\, e^{-2\pi i \xi\cdot at}.$$

For $\psi \in L^2(\mathbb{R}^n)$, the *continuous wavelet transform* W_ψ associated with ψ and G is defined by

(13)

$$\begin{aligned}(W_\psi f)(a,t) &= \langle f, \psi_{(a,t)}\rangle\\ &= |\det a|^{-\frac{1}{2}}\int_{\mathbb{R}^n} f(x)\,\overline{\psi(a^{-1}x-t)}\,dx,\end{aligned}$$

and it maps $f \in L^2(\mathbb{R}^n)$ into a space of functions on G. For D a closed subgroup of $GL(n,\mathbb{R})$, $H = \{(a,t) : a \in D, t \in \mathbb{R}^n\}$ is a closed subgroup of G and the left Haar measures on H are the product measures $d\lambda(a,t) = d\mu(a)\,dt$, where μ is a left Haar measure on D. In the special case where D is a discrete subgroup of $GL(n,\mathbb{R})$ such as $\{u^j : j \in \mathbb{Z}\}$, for some $u \in GL(n,\mathbb{R})$, we can take μ to be the counting measure on D.

We then seek conditions on D and μ for which restricting $W_\psi f$ to H gives an isometry from $L^2(\mathbb{R}^n)$ into $L^2(H, d\lambda)$ or, equivalently, we have a continuous reproducing formula

$$f = \int_H \langle f, \tau_{(a,t)}\psi\rangle\,\tau_{(a,t)}\psi\,d\lambda(a,t),$$

for each $f \in L^2(\mathbb{R}^n)$. As shown in [4], this holds if and only if

$$(14)\qquad \int_D |\hat{\psi}(a^*\xi)|^2\,d\mu(a) = 1, \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

in which case ψ is a *continuous wavelet* (with respect to D). In the special case $D = \{aI_n : a > 0\}$, where I_n is the $n\times n$ identity matrix and $d\mu(aI_n) = \frac{da}{a}$, the expression (14) reduces to the classical *Calderón condition*

$$(15)\qquad \int_0^\infty |\hat{\psi}(a\xi)|^2\,\frac{da}{a} = 1, \quad \text{for a.e. } \xi \in \mathbb{R}^n,$$

which is easy to satisfy. Indeed, when $n = 1$, the set

$$\{g \in L^2(\mathbb{R}) : \ c\,g \text{ satisfies (15) for some } c > 0\}$$

is dense in $L^2(\mathbb{R})$ (a similar observation holds for $n > 1$). When $D = \{u^j : j \in \mathbb{Z}\}$, for $u > 1$, one can show that if $\{\psi_{(u^j,k)} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ is a Parseval frame, then ψ is also a continuous wavelet with respect to D. Conversely, (15) is only necessary but not sufficient for $\{\psi_{(u^j,k)} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ to be a Parseval frame.

For $\psi \in L^2(\mathbb{R}^n)$, the *modified continuous wavelet transform* $\widetilde{W}_\psi$ associated with ψ and G is defined by

$$\begin{aligned}(16)\quad (\widetilde{W}_\psi f)(a,t) &= (W_\psi f)(a, a^{-1}t)\\ &= |\det a|^{-\frac{1}{2}}\int_{\mathbb{R}^n} f(x)\,\overline{\psi(a^{-1}(x-t))}\,dx,\end{aligned}$$

and it also maps $f \in L^2(\mathbb{R}^n)$ into functions on $G = D \times \mathbb{R}^n$. The modification arises from using the analyzing function $|\det a|^{-\frac{1}{2}}\psi(a^{-1}(x-t))$ rather than $|\det a|^{-\frac{1}{2}}\psi(a^{-1}x - t)$, and it simplifies the problem of estimating the asymptotic decay properties of the continuous wavelet transform.

Indeed, for $G = D \times \mathbb{R}^n$, where $D = \{aI_n : a > 0\}$, let us consider the modified continuous wavelet transform

(17)

$$\begin{aligned}(\widetilde{W}_\psi f)(a,t) &:= (\widetilde{W}_\psi f)(aI_n, t)\\ &= a^{-n/2}\int_{\mathbb{R}^n} f(x)\,\overline{\psi(a^{-1}(x-t))}\,dy.\end{aligned}$$

A fundamental property of this transform is its ability to characterize the local regularity of functions. For example, let f be a bounded function on $\mathbb{R}$ which is Hölder continuous at x_0, with exponent $\alpha \in (0,1]$; that is, there is $C > 0$ for which

$$|f(x_0+h) - f(x_0)| \le C|h|^\alpha.$$

Suppose that $\int_{\mathbb{R}}(1+|x|)|\psi(x)|\,dx < \infty$ and that $\hat{\psi}(0)=0$. Since the last condition implies that $\int_{\mathbb{R}}\psi(x)\,dx = 0$, then

$$(\widetilde{W}_\psi f)(a,t) = a^{-1/2}\int_{\mathbb{R}}(f(x)-f(x_0))\overline{\psi(a^{-1}(x-t))}dx.$$

Thus, using the Hölder continuity and a change of variables, we have

$$|(\widetilde{W}_\psi f)(a,t)| \le a^{-1/2}\int_{\mathbb{R}}|f(x)-f(x_0)|\,|\psi(a^{-1}(x-t))|\,dx \tag{18}$$

$$\le C\,a^{\alpha+1/2}\int_{\mathbb{R}}|y+a^{-1}(t-x_0)|^{\alpha}\,|\psi(y)|\,dy. \tag{19}$$

This shows that, at $t=x_0$, the continuous wavelet transform of f decays (at least) like $a^{\alpha+1/2}$, as $a\to 0$. Under slightly stronger conditions on ψ, one can show that the converse also holds, hence providing a characterization result. It is also possible to extend this analysis to discontinuous functions and even distributions. For example, if f has a jump discontinuity at x_0, then one can show that the continuous wavelet transform of f decays like $a^{1/2}$, as $a\to 0$, and similar properties hold in higher dimensions (cf. [22]).

While the continuous wavelet transform (17) is able to describe the local regularity of functions and distribution and detect the location of singularity points through its decay at fine scales, it does not provide additional information about the *geometry* of the set of singularities. In order to achieve this additional capability, one has to consider wavelet transforms associated with more general dilation groups.

For example, in dimension $n=2$, let M be the subgroup of $GL(2,\mathbb{R})$ of the matrices

$$\left\{ m_{a,s} = \begin{pmatrix} a & -a^{1/2}s \\ 0 & a^{1/2} \end{pmatrix} : a>0, s\in\mathbb{R} \right\},$$

and let us consider the corresponding generalized continuous wavelet transform

$$(\widetilde{W}_\psi f)(a,s,t) := (\widetilde{W}_\psi f)(m_{a,s},t) = a^{-3/4}\int_{\mathbb{R}^2} f(x)\,\overline{\psi(m_{a,s}^{-1}(x-t))}\,dx, \tag{20}$$

where $a>0$, $s\in\mathbb{R}$, and $t\in\mathbb{R}^2$. It is easy to verify that we have the factorization $m_{a,s} = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}$; that is, $m_{a,s}$ is the product of an anisotropic dilation matrix and a shear matrix. As a result, the analyzing function $\psi_{a,s,t} = a^{-3/4}\psi(m_{a,s}^{-1}(x-t))$ associated with this transform ranges over various scales, orientations, and locations, controlled by the variables a, s, t respectively. This is similar to the discrete shearlets in the section "Wavelets in Higher Dimensions". The transform $(\widetilde{W}_\psi f)(a,s,t)$ is called the *continuous shearlet transform* of f.

Thanks to the properties associated with dilation group M, the continuous shearlet transform is able to detect not only the location of singularity points through its decay at fine scales but also the geometric information of the singularity set. In particular, there is a general characterization of step discontinuities along 2D piecewise smooth curves, which can be summarized as follows [16]. Let $B=\chi_S$, where $S\subset\mathbb{R}^2$ and its boundary ∂S is a piecewise smooth curve.

- If $t\notin\partial S$, then $\widetilde{W}_\psi B(a,s,t)$ has rapid asymptotic decay, as $a\to 0$, for each $s\in\mathbb{R}$. That is,
$$\lim_{a\to 0} a^{-N}\widetilde{W}_\psi B(a,s,t) = 0, \quad \text{for all } N>0.$$
- If $t\in\partial S$ and ∂S is smooth near t, then $\widetilde{W}_\psi B(a,s,t)$ has rapid asymptotic decay, as $a\to 0$, for each $s\in\mathbb{R}$ unless $s=\tan\theta_0$ and $(\cos\theta_0,\sin\theta_0)$ is the normal orientation to ∂S at t. In this last case, $\widetilde{W}_\psi B(a,s_0,t)\sim a^{\frac{3}{4}}$, as $a\to 0$. That is,
$$\lim_{a\to 0} a^{-\frac{3}{4}}\widetilde{W}_\psi B(a,s,t) = C \ne 0.$$
- If t is a corner point of ∂S and $s=\tan\theta_0$ where $(\cos\theta_0,\sin\theta_0)$ is one of the normal orientations to ∂S at t, then $\widetilde{W}_\psi B(a,s_0,t)\sim a^{\frac{3}{4}}$, as $a\to 0$. For all other orientations, the asymptotic decay of $\widetilde{W}_\psi B(a,s,t)$ is faster and depends in a complicated way on the curvature of the boundary ∂S near t [16].

Similar results hold in higher dimensions and for other types of singularity sets.

Note that, in the definition of $\widetilde{W}_\psi B(a,s,t)$, we are taking the inner product of f with the *continuous* shearlets $T_t D_{m_{as}}^{-1}$. On the other hand, the discrete shearlets in "Wavelets in Higher Dimensions" involve the reverse order of operators. Despite this fact, the decay properties of the continuous shearlet transform are related to the approximation properties of discrete shearlets.

Various Other Wavelet Topics, Applications, and Conclusions

The number of researchers who have worked and are working on wavelets is very large. This field and its applications are enormous. We could not cover all the topics that are most important and interesting in such a short article. We make no claim that we have chosen to cover all "the most important" topics on wavelets. In this article we have defined wavelets to be elements of the Hilbert space $L^2(\mathbb{R}^n)$. In fact, we can extend the wavelet techniques we described to the Banach spaces

$L^p(\mathbb{R}^n)$, $p \neq 2$. As usual, the flavor for $1 < p < 2$ is very different from $p \geq 2$. The roles played by $\mathbb{R}$ and the 1-torus $\mathbb{T}$ were shown to extend to higher dimensions. It is well known that the harmonic analysis involving $(\mathbb{Z}, \mathbb{T})$ extends to the setting $(G, \widehat{G})$ where G is a locally compact Abelian group and $\widehat{G}$ its dual. Wavelet theory extends to $(G, \widehat{G})$ and other abstract settings.

Wavelets continue to stimulate and inspire active research going beyond the area of harmonic analysis, where they were originally introduced. While during the 1980s and 1990s most of wavelet theory was devoted to the construction of "nice" wavelet bases and their applications to denoising and compression, during the last decade wavelet research was focused more on the subject of approximations and the so-called *sparse* approximations. As we mentioned above, several generalizations and extensions of wavelets were introduced with the goal of providing improved approximation properties for special classes of functions where the more traditional wavelet approach is not as effective. This research has stimulated the investigation of redundant function systems (that is, frames which are not necessarily tight) and their applications using techniques coming not only from harmonic analysis but also from approximation theory and probability. The emerging area of compressed sensing, for example, can be seen as a method for achieving the same nonlinear approximation properties of wavelets and their generalization by using linear measurements defined according to a certain clever strategy.

Some fundamental ideas from wavelet theory, most notably the multiresolution analysis, have appeared in other forms in very different contexts. For example, the theory of diffusion wavelets [11] provides a method for the multiscale analysis of manifolds, graphs, and point clouds in Euclidean space. Rather than using dilations as in the classical wavelet theory, this approach uses "diffusion operators" acting on functions on the space. For example, let T be a diffusion operator (e.g., the heat operator) acting on a graph (the graph can be a discretization of a manifold). The study of the eigenfunctions and eigenvalues of T is known as spectral graph theory and can be viewed as a generalization of the theory of Fourier series on the torus. The main idea of diffusion wavelets is to compute dyadic powers of the operator T to establish a scale for performing multiresolution analysis on the graph. This approach has many useful applications, since it allows one to apply the advantages of multiresolution analysis to objects that can be modeled as graphs, such as chemical structures, social networks, etc.

In order to describe the more recent applications inspired by wavelets, we quote Coifman from [9, p. 159]:

> Over the last twenty years we have seen the introduction of adaptive computational analytic tools that enable flexible transcriptions of the physical world. These tools enable orchestration of signals into constituents (mathematical musical scores) and opened doors to a variety of digital implementations/applications in engineering and science. Of course I am referring to wavelet and various versions of computational Harmonic Analysis. The main concepts underlying these ideas involved the adaptation of Fourier analysis to changing geometries as well as multiscale structures of natural data. As such, these methodologies seem to be limited to analyze and process physical data alone. Remarkably, the last few years have seen an explosion of activity in machine learning, data analysis and search, implying that similar ideas and concepts, inspired by signal processing might carry as much power in the context of the orchestration of massive high dimensional data sets. This digital data, e.g., text documents, medical records, music, sensor data, financial data, etc., can be structured into geometries that result in new organizations of language and knowledge building. In these structures, the conventional hierarchical ontology building paradigm merges with a blend of Harmonic Analysis and combinatorial geometry. Conceptually these tools enable the integration of local association models into global structures in much the same way that calculus enables the recovery of a global function from a local linear model for its variation. As it turns out, such extensions of differential calculus into the digital data fields are now possible and open the door to the usage of mathematics similar in scope to the Newtonian revolution in the physical sciences. Specifically we see these tools as engendering the field of mathematical learning in which raw data viewed as clouds of points in high dimensional parameter space is organized geometrically much the same way as in our memory, simultaneously organizing and linking associated events, as well as building a descriptive language.

Let us illustrate three examples that will give the reader a more concrete idea of what Coifman asserts (see also [10]).

(a) In the oil exploration and mining industry, one needs to decide where to drill or mine to greatest advantage for finding oil, gas, copper, or other minerals. This involves an analysis of the composition and structure of the soil in a certain region. From such analysis one would find properties that optimize where these resources are most likely to be found.

(b) Suppose that we would like to decompose a large collection of books into subclasses of "similar" books, e.g., novels, histories, physics books, mathematical books, etc. Possibly we also want to assign "distances" between these subclasses. It is not unreasonable that the distributions of many particular words contained in each book can identify various kind of books so that they can be assigned in a specific subclass.

(c) In medical diagnostics, important information can be gleaned from the analysis of data obtained from radiological, histological, chemical tests, and this is important for arriving at an early detection of potentially dangerous tumors and other pathologies.

What is surprising is that the analysis of these very different types of data can be performed very efficiently using the type of mathematics based on the ideas presented in this paper. The practical impact of these ideas in applications is remarkable. For example, several hospitals have adopted medical diagnostics methods that were developed by Coifman and his group using wavelet-based methods.

In conclusion, we want to stress that wavelets is a huge field. Many have helped to create it. We want to state, however, that Yves Meyer has contributed and introduced many ideas that were most important in its creation. The material in the first chapter of [21] describes many constructions which are due to him and the ideas that paved the way for many of the topics (e.g., the MRA) we have presented in this paper.

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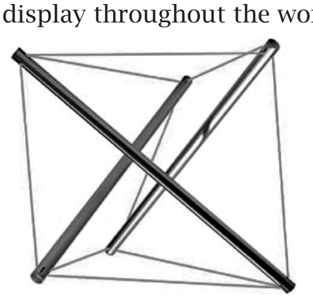
WHAT IS... a Tensegrity?

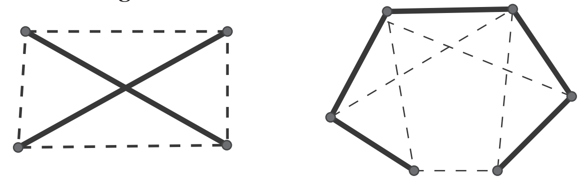
Robert Connelly

In the late 1940s, a young artist named Kenneth Snelson showed some of his string-and-stick sculptures to R. Buckminster Fuller. Out of this interaction the term "tensegrity" was born. These sculptures were quite surprising. The sticks were suspended in midair and supported by thin wires that were almost invisible. Fuller chose the word tensegrity to describe such structures because of their *tensional integrity*, and it has stuck throughout the years. The following shows a picture of one of the first small tensegrities that Snelson made with just three sticks (taken from his Web page). Since then, Snelson has made numerous other tensegrity sculptures, many quite large, and they are on display throughout the world.



But why do these structures hold up? Why are they rigid? What does it mean to be rigid? How can we model them mathematically?

The natural model is to define a tensegrity as a finite graph G whose vertices $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$, the *configuration*, are points in some Euclidean space $\mathbb{E}^d$ with two types of edges labeled *cables*, corresponding to the strings, and *struts*, corresponding to the sticks. The whole tensegrity is denoted as $G(\mathbf{p})$. The cables are constrained not to get longer, while the struts are constrained not to get shorter. A tensegrity $G(\mathbf{p})$ is defined to be *rigid* if, for any other configuration $\mathbf{q}$ in $\mathbb{E}^d$ sufficiently close to the configuration $\mathbf{p}$ and satisfying the cable and struts constraints of $G(\mathbf{p})$, $\mathbf{q}$ is rigidly congruent to $\mathbf{p}$. Some examples are below. Note that cables and struts can cross with no effect. A *flexible* tensegrity is one that is not rigid, and it necessarily has a smooth motion that is not a rigid motion of the whole tensegrity.



Struts are shown as solid line segments and cables as dashed line segments.

Static Rigidity

There are two principal methods to show rigidity of a tensegrity. Both methods involve the notion of a *stress* in the structure, which, for mathematical purposes, is just a scalar $\omega_{ij} = \omega_{ji}$ associated to every cable or strut connecting the ith vertex $\mathbf{p}_i$ to the jth vertex $\mathbf{p}_j$. We set $\omega_{ji} = 0$ when $\{i, j\}$ is not an edge of G. A stress is *proper* if $\omega_{ij} \geq 0$ for a cable and $\omega_{ij} \leq 0$ for a strut. The first method is derived from the linearization of the distance constraints. At each vertex $\mathbf{p}_i$ consider a force vector $\mathbf{F}_i$, the *equilibrium load*, such that these forces do not have any linear or angular momentum on the configuration. This means, summing over all i, that

$$\sum_i \mathbf{F}_i = 0 \quad \text{and} \quad \sum_i \mathbf{F}_i \wedge \mathbf{p}_i = 0.$$

In dimension 2 or 3 the wedge product can be replaced by the usual cross product. A tensegrity $G(\mathbf{p})$ is *statically* rigid if every equilibrium load $\mathbf{F} = (\mathbf{F}_1, \dots, \mathbf{F}_n)$ can be resolved by a proper stress

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$\omega = (\dots, \omega_{ij}, \dots)$ in the sense that, for each i, $\omega_{ij} = \omega_{ji}$ and

$$\mathbf{F}_i + \sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0.$$

This condition is then equivalent to there being a solution to a system of linear equations with linear inequality constraints on the stress ω, namely, a linear programming feasibility problem. In particular, if there are too few cables and struts, there will always be an equilibrium force that cannot be resolved by a proper stress. Suppose a tensegrity with n vertices and m cables and struts is statically rigid in dimension d. There are dn coordinate variables, and $d(d+1)/2$ is the dimension of the equilibrium loads (assuming the configuration does not lie in a lower-dimensional subspace). Then $m \geq dn - d(d+1)/2 + 1$. (The $+1$ arises due to inequality constraints.) In dimension 2, $m \geq 2n - 2$, and in dimension 3, $m \geq 3n - 5$. It is a basic result that, for a tensegrity, static rigidity is sufficient but not necessary for rigidity.

Prestress Stability

For example, the Snelson tensegrity in the first figure has $n = 6$ and $m = 12 < 3n - 5$, so it is not statically rigid. Nevertheless, it is rigid. Indeed, many of the tensegrities constructed by artists are statically underbraced. Although, from an engineering perspective, nonstatically rigid structures are often considered too soft to be dependable, we still want to detect and analyze such tensegrities. Basically, we can ensure that the tensegrity is rigid if the configuration minimizes an underlying energy function, which depends only on the lengths of the cables and struts, and if the minimizing configuration is unique up to rigid motions of the whole space. For example, static rigidity can be regarded as achieving a local minimum for reasonable choices of an energy function. Although the Snelson tensegrity and others like it have external loads that cannot be resolved at the given configuration, the tensegrity deforms slightly to resolve them. Indeed, even a statically rigid structure will deform under any nonzero load. When an energy function at the given configuration is at a local minimum due to the second derivative test, as in analysis, then the tensegrity is said to be *prestress stable*.

A *self-stress* for a tensegrity is a stress that resolves the zero load. Suppose that ω is a proper self stress for a tensegrity $G(\mathbf{p})$. Fix ω. Define a quadratic form $E(\mathbf{p})$ on the space of all configurations $\mathbf{p}$ in $\mathbb{E}^d$ by

$$E(\mathbf{p}) = \sum_{i<j} \omega_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2.$$

It turns out that the matrix of E is the Kronecker product of d copies of an n-by-n matrix Ω with identity, where n is the number of vertices of the graph G, since the energy decouples into the value on each coordinate separately. For $i \neq j$, the i, jth entry of Ω is $-\omega_{ij}$, while the diagonal entries are such that the row and column sums of Ω are 0.

The stress energy E and the associated stress matrix Ω have some interesting properties.

1) If ω is a proper self-stress for the tensegrity $G(\mathbf{p})$ and the vertices of $\mathbf{p}$ span a d-dimensional (affine) linear subspace of $\mathbb{E}^d$, then the kernel of the associated stress matrix Ω is at least $(d+1)$-dimensional.
2) If ω is a proper self-stress for the tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$, the kernel of the associated stress matrix Ω is $(d+1)$-dimensional, and $G(\mathbf{q})$ is another tensegrity for another configuration $\mathbf{q}$ with the same self stress ω, then $\mathbf{q}$ is an affine image of $\mathbf{p}$.
3) If ω is a proper self-stress for the tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$, the kernel of the associated stress matrix Ω is $(d+1)$-dimensional, Ω is positive semidefinite, and $G(\mathbf{q})$ is a tensegrity for a configuration $\mathbf{q}$ with cables no longer and struts no shorter, then $\mathbf{q}$ is an affine image of $\mathbf{p}$.

Note that Property 3 does not quite say that the tensegrity is rigid, just that it is rigid up to affine motions, and they must preserve the lengths of the edges $\{i, j\}$, where $\omega_{ij} \neq 0$. Nevertheless, this is close enough in many circumstances. For a tensegrity $G(\mathbf{p})$ with a self-stress ω, its *stressed edge directions* are lines through the origin determined by the vectors $\mathbf{p}_i - \mathbf{p}_j$, where $\omega_{ij} \neq 0$. Think of these lines as points in the projective space $\mathbb{RP}^{d-1}$ of dimension $d-1$. We say that the stressed edge directions of a tensegrity $G(\mathbf{p})$ with self-stress ω *lie on a conic at infinity* if, as points in $\mathbb{RP}^{d-1}$, they lie on a conic. Then it is a pleasant exercise to show that, if the stressed edge directions of a tensegrity $G(\mathbf{p})$ do not lie on a conic at infinity, then there is no affine map of the configuration $\mathbf{p}$ that satisfies the cable and strut constraints on the stressed edges other than a rigid motion of the whole space $\mathbb{E}^d$, a congruence.

Putting this all together, we get the following basic result.

Theorem 1. *Suppose that $G(\mathbf{p})$ is a tensegrity in $\mathbb{E}^d$ with n vertices and a proper self-stress ω with associated stress matrix Ω, and the following hold:*

1) *The rank of Ω is $n - d - 1$.*
2) *Ω is positive semidefinite.*
3) *The stressed directions of $G(\mathbf{p})$ do not lie on a conic at infinity.*

If $\mathbf{q}$ is another configuration in any $\mathbb{E}^D \supset \mathbb{E}^d$ satisfying the cable and strut constraints of $G(\mathbf{p})$, then $\mathbf{q}$ is congruent to $\mathbf{p}$.

I call any tensegrity that satisfies the three conditions of Theorem 1 *superstable*, and this sort of stability is a strong example of a structure being prestress stable. Quite a few of the tensegrities of Snelson and other artists are superstable. From a geometric point of view, the very strong conclusion of Theorem 1 about global reconfigurations is also interesting.

Global Rigidity

An edge of a tensegrity can also be regarded as both a cable and a strut, in which case it is called a *bar*; and if all the edges are bars, it is called a *bar framework*. In other words, bars are not permitted to change their lengths at all. From an engineering perspective, a tensegrity is just a bar framework where some bars can support only tension—these are the cables—and others can support only compression—these are the struts. A bar framework, or tensegrity, $G(\mathbf{p})$ is called *globally rigid* in $\mathbb{E}^d$ if any other configuration $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ in $\mathbb{E}^d$ that satisfies the constraints of $G(\mathbf{p})$ is congruent to $\mathbf{p}$. While Theorem 1 gives conditions for a tensegrity to be globally rigid in all dimensions, the following result (see [3]) is for bar frameworks in a fixed $\mathbb{E}^d$. A configuration $\mathbf{p}$ in $\mathbb{E}^d$ is *generic*, which means it is typical, if the coordinates of all of the points do not satisfy any nonzero polynomial with integer coefficients.

Theorem 2. *A bar framework $G(\mathbf{p})$ with $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ generic is globally rigid in $\mathbb{E}^d$ if and only if G is the complete graph (all vertices connected to all others) on $d+1$ or fewer vertices or it has a self-stress ω and corresponding stress matrix Ω of rank $n-d-1$.*

There has been a lot of activity applying the ideas here to packing, granular materials, and point location in computational geometry, for example. One instance is that packings of spherical disks in a polyhedral container can be regarded as tensegrities with all struts connecting centers of touching disks to each other and the boundary of the container. Some of the results described here can be found in [1], [2], [3].

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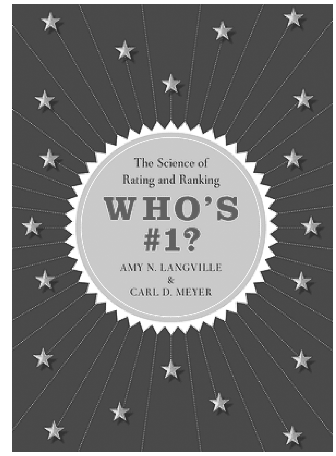
Who’s #1? The Science of Rating and Ranking

Reviewed by Andrew I. Dale



Who’s #1? The Science of Rating and Ranking
A. N. Langville and C. D. Meyer
Princeton University Press, February 2012
US$29.95, 266 pages
ISBN-13: 978-06911-542-20

The XXX Olympiad of the modern era has recently taken place, and readers will be fully aware of the considerable enthusiasm, verging at times on the perfervid, with which the outcomes of each day’s events were anticipated. Countries were ranked according to the number of medals won, but what would the ranking table be if the competing countries were ranked by population size or GDP[1] (Gross Domestic Product) or by the number of athletes in each team? According to an alternative medals table published in *The Guardian* (a British newspaper), the USA, which gained the greatest number of medals (104), has a GDP rank of 66 (Grenada is ranked 1), a population rank of 47 (again Grenada is 1), and a team size rank of 5 (Grenada 9).

Daily and constantly one is faced with having to choose one or more “things” (for want of a better word) from several alternatives. Such a choice may be made in an almost irrational way or it may be necessary to carefully weigh up the merits (including all relevant factors) of the different options. Our desire, however, may be seemingly simpler: to rank the alternatives according to some measure of “importance” or “bestness”, or we may even want to go so far as to choose, say, not just the “best” option, but a subset of options containing the “best” one (though this is really a question of *selection* rather than *ranking*). It is of course necessary in such a situation to know that there *exists* a best option and in such a case to be able to identify it. If we are sophisticated enough, we may well feel the need, after a statistical experiment, for statistical procedures to enlighten us to the probability (and possible consequences) of an erroneous choice.

The intended readership of *Who’s #1?*, Langville and Meyer claim in their preface, comprises “sports enthusiasts, social choice theorists, mathematicians, computer scientists, engineers, and college and high school teachers of classes such as linear algebra, optimization, mathematical modeling, graph theory” [p. xiii]. I suspect that the book will be of most interest to one who is already aware of some of the many different methods used in sports to rank football, basketball, etc., teams. To the results of which of these ranking methods can the fan attach most credence, and how do the different rankings compare? Langville and Meyer have presented a most useful work on this subject, exposing the reader to many different methods of ranking, discussing their origins with reference to much original work and exploring the ranked lists obtained by different methods.

While of course the choice of the method of ranking and the input data need careful thought, Langville and Meyer note the necessity for the making of occasional snap decisions: “Evolution has …rewarded those who make quick comparisons” [p. 3]: those who thought more slowly (or even those who were quick but incorrect) were no doubt removed from the gene pool by the swifter-thinking (or swifter-moving) predator!

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[1] *Before one places too much faith in the GDP, one may want to read a recent article in* Significance *in which Marks has argued cogently that the GDP “ignores all the important measures of human life”* [5, p. 27].

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Who's #1? is not a text on the *theory* of ranking methods (the authors wisely and correctly distinguish between *ranking* and *rating*). Rather, it is a discussion of various methods from the simple to the more complicated, with illustrations drawn from various fields (mainly American football), often with convenient summarizing sections. The second chapter, for example, deals with Kenneth Massey's original method for ranking college football teams, where the difference in ratings between two teams is used to predict the margin of victory in a future contest between these two. Things get more complicated and methods more refined as we go through the book, even the matter of ties being covered. The last chapter discusses not only the obtaining of but also the appropriate computer entering of data for analysis.

According to Langville and Meyer, the mathematical background required for an easy read consists mainly of elementary linear algebra and some optimization. They suggest that the book can be read for enjoyment even by those who lack these skills, who can implement the requisite techniques by the use of appropriate software, though those relying on such software must always be aware of possible pitfalls in the interpretation of results obtained in this way. Much use is expeditiously made here of the Perron-Frobenius theory to extract a unique ranking vector in a number of cases. The mathematics is by no means "heavy", and the interested reader will be able to follow Langville and Meyer without going into the deeper algebra.

While the methods discussed are, as stated, based on matrix analysis and optimization, Langville and Meyer do point out that there exist methods of more particular use in subjects such as game theory or statistics. In fact, there are techniques required in *Who's #1?* that I would classify as coming from statistics rather than linear algebra: for instance, on p. 10 Langville and Meyer write of the normal equations, minimum variance, and linear unbiased estimates, and the last of these certainly requires some knowledge of statistics for its proper understanding.

Between each two successive chapters there is a page headed "By the Numbers": for instance, we read on page 8 that the attendance at American college football games in 2010 was 50,000,000. The reader interested in such things might like to look at Watson and Yip [8] on the estimation of the sizes of crowds.

Flaws in the various methods are also indicated where necessary, and rankings obtained by different methods on the same data set are compared. For example, some methods for obtaining rankings use data different from others or perhaps weigh the same data differently (the Colley method, for instance, ignores game scores, but as Langville and Meyer ask, is this a strength or a weakness?).

In Chapter 10 Langville and Meyer discuss user preference rankings, the sort of thing that might arise in preference scores given to products advertised on the World Wide Web. But do respondents rank from the top down (five stars, four stars, etc.) or from the bottom up? This might well make a difference, as Luce [4, §2F] has suggested.

Chapter 16 is concerned with trying to discover which of the many methods discussed thus far is best. Here we find use of the well-known statistical tools *Kendall's tau* and *Spearman's footrule*. Useful here is the bipartite graph for a quick visual assessment of the comparison between methods.

There is some discussion of Arrow's Impossibility Theorem on the nonexistence of a perfect voting system. Perhaps more correctly, Arrow showed that the existence of a perfect voting system is incompatible with the requirement that certain reasonable criteria (e.g., collective rationality, nondictatorship, independence of irrelevant alternatives) be met. (This result was given in Arrow [1, Chap. V] as the general *possibility* theorem for social welfare functions.) Incidentally, Rescher has noted that "It is one of the tasks for the *social* order embracing different individuals to find a means of resolving a unified result out of a mass of potentially divergent individual preferences" [6, p. 99]. And though we may feel *preference* and *value* to be intimately connected, Rescher has suggested that "preference is too gross an instrument to capture the subtle nuances of value" [6, p. 109].

In their discussion of Keener's Method (Chapter 4) Langville and Meyer introduce a_{ij}, the value of some statistic or attribute that is thought to be a good basis for comparing two teams i and j. If S_{ij} is the number of points scored by i against j, then one's first choice might be

$$a_{ij} = \frac{S_{ij}}{S_{ij} + S_{ji}}.$$

They go on, however, to say that Keener indicated that a more appropriate measure would be

$$a_{ij} = \frac{S_{ij} + 1}{S_{ij} + S_{ji} + 2}$$

and add "The motivation for this is Laplace's rule of succession" [p. 31]. However reminiscent this formula might be of the latter rule, I find no such identification in Keener [3], and the resemblance is perhaps more apparent than real.

There are a few points to which one may well take exception. For example, on page 1 we read "because you scored in the 95th percentile": one who prefers the correct technical use of Francis Galton's carefully defined term may like to read

Senn's [7] comments. Further, there are a few errors in the index: for instance, we have "Saaty, T. L." and "Saaty, Thomas" (the references being to different pages); there are entries for "Saverin, Eduardo" and "Eduardo Saverin"; "Bhlmann" should be "Buhlmann" (also on p. 226, and here 1962 should be 1963). There are also some inconsistencies: "Kendall's tau" or "Kendall tau" and "Spearman's footrule" or "Spearman footrule"? Finally, in the bibliography, reference [47] is out of place, and the paper cited in reference [80] in fact occupies pages 155–169. Perhaps such things are bound to happen in a jointly written book with a computer-generated index.

The reader who is not *au fait* with American football will find enough discussion here of rankings in other areas to whet his/her appetite for more. Langville and Meyer refer frequently to the Netflix system for the online renting of films whose popularity has been rated. Recently a large prize was offered to anyone who could make a ten percent improvement in the company's own recommendation system: the question has undergone statistical investigation by Feuerverger et al. [2]. As an example of other recent work we mention the results of the British Film Industry magazine *Sight & Sound*'s recently published 2012 poll (846 respondents). For the first time since 1962 in the ten-yearly report, *Citizen Kane* was knocked out of first place by Alfred Hitchcock's 1958 film *Vertigo*.

When I started this book I knew very little about American football. I was little the wiser after finishing it, but I had an excellent understanding of various methods used in the obtaining of the ranking of teams and their interrelationships. Langville and Meyer are to be commended for this collection, and anyone who is more conversant with North American sports than I am will most certainly be stimulated by reading *Who's #1?*

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About the Cover

2013 marks the 125th anniversary of the American Mathematical Society. The first meeting of the Society was held on Thanksgiving Day in 1888. When it was founded, the Society had dedicated mathematicians as members, but it had no staff.

In 2013, the Society has approximately 30,000 individual members and 570 institutional members supported by a staff of over 200. This month's cover shows staff photos from the AMS's four facilities.

The Ann Arbor staff, numbering approximately 75, creates and maintains the Mathematical Reviews database and MathSciNet. Their job is a big one, staying abreast of the mathematical research literature in nearly 2,000 journals, plus research monographs and numerous refereed conference proceedings.

The AMS staff members in Pawtucket, Rhode Island, print and distribute all of the books published by the AMS, all of the AMS research journals that have print editions, and materials that are widely distributed as part of the Society's public awareness program.

The Washington Office staff is small in number, but has a very big footprint in its leadership of government relations and public policy. Its mission is advocacy and the advancement of the research and educational interests of the sciences and all areas of mathematics.

The Society's Providence headquarters has the largest and most diverse staff. Approximately 115 staff members cover meetings, professional programs, membership services, organizational infrastructure for finance and information technology, acquisition and prepress work for AMS publications, and support of Society governance. The original focus of the Society in 1888, meetings, has grown today to include support of eight sectional meetings, the national meeting as part of the January Joint Mathematics Meetings, and one international meeting every year. The publishing program encompasses thirteen journals and a broad book program.

The Society's activities and its staff have come a long way from the inspired beginning in 1888.

—Donald E. McClure
Executive Director

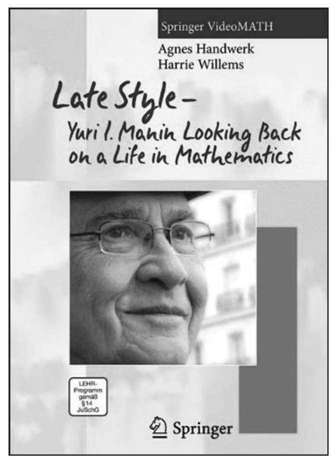
Late Style—Yuri I. Manin Looking Back on a Life in Mathematics

Reviewed by Gunther Cornelissen

Late Style—Yuri I. Manin Looking Back on a Life in Mathematics
Springer VideoMATH, 2012
US$39.95, Duration: 60 minutes
ISBN-13: 978-3642244827
Directed by Agnes Handwerk and Harrie Willems

The expression "Late Style" derives from Adorno's 1937 essay on the *Spätstil* of Beethoven. The phrase was used as the title of a course at Columbia University and later of a posthumous book by Edward Said, who explained that "Late Style" refers to "the way in which the work of some great artists and writers acquires a new idiom towards the end of their lives." Here one is viewing, however, "artistic lateness not [necessarily] as harmony and resolution, but as intransigence, difficulty, and unresolved contradiction." With the subtitle "Looking Back on a Life in Mathematics", it is also the chosen title for a documentary about Yuri Manin.

Yuri Manin's (1937–) variegated works span much of diophantine geometry (including a proof of the Mordell conjecture over function fields, and pioneering the use of modular symbols, p-adic automorphic forms, and noncommutative geometry), algebraic geometry (Fano varieties, cubic surfaces), and mathematical physics (he is the "M" in the ADHM construction of instantons, was at the cradle of quantum computing and quantum cohomology, etc.). He wrote influential textbooks on mathematical logic, homological algebra, cubic forms, gauge theory, and noncommutative



algebraic geometry. He supervised more than fifty Ph.D. students, many of whom went on to do great things. Apart from that, Manin published not only in the philosophy of mathematics and physics but also in socio-linguistics and Jungian psychology, and he wrote a few collections of poems and literary reviews. This nonmathematical part of his works has now been preserved by the publication of his book *Mathematics as Metaphor*; an English translation was published by the AMS, unfortunately leaving out the literary works. In a foreword to this book, Freeman Dyson identified Manin as the prototype of a "Bird", a scientist possessing overview and acting as a painter of large landscapes, by contrast calling himself a "Frog", more the sort of animal that digs around locally in the mud (thus downplaying his own tremendous contributions to science).

From the current documentary, we pick up two related viewpoints. Don Zagier, one of several mathematicians who appear in the documentary, sees the work of Manin guided by the following principle: if one detects a similar structure in seemingly different areas, one should search for an actual mathematical structure that bridges these fields. In the movie, Anatoly Vershik describes Manin as a product of the Soviet system, in which many great minds that in freedom would choose

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literature, poetry, or psychology were scared away from such pursuits by the need to embed their thoughts in Communist Party ideology. These minds turned to mathematics, where no such embedding was required, thereby enlarging the pool of different styles of doing mathematics. I very much view mathematics as possessing (or even being) a variety of different ways of speech. Some of these are philosophical or poetical, while others, equally valuable, are visual, formalistic, or computational. In this sense, Vershik's description reveals Moscow mathematics as a kaleidoscopic potpourri of unparalleled variation. In an interview in the *Berlin Intelligencer* (published for the International Congress of Mathematicians held in Berlin in 1998), Manin described one of his life values by the Renaissance term *varietà*, and we see a coherent reflection of this value in his work, which is always imbued with his own distinctively "literary" style. Manin reflects on all of this in a fascinating interview with David Eisenbud that can be seen on the website of the Simons Foundation's "Science Lives" project. Despite the fact that the interview is artificially cut up into fragments of ten minutes or less, I would recommend anyone to watch it.

My feelings about the movie under review are less positive. It is less contemplative, more biographical in the narrow sense, and also does not explain any actual mathematics—not that this is necessarily bad. In it, we see Manin go back to his early university education in Moscow, and we experience his rise to fame and his visits to Paris, where he interacted with Grothendieck. We also learn about his deliberate choice to sign the letter to release Alexander Esenin-Volpin from involuntary psychiatric imprisonment, which led to the withdrawal of Manin's teaching rights and travel permits. We see hints of the singularly lively scientific and cultural life in Moscow after the Stalin era, and we discover how Manin's life was affected by the destruction of the iron curtain, followed by his subsequent travels, and his settlement in Germany. After this, we switch to scenes from his earliest youth in Crimea, including the story of his father, who died young in the Second World War, and his mother, a victim of the "rootless cosmopolitanism" action, an anti-Semitic campaign during the Stalin era.

The movie describes the life of an outstanding and scrupulous scientist in a tumultuous environment. This scientist, Manin, is a man of subtle ideas, not an active dissident, though not without influence. One aspect of his personality, "inner freedom", is described in the movie by Beilinson as "light breathing", apparently referring to the story of the same name by Ivan Bunin. In the story, Bunin describes one of his female characters as "afraid of nothing—neither of ink stains on her fingers, nor of her face flushing; neither of her hair being untidy, nor of exposing her knee when running. Without any concerns and efforts, and somehow unnoticed, she came to possess all that set her so far apart from the rest of the schoolgirls in the last two years—gracefulness, elegance, litheness, and a clear brilliance in her eyes." But it seems "light breathing" does not create captivating movie characters. By contrast, when Grothendieck enters the scene—certainly a person of rather heavy breathing, if I may use this expression—he instantly becomes the dominant character even though physically absent.

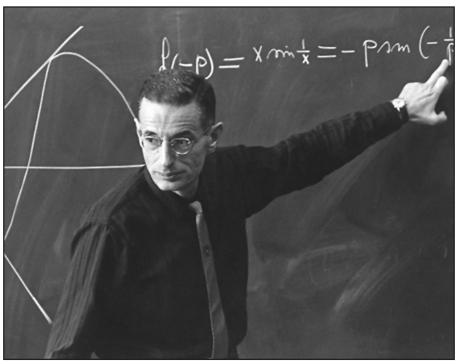
The first reaction to the movie by some of my nonmathematical friends was that they found it unclear why they should watch a documentary about *this* person. This shows the project largely failed in revealing Manin as what he is: one of the most interesting mathematicians of our time. My first impression was that the movie missed its target, but I now believe that, rather than missing the target, the target itself went missing: the depiction of Manin as a character in this rendition of his life is ineffective. This stands in great contrast with the wealth of scientific, artistic, and literary ideas that he has generated. Actually, Manin's own thoughts about his life make an interesting meta-subject.

Also, the technical execution of the (obviously low-budget) project leaves something to be desired, despite the interesting musical score of Tom Willems. For example, many transitions are images of Manin and his wife walking away from the camera in some outdoor scenery. Actually, the moviemakers seem to have a preference for depicting—preferably old—people from the back and for the use of backlighting and shaky handheld cameras, which at first I found amusing but, as the film progressed, irritated me more and more. The movie contains pictures of and short interviews with quite a few celebrated mathematicians; recognizing them is pleasant, at least for the In Crowd, but maybe it is just too many faces and too little discourse.

As I mentioned before, I believe Manin is exactly the person who could arrive at a very interesting reflection upon his own life, but such a reflection is only alluded to in the present movie. Maybe it was just too early, for what would the embodiment of such a reflection be? Surely Yuri Manin is not the Beethoven of mathematics; from the aforementioned book of Said, we extract musicologist Ruth Subotnik's description of Beethoven realizing, as an older man facing death, that his work proclaims that "no synthesis is conceivable." By contrast, everything seems to point to Manin's continuing interest to see mathematics, and the whole intellectual project of humanity, as one big superstructure to be revealed, analyzed, described, painted, sung about, and calculated in. Let us hope that he will be with us much longer to help us uncover many more untrodden paths—that his *Late Style* is still to come.

Ernst Snapper (1913–2011)

Joseph Buckley



Used with permission of James Snapper.

Ernst Snapper

Ernst Snapper was born in Groningen in the Netherlands in 1913. He came to the United States in 1938 to study at Princeton and received his Ph.D. in 1941 under the direction of J. H. M. Wedderburn. He remained at Princeton as an instructor until 1945, when he was appointed assistant professor at the University of Southern California. During his USC years, Snapper held two visiting appointments—at Princeton, 1949–1950, and at Harvard, 1953–1954. He was promoted to full professor in 1953.

In 1955 Snapper was named the Andrew Jackson Buckingham Professor of Mathematics at Miami University of Ohio. A few years later, in 1958, he accepted a position as professor at Indiana University. Then, in 1963, he moved to Dartmouth College, where he was named the Benjamin Pierce Cheney Professor of Mathematics in 1971, a position he held until his retirement in 1979.

Snapper's research made significant contributions in commutative algebra, algebraic geometry, cohomology of groups, character theory, and combinatorics.

An early sequence of papers extended the Steinitz field theory to completely primary rings using ideas from the work of Krull. During his visits at Princeton and Harvard, Snapper studied algebraic geometry and the homological and sheaf-theoretic methods of Serre and Grothendieck. Later he applied those methods in several important papers on the polynomial properties of the Euler characteristic associated with divisor classes of an irreducible normal projective variety. He continued using homological methods in a sequence of papers in which he extended the classical cohomology of groups to the cohomology of arbitrary permutation representations of finite groups. Snapper then applied these methods to obtain a classical result on Frobenius kernels.

In the area of combinatorial mathematics, Snapper extended de Bruijn's theory of the cycle index of a finite group to that of an arbitrary permutation representation. A subsequent paper coauthored with Arunas Rudvalis extended this cycle index to a generalized cycle index of a permutation representation paired with a class function. They then obtained the theorem of Frobenius that every simple character of the symmetric group is an integral linear combination of transitive permutation characters.

In 1971 Snapper coauthored the text *Metric Affine Geometry* (Academic Press) with Robert J. Troyer, a text for upper-level undergraduates and graduate students. It reflected their conviction that geometry for both high school and college students should be based on a foundation of linear algebra. Their book was inspired by the classic text *Geometric Algebra* by E. Artin.

Snapper was an outstanding lecturer much in demand by the MAA, and he taught in numerous summer institutes for both high school and college mathematics teachers. He also had a long-time interest in the foundations of mathematics. For his beautiful paper "The three crises in mathematics: Logicism, intuitionism and formalism", Snapper was awarded the Carl B. Allendoerfer Award from the MAA in 1980.

During his career Snapper mentored fifteen Ph.D. students. His students valued his lucid lectures, which frequently challenged them with open questions. Both at Indiana University and at Dartmouth College, Snapper and his wife, Ethel, hosted numerous gatherings for visiting speakers, which were well attended by faculty and graduate students. When graduate students were unable to go home during vacations, the Snappers invited the students to their home for holiday meals. Snapper loved outdoor activities, from sailing on the lakes near Bloomington to walking in the woods near Hanover and hiking in the White Mountains of New Hampshire, and he enjoyed many adventurous summer trips with his two sons.

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When 7,000 Mathematicians Come to Boston

Alexi Hoeft

Most of my nine-hour train ride to Boston had been spent meticulously putting together a minute-by-minute schedule for the four-day Joint Mathematics Meetings, sketching out every talk, event, and social gathering I was going to attend. That was my naïve attempt to get the most out of my first JMM, which also happened to be the largest mathematics meeting in history, with over seven thousand attendees. But apparently such a rigid schedule is no way to approach a conference of this size, and my plan deteriorated during the first half hour.

My inaugural session's first talk had had an intriguing title, but when the speaker was a no-show and we sat in silence until the next scheduled talk, my JMM experience was off to quite the anticlimactic start. I slipped out of the session and went into the neighboring room, listening in on a PDE session and flipping through the program trying to regroup and find a 9 a.m. talk. "Hodge Theory and the Netflix Problem" in the unusual-sounding session of Generalized Cohomology Theories in Engineering Practice eventually won out, and I was rewarded with a standout talk. Lek-Heng Lim of Chicago explained how algebraic topology provided tools the applied Netflix Problem needed to create a global ranking (of movies in the specific case of Netflix) based only on sparse local rankings given by individual users in a system where transitivity fails in general. Fully warmed up and with my schedule officially abandoned, I was ready to start my ad hoc approach to the JMM in earnest.

My string of good talks continued when intuition led me to the Alan Turing session on the afternoon of the first day. It was a standing-room-only crowd gathered to listen to an impressive lineup of speakers: Marvin Minsky, the father of Artificial Intelligence; Martin Davis, expert on Hilbert's 10th Problem; and Andrew Hodges, biographer of Alan Turing. I sat crowded in the back for the first talk and settled into the rhythm of the session, taking notes on Martin Davis's talk. My favorite anecdote that Davis shared was that von Neumann's letter of recommendation for Turing spoke only of the latter's work with almost periodic orbits—and nothing at all about his results in logic—perhaps, according to Davis, because "[von Neumann] never forgave logic for what Gödel did to him."

During the huge exodus for the break between talks, I wound my way around and over people to the front of the room and strategically chose a seat in the front row, the middle of three vacant seats by the projector; this was too good a session to be stuck in the back corner of the room. But I was in for a surprise. The seat to my right filled with a session newcomer, whom I filled in with the details of Martin Davis's talk. Then the seat to my left was filled by none other than Davis himself. What a feeling it was to be bumping elbows with Martin Davis for the last two talks of the session.

Andrew Hodges spoke next and, probably because I will be entering a Ph.D. program in the coming years, two tidbits caught my attention: Turing had gotten a professorship without a Ph.D., and Turing's surprisingly broad academic background seemed to have heavily contributed to his research success and creativity. The former was just a sign of the times and how much smaller and more intimate the academic community was during Turing's era. But I was curious to what extent that second point translated to today's math climate. My impression of what current grad students are encouraged to do seemed contrary to Turing's wide-ranging research scope. Instead of working toward seeing the big picture of mathematics and the connections between fields, the name of the game in grad school seems to be passing quals and then specializing as quickly as possible to get into a small enough niche for a Ph.D. thesis.

I wanted to hear what others in the room thought about the narrow focus of the modern math thesis versus the broad thinking of characters like Turing, so I brought up the topic for discussion in Hodges's Q&A session. The audience became

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animated and gave varied and pointed opinions. A few thought it was only the place of geniuses like Turing to research broadly in this way and "mere mortals" should stick to their narrow niches. Others were passionate about the importance of broad thinking as an ingredient in healthy mathematics research. Still others saw a broad research scope as possible only in the bygone days of a smaller, less developed, world of mathematics. One practical response I got was that it is mandatory to specialize quite narrowly for the Ph.D. thesis, after which it is a good idea to branch out and keep about three parallel research lines going.

I got a full range of answers during the prolonged discussion, which turned out to be essentially an in-person MathOverflow thread with a special JMM flavor. Even in the days after the session, the discussion continued, as attendees would find me and ask, "You were the one who asked that question in the Turing session, right?" and away we'd go on some conversation.

The meetings' keynote speaker was Berkeley's Edward Frenkel, who gave a three-part lecture series on the Langlands program. Frenkel's talent in highly technical, deep areas of mathematics and his having earned a Harvard Ph.D. in one year are both obvious sources of inspiration, particularly for a student such as me with a thesis on the horizon. But I sought him out more specifically for two other unusual qualities: the cross-disciplinary nature of his research and his position as an accessible public figure representing mathematics to the world at large.

After his third and final keynote address, I joined a conversational circle of Frenkel and three others in front of the stage, and it was quickly obvious they were experts familiar with his line of research. I nodded along, enjoying the debates and clarifications on small technical points and waited for them to get through all they wanted to say. I was by no means contributing mathematically to the conversation, but Frenkel had the decency to include me in his eye contact sequence as he elaborated on this topic and that.

When their conversation had petered out, it was my turn, and others in the circle lingered to hear what I had to say. We got to talking about what I saw as a main thread running through the meetings: that many of the best recent results were due in part to broad thinking and the cross-fertilization of disciplines. The JMM had been full of illustrations of this trend: Frenkel on moving between the worlds of number theory and the geometry of Riemann surfaces via results from curves over a finite field; Joseph Silverman on dynamics in number theory; Lek-Heng Lim using cohomology's Hodge Theory to attack the Netflix Problem; Larry Guth on applying the polynomial method to knock down old questions in combinatorics; Hee Oh ending up deep in dynamics and hyperbolic geometry and even applying the Poincaré Extension Theorem in her study of Apollonian circle packings; and Allen Knutson's unusual explanation of how he was decomposing varieties so they could be studied with combinatorial tools, which he memorably demonstrated with juggling patterns performed on stage throughout his talk.

The topic of our conversation in front of the stage then shifted to the all-too-neglected task of connecting the math world back to the public. Frenkel has done some unusual work producing a math-oriented artistic film, and he is also skilled at speaking accessibly to general math audiences and the general public alike. Somehow, math has lost the public's attention, and a festering disconnect between our world and theirs has resulted in the public thinking that mathematics is arithmetic and mathematicians are calculators. Wouldn't it be possible to fix that image by sharing with the nonmath populace some of our most beautiful results in a nontechnical way?

In fact, perhaps we should start within the math community and fix the frustrating trend that has led the seminars in many departments to become so technical as to be largely inaccessible to anyone who did not coauthor the research being presented. But, while there may be a need for improvement in our communication within and outside the world of mathematics, the JMM featured some refreshingly intelligible talks, so if those data points are a representative sample, then the future looks hopeful.

Other priceless moments rounded out my JMM experience, including chatting with the president of the AMS, Eric Friedlander, about topics ranging from Virginian accents to a women's postbaccalaureate program at Smith College; visiting MIT and Harvard in person for the first time; hosting my department's table at the grad school fair and speaking with prospective students; attending Princeton valedictorian John Pardon's talk on his solution to an old question posed by Gromov on knot distortion; reaching for a Springer Ricci flow book at the same time as another mathematician, who in turn told me what Richard Hamilton is like as a person; having a chance meeting in the Press Room with Princeton Ph.D.-turned-math-writer Dana Mackenzie; and wandering around Boston to get a taste for the city's intellectual personality.

I left the meetings several lessons the wiser. When given only four days to mingle with seven thousand mathematicians, it turns out the best way to make the most of that short time is by using snap judgment and spur-of-the-moment instinct. Keep an eye out for opportunities, grab them when they present themselves, and mingle constantly, since you can never guess the fascinating stories an ordinary-looking mathematician keeps locked up in his head.

AWM Institutes New Research Prizes for Early-Career Women

Ruth Charney

Here is your quiz for today: Name five women who have won major prizes for their mathematics research. Having trouble? That's not surprising. A recent study [1] by the Association for Women in Science (AWIS) shows that, in most scientific disciplines, the number of women receiving prizes for research is not consistent with the representation of women in the research community. There are multiple plausible explanations for this imbalance, and a variety of approaches have been proposed for addressing the issue. Clearly, one key strategy is to increase the pool of women considered for such prizes.

The Association for Women in Mathematics (AWM) is pleased to announce a new series of research prizes. These prizes will focus on women at the beginning stages of their careers. Early-career prizes (such as the AMS Centennial Fellowship) can significantly affect the career trajectory of recipients. Receiving such a prize may lead to promotions, job offers, speaking invitations, and new collaborations. In addition, prizes beget prizes. Highlighting outstanding work by young women increases their chances of being nominated for major prizes in the future. Thus, the new AWM prizes will serve both to celebrate the increasing contributions of women to mathematical research and to identify potential nominees for future prizes. The prize recipients will also provide inspiring role models for the next generation of women mathematicians.

The goal of this initiative is to institute four new research prizes. They will be organized by field, with four broadly defined areas: analysis, algebra/number theory, geometry/topology, and applied mathematics. Two of the four prizes will be awarded each year, with each topic appearing in alternate years. The awards will be aimed at early-career women, generally pretenure or within ten years of receiving their Ph.D. The first two of these prizes, the **AWM-Sadosky Research Prize in Analysis** and the **AWM-Microsoft Research Prize in Algebra and Number Theory**, have been funded, and the inaugural awards for these two prizes will take place at the Joint Mathematics Meetings in Baltimore, Maryland, in January 2014.

The AWM-Sadosky Research Prize in Analysis is named for Cora Sadosky (1940–2010), a former president of AWM. It is made possible by generous contributions from Cora's husband, Daniel J. Goldstein; daughter, Cora Sol Goldstein; and friends Judy and Paul S. Green. Sadosky was president of AWM from 1993 to 1995 and a long-time faculty member at Howard University. Born in Argentina in 1940, she received her doctoral degree in mathematics from the University of Chicago in 1965 and wrote over fifty papers in harmonic analysis and operator theory. A strong advocate for women in mathematics and active in promoting the greater participation of African-Americans in mathematics, Sadosky served as a member of the Human Rights Advisory Committee of the Mathematical Sciences Research Institute as well as the AMS Committee on Human Rights. She was also a member-at-large of the AMS Council (1995–1998) and served on two of the main policy committees of the AMS: the Committee on Science Policy (1996–1998) and the Committee on the Profession (1995–1996).

As described by Daniel and Cora Sol Goldstein and Judy Green, "Cora loved mathematics and hated discrimination. She saw mathematics as an object of supreme beauty and empowerment and was passionately committed to ensuring that everyone in society would have total access to

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mathematics. Cora believed firmly in equality and freedom, and while she often spoke specifically of 'the right of women to mathematics,' she fought with intelligence and perseverance to increase the opportunities and possibilities in mathematics for all people." AWM is proud to honor her memory through this new prize.

The AWM-Microsoft Research Prize in Algebra and Number Theory is made possible by a generous contribution from Microsoft Research through the efforts of Kristin Lauter, a principal researcher and head of the Cryptography Research Group at Microsoft Research. Lauter is a cofounder of the Women in Numbers Network, a research collaboration community for women in number theory. She received an AWM mentoring grant in 1999 to work with Jean-Pierre Serre at the Collège de France and was the 2008 recipient of the Selfridge Prize in Computational Number Theory. She currently serves on the AWM Executive Committee.

Lauter observed, "The high-tech industry has started to funnel significant resources to promote the advancement of women in science, recognizing that achieving gender parity in science is a national priority and is crucial for the future success of the industry. The AWM Research Prizes are an exciting way to recognize the work of young female researchers in mathematics, focus attention on their contributions, and encourage the next generation of research leaders. This type of recognition can be important to ensuring future success in their careers, both in being awarded tenure and in securing grants to support ongoing research. It is also important to help create positive role models for younger women considering a research career in mathematics." AWM is grateful to Kristin Lauter and Microsoft Research for the opportunity to create this prize.

These new prizes for early-career mathematicians complement AWM's existing prizes, which recognize the achievements of women in mathematics at many levels as well as efforts to encourage young women to enter the field. The following awards are presented annually.

- The Alice T. Schafer Prize, named after one of the original founders of AWM, recognizes an undergraduate who has demonstrated excellence in mathematics.

- The Gweneth Humphreys Award recognizes a mathematics teacher (female or male) who has encouraged female undergraduate students to pursue mathematical careers or the study of mathematics at the graduate level.

- The Louise Hay Award recognizes outstanding achievements in any area of mathematics education.

- The Ruth I. Michler Memorial Prize provides a fellowship for a recently tenured woman to spend a semester focusing on her research at Cornell University.

Three other AWM awards are associated with annual lecture series.

- The Noether Lecture honors women who have made fundamental and sustained contributions to the mathematical sciences. These lectures are presented at the Joint Mathematics Meetings each January.

- The Sonia Kovalevsky Lecture highlights significant contributions of women to applied or computational mathematics. It is awarded jointly by AWM and the Society for Industrial and Applied Mathematics (SIAM), and the lecture is given at the SIAM Annual Meeting.

- The Etta Z. Falconer Lecture, established in memory of Falconer's profound vision and accomplishments in enhancing the movement of minorities and women into scientific careers, honors women who have made distinguished contributions to the mathematical sciences or mathematics education. It is awarded jointly by AWM and the Mathematical Association of America (MAA), and the lecture is presented at MathFest each summer.

AWM is excited to add the two new prizes discussed above to this list.

So here is your next quiz question: Name five women who have the potential to win major prizes in the future. Are any of them eligible for the Sadosky Prize in Analysis or the Microsoft Prize in Algebra and Number Theory? AWM welcomes nominations for these prizes. The deadline for nominations for both prizes is **February 15, 2013**. Please visit `www.awm-math.org` for more details.

Inquiries about funding the remaining two prizes in geometry/topology and applied mathematics are also welcome! Please see `https://sites.google.com/site/awmmath/home/announcements/awmresearchprizes`.

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My Summer at *Scientific American*

Evelyn Lamb

Each year the AMS sponsors a fellow to participate in the Mass Media Fellowship program of the American Association for the Advancement of Science (AAAS). This program places science and mathematics graduate students in summer internships at media outlets. In this article the 2012 Fellow describes her experiences during her fellowship at *Scientific American*. For information about applying for the fellowship, see the "Mathematics Opportunities" section in this issue of the *Notices* or visit the website `http://www.ams.org/programs/ams-fellowships`. The application deadline is **January 15, 2013**.

Writing for a popular science news outlet was the perfect way for me to spend the summer after finishing my Ph.D. in math at Rice University. I walked into the *Scientific American* offices not really sure what to expect. By the end of my first day, I had successfully proposed an article about geometry labs to my editor, found a paper about "natural" selection in music that I wanted to cover, and was knee deep in freshman-year physics for an article about optics. It was a bit daunting but exhilarating as well. Over the course of the summer I wrote quite a bit about math, but I also wrote about a new battery design, the genome of a fungus used in making miso, prostate cancer screening, and much more.

In many ways, working for a news outlet is the polar opposite of doing math research. Some news pieces took several weeks to prepare, but most of them were written two or three days after I found out about them. I got many of my ideas from press releases, which we get a few days before the public. (In an attempt to level the playing field and prevent hastily written, inaccurate news stories, journals and universities give information to the press before it can be released to the public on the condition that the reporters will wait until the "embargo" to break the news.) When working on a piece that is under embargo, the goal is to publish the piece at the embargo time, if possible. This meant very quick assimilation of new material, often in a field where I had almost no experience. I found that my work often had an ebb and flow, with many assignments cropping up in a short time followed by a less busy spell, when I would take time to survey the territory, work on long-term projects, and poke around for fun new stories.

One of the challenges of the job was recognizing whether a particular study was newsworthy or not. Press releases and embargoes have a purpose, but they often serve to create a false sense of urgency about a paper. I am glad that my editor, while appreciating coverage of embargoed news, also encouraged me to look beyond press releases for news. Finding stories was often a challenge, not because science papers are rare, but because they are so abundant. It was like drinking out of a fire hose, and I worried about covering one story when I should have been covering something more important. My editor encouraged me to pitch as many stories as I wanted, and she accepted most of what I proposed.

Scientific American has a variety of different types of content, and I was able to experiment with many of them. In addition to traditional news stories, I wrote blog posts, podcasts (which I also voiced), images with extended captions, and slide shows. Some of my articles made it into the print magazine as well. I really enjoyed exploring the different media. Some stories made more sense in one format than another, and blogging even gave me a chance to write a few opinion pieces.

In late July political scientist Andrew Hacker wrote an op-ed piece called "Is Algebra Necessary?" for the *New York Times*, putting forth the idea that algebra education is an unnecessary waste of time and talent. My husband and I found the article ridiculous and spent much of the evening talking about it. The next day I wrote a blog post about it, and it took off. In addition to getting dozens of comments, "likes", and "shares", it led to an appearance on the PRI show "The Takeaway" later that week. Although I wish I could have been more eloquent on the radio, even at 7 a.m., the experience was positive. While all of that was happening,

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it finally dawned on me that I was now a member of the media. I would have had the exact same opinions of Hacker's piece whether I had been writing for a major news outlet or not. All of a sudden I was in a position not just to talk to my husband about it over dinner, but to have thousands of people read it as well. It's still a little surreal to realize that my opinions on math education were so widely read.

The idea that the public hates math and is not interested in learning about it is widespread in the mathematical community, but I think some of that attitude might be defensive on our part. We often have trouble communicating math, and we are all too willing to believe that others are hostile to us. Among the comments on my algebra education post were dozens from musicians, artists, real estate agents, and others who don't work in STEM careers telling me how glad they were to have learned algebra or how useful math had been in their lives. This and my other math articles showed me that there is a market for good, accessible math writing. People are inherently curious about numbers and shapes. Data visualization and mathematically inspired art are immensely popular. The challenge is to find ways to summarize dense research-level math for a general audience. It's difficult but rewarding when it works.

I found that writing about a subject I know so well was a double-edged sword. I had the mathematical background to assimilate information quickly, but I didn't always have perspective about what would be difficult for a nonmathematical audience. I would often work on a piece and not realize that a reference to the complex plane or even the use of the word "function" needed to be explained.

One issue I noticed was that members of my audience weren't nearly as comfortable not understanding new concepts as mathematicians are. A mathematician is likely to accept a statement along the lines of, "The researchers study mathematical objects called fluxtrons. Specifically, they are trying to determine whether all fluxtrons have the gromlick property." Most people would want a deep understanding of fluxtrons and the gromlick property before continuing, whereas mathematicians are often content to read the rest of the story, later going back to learn about fluxtrons and gromlicks if they decide they want to know more. As John von Neumann said, "In mathematics, you don't understand things. You just get used to them." There were several times when an editor would comment that I needed to define what something was, while I just assumed readers would know that the point wasn't the object but how it was used later. In those situations, my initial impulse was always to walk over to the editor's desk and explain to him or her what I meant. I had to keep reminding myself that I can't lean over each reader's shoulder to clarify a point. If an editor tells me something is confusing, my readers will need more clarification.

I have no doubt that my work this summer will have payoffs in my teaching. After seven years of being elbow deep in mathematicians, I was reminded of what concepts are hard for people. The algebra education post and its aftermath also gave me a lot of food for thought about the purpose of math education and what methods work or don't. I will probably still have problems getting through to some of my students, but now I have more ideas about how to get them interested.

My mathematical training helped me understand and assimilate new scientific material quickly. In the case of physics, much of the content itself is based on mathematics, and I could pick it up fairly easily. My first assignment, a story about optics, was full of vectors and derivatives. In other fields, almost all science papers contain descriptions of study methods and the statistical results of the experiments. I was able to dive in and see past the hype that sometimes comes in press releases. The fact that I can evaluate the statistics in a paper with ease is a boon to me in my efforts to present accurate information about the significance of scientific results.

Interviews ended up being some of the most rewarding parts of my job. I am a naturally phone-averse person, and picking up the phone for the first interview was a challenge. But researchers were generous with their time and eager to share their work with me. Several scientists told me that I had written the first article about their work to get all the details correct. My talks with mathematicians were especially nice. For my slide show about the geometry labs at the University of Maryland, University of Texas Pan-American, and University of Illinois Urbana-Champaign, I interviewed people in my own field about their remarkable undergraduate research and outreach programs. Talking with them about their passion for undergraduate research and how they started their work was truly inspiring. I would recommend that all mathematicians take some time to interview a colleague about his or her teaching, outreach, or what they love about math. You'll see them in a different light, and you might get excited about working on a new project.

In addition to research and writing, I found myself immersed in the social network of science writing. I had never used Twitter, but I started an account this summer in order to keep up with science and math news. I use Twitter to promote my own work and to point out good math content that I see. Content that I find on a math blog and tweet about often ends up getting spread to a wider group of people who are interested in science in general but perhaps don't know much about math. I also try to share content related to my stories as a way to give some extra information to those who are interested. When I created a slide show of art from Bridges, a math-art conference, I tweeted a link to an applet created by David Chappell, one of the featured artists. Using Twitter this way gave me the chance to give more content to those who were interested without overwhelming casual readers. I am still learning how to use social media effectively for math communication, but I think it has the potential to be a valuable resource.

I loved the variety of my assignments and the challenge of learning new science quickly, but I missed the process of discovery itself. In some ways, I feel like two different people. One of me wants to take any possible opportunity to educate people about all kinds of math, while the other one wants to discover new theorems. During the 2012–2013 academic year, since I do not have an academic position, I am continuing to write for *Scientific American* as a freelancer while also attending math conferences and working on my own research. In August 2013, when I begin my postdoc at the University of Utah, I plan to continue science and math communication for a general audience through blogging and freelance writing. Writing has joined research and teaching as an important and rewarding part of my career.

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"A Modest Proposal" Research Base

Max Warshauer

Jonathan Swift's original "A Modest Proposal" [1] was a thought-provoking satire designed to help deal with a food famine and its root causes of overpopulation. Schoenfeld's "Modest Proposal" [2] is designed to deal with a different kind of famine, namely, an intellectual famine brought about by not incorporating real sense making into the teaching of mathematics.

We extend Schoenfeld's proposal by suggesting that all students can do mathematics at a high level if properly challenged and engaged, and provide a collection of research questions to examine this proposition. Evidence abounds that students are having great difficulties in learning algebra [6]. We realize algebra is not a synonym for "sense making", and indeed algebra can be taught in a procedural way that does not encourage sense making at all. The reason that we focus on algebra as a proxy for sense making is that, when properly taught, algebraic reasoning provides students a powerful tool for explaining ideas rigorously and precisely.

Over the past six years a group of mathematics faculty has developed a curriculum, Math Explorations [13], designed to prepare all students for algebra by grade 8 or earlier, weaving in the type of sense making that Schoenfeld so elegantly describes. Students are engaged in exploring problems deeply and learn to explain simple ideas, such as adding and subtracting integers, with visual models. Variables and algebra are integrated throughout. Here we shall describe research questions that have arisen and propose a framework for further study.

Consider Suzuki's work on talent development in music [3], [4]. Suzuki observed that "any child is able to display highly superior abilities if only the correct methods are used in training." A method for teaching mathematics with the Suzuki method was described by the author previously in [5]. One of Suzuki's fundamental tenets is that, if one begins learning music when older, this may be a harder task than if one learns music as a child,

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since much time must be spent "unlearning" misconceptions. Suzuki observed [3],

> If they have learned the wrong fa by hearing it five thousand times, one must make them listen to the right fa six thousand or seven thousand times.

Reasons for suggesting that all students can succeed in algebra when younger if given the opportunity include:

1. The National Math Panel [6] observed:

> Teachers and developers of instructional materials sometimes assume that students need to be a certain age to learn certain mathematical ideas. However, a major research finding is that what is developmentally appropriate is largely contingent on prior opportunities to learn. Claims based on theories that children of particular ages cannot learn certain content because they are "too young", "not in the appropriate stage", or "not ready" have consistently been shown to be wrong.

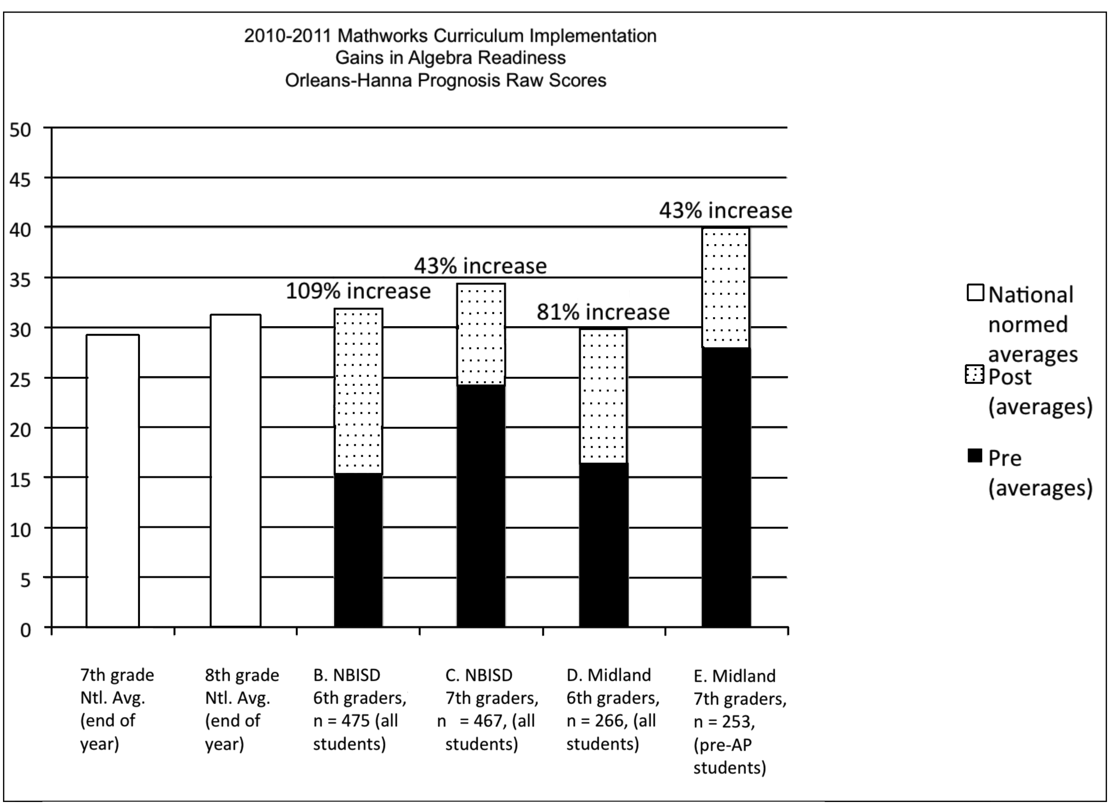
2. If waiting to learn music till later makes the task significantly harder, would not the same logic suggest that waiting to learn "algebra" till later might also be a much more challenging task?

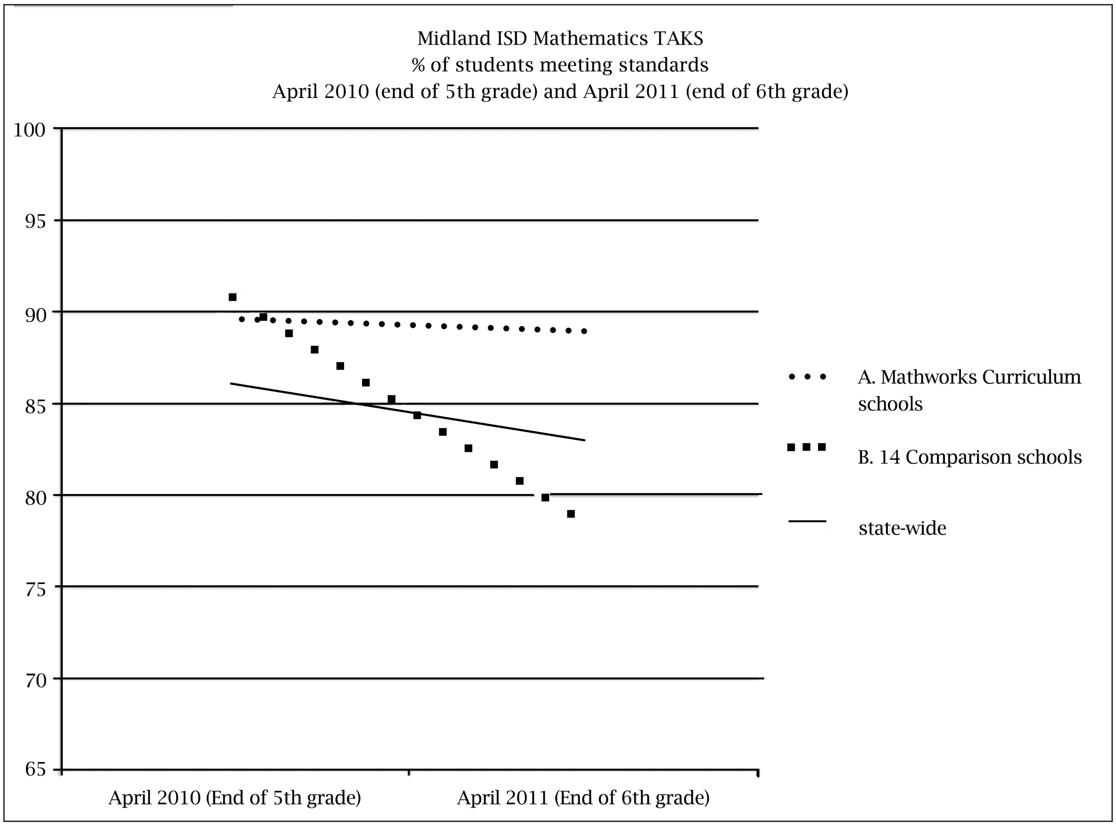
3. Is waiting till ninth grade to teach algebra part of the problem? The NCTM *Standards* [12] advocates weaving in algebraic ideas throughout elementary and middle school.

The mathematics curriculum presently used in many middle schools, as Schoenfeld observes, is having the exact effect that Suzuki describes—it is dulling the students' minds. They are being taught that mathematics is not about making sense, that math is a mysterious, incomprehensible topic. We hypothesize that not teaching sensemaking throughout middle school may be training students with attitudes about mathematics that are difficult to overcome. As Suzuki points out, once misconceptions are ingrained in young students, it can be very difficult to "undo" this poor training.

One argument that schools often make for not allowing all young students to begin algebra is that some students are not up to the challenge. Evidence from the Charles A. Dana Center suggests the opposite. In a study [8] of 378 schools and 26,363 students who took the Algebra I EOC in 1997 and 26,334 who took the Algebra I EOC in 1998, the following was a key finding:

> Teachers at schools with improving scores on the Algebra I EOC exam stated repeatedly that they believed that all their students could perform in algebra and successfully pass the exam—this critical issue was identified as an "Algebra for All" vision. In





contrast, the schools with declining scores reported that their students worked very hard but suggested that there should be mathematics classes less difficult than Algebra I available for some students.

By waiting to introduce some students to algebra, is it possible that we are sustaining and perpetuating an educational path that is destined to failure?

One measure of student ability to do algebra is the Orleans-Hanna Algebra Prognosis test. Each summer we give the Orleans-Hanna algebra prognosis test as a pre- and posttest in an intensive two-week Junior Summer Math Camp for rising fourth- to eighth-grade students. Students completing level 3 of our summer camp score on average above eighth-grade level. The results from pilots of the Math Explorations are even more convincing. After using this curriculum for one year, seventh-graders scored on average over eighth-grade level in their preparation for algebra, as measured by the Orleans-Hanna algebra prognosis test, and over 65 percent of the sixth-graders scored over eighth-grade average; see the chart below.

The group of students using Math Explorations also scored significantly higher than comparison groups on the state math assessment (TAKS). We compared four schools using Math Explorations to fourteen similar schools in the same district. All student groups entered the school year with approximately the same passing percentages on TAKS. By the end of their sixth-grade school year, students in the fourteen comparison schools experienced a sharp drop in passing percentages, while students using Math Explorations experienced no such decrease.

This brings us to the central question raised by this article: What are the keys to teaching mathematics with real sense making? We suggest a collection of research questions to address this fundamental question:

1. Is sense making as described by Schoenfeld correlated to providing students with algebra and algebraic models as a tool? Is there evidence that teaching sense making without algebra is more or less effective than teaching the same concepts with algebra?

2. Can all students succeed in algebra when young (grade 8 or earlier)? Several studies [9], [10], [11] develop a research base for early algebra and suggest that much more can be done.

3. Is the data from multiple pilots of Math Explorations the result of the curriculum, teaching, teacher training that includes observing math camps, school environment, faculty support, or a combination of these or other factors?

4. If students wait until the ninth grade to begin algebra, does this yield significantly worse results than beginning algebra in eighth grade (or younger)?

5. Is there a correlation between students who are placed into early algebra and the factors of ethnicity, race, or gender? Is "early algebra" for "some" one of the root causes of the gaps that we are seeking to close?

6. If we begin teaching students algebra in grades 6–8, will these students complete all of their math requirements earlier in high school and stop taking further mathematics? Do students who take algebra when younger do worse in subsequent courses and college than students who began taking algebra later? Longitudinal studies are needed.

These are testable questions. More work needs to be done. The key is to find schools that are willing to challenge all of their students. To convince a school to offer Algebra I for all students in grade 8 or earlier may require evidence of the kind we are proposing to gather, which is like a Catch-22.

However, there are encouraging developments. Programs such as the TEX Prep Program, begun by Manuel Berriosabal at University of Texas-San Antonio, have opened the door of opportunity for many Hispanic students. That program may be a fertile ground for research about young students succeeding in early algebra. The Math Explorations curriculum described above weaves in algebra for all students in grades 6–8 and is currently being tested at several sites. These are samples of what can be done. The challenge now will be to gather more evidence and to continue our efforts to engage all students in real "sense making", as suggested by Schoenfeld, while giving a rigorous definition of what this means in terms of introducing all students to algebra and higher-level mathematics.

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Experience with a Free Electronic Journal: *Theory and Applications of Categories*

Robert Rosebrugh

The word "free" in the title intentionally has two meanings: free of cost and in a state of liberty. The thesis of this note is that mathematics journals should be free and that this can be achieved much more easily than might be supposed. After describing how one free journal began and thrived, we make some observations about our thesis.

For me, the "serials crisis" began in earnest when I was hired at a small undergraduate university (Mount Allison) in 1981. A commitment to research was complicated by a university library containing none of the journals needed to keep up with my field. In 1981 much communication of new work was done through circulation of preprints and conference talks, but to access journals for archival material required a 125-mile journey to Dalhousie University in Halifax, Nova Scotia—a journey made many times. During the 1980s advocating improvements to library holdings was ineffective because of the severe cost pressure from the soaring prices of serials. Soon the Dalhousie library began to cut journals, and access to the literature became more fragile.

Luckily, in the 1980s many of us began to use email and Knuth's wonderful TEX. Change became possible. In January 1990 I emailed about twenty-five colleagues in category theory, proposing that we set up an electronic journal based on TEX and using email and ftp. The response to my proposal varied from enthusiasm through indifference to scathing. We didn't proceed. In retrospect, the enthusiasts and I were right. There was no technical obstacle then to starting a free electronic journal. The only significant technical difference between then and now is the explosion called the Web. Of course, there is an important nontechnical difference: electronic journals are common and accepted today.

Through the early 1990s the serials crisis became more acute as commercial journal publishers raised prices precipitously. In April 1994 some students showed me a Web browser called Mosaic. It was an unforgettable moment—the time was suddenly right to propose a journal again. At the same time, several other subject area electronic journals were starting up, notably the *Electronic Journal of Combinatorics* (EJC), the

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Electronic Journal of Differential Equations (EJDE), the *Electronic Transactions on Numerical Analysis* (ETNA), and the *New York Journal of Mathematics* (NYJM). Seeing them made me regret not going ahead in 1990. That June I wrote to a dozen colleagues among the leaders in my field. They were invited to become founding editors of a new electronic journal. Everyone agreed immediately. Something had changed. At our international conference in France in July we met and sketched plans for the journal. All of the editors agreed that the highest conventional editorial standards must be met, and all were determined that our venture would be highly respected.

The first dozen editors elected another group, and the journal started with twenty-four editors from twenty-one universities (including Buffalo, Cambridge, McGill, Milan, Paris, Rutgers, and Sydney) in ten countries. Our TEX expert, Michael Barr, soon produced a pleasing LATEX style for the journal that is based on the "article" style: we'd decided to look like a paper journal. Translating the widely varying author submissions to the house style was time intensive at first, but the journal does have a more consistent "look and feel" than many early electronic math journals. It was also decided that the content would be defined by dvi files. Largely, it still is.

In January of 1995 the new journal, *Theory and Applications of Categories* (TAC), was announced. Articles were submitted, and soon one was refereed and accepted. It was entitled "Oriented singular homology". The revised version was received on April 4, 1995, and the article was published on April 6. There were 9 articles and 178 pages that first year—just about right for the limited skills of and time available to the managing editor.

We also had to consider archiving. The National Library of Canada was then investigating storage of electronic serials, and TAC was in the first group it archived. Also, in June of 1995 the European Math Society set up what became the *European Mathematical Information Systems* (EMIS), and TAC joined. EMIS began actively archiving about a dozen journals, some of which were purely electronic. Many journals are now archived by EMIS and mirrored on five continents. The archive is secure.

Early growth in TAC's number of articles and pages was slow. By 2000 TAC was publishing fifteen articles, totalling 338 pages. In 2001 there was stronger growth. That volume had 23 papers of nearly 600 pages, and a special issue was also published. Over the last decade the number of articles published annually has been stable in the mid-twenties. Occasionally several more articles appear in a special issue. The 275 articles TAC has published in the last decade represent about a seventh of the items in the category theory MSC category, and that number is in the same range as the main commercial journal in the subject area.

For several reasons some of the classic papers in category theory were published where they are inaccessible even to those with good library resources. Some of these invaluable articles circulated as dog-eared Xerox copies. In 2002 TAC launched a unique service, a series called Reprints in Theory and Applications of Categories. Criteria for inclusion are strict, and the series now includes about twenty reprints of articles, theses, and out-of-print books.

The editorial board has become slightly larger, and as some editors have left there has been renewal. Happily, several younger colleagues have been elected. The current editors are even more international than the 1995 group. Thanks to the editors' vigilance, the quality goals we started with have always been met. For anyone who simply prints the published articles, TAC is not visibly distinguishable from a paper journal. However, the Web pages provide quick, easy, and free access. The front page remains very simple and elegant: pages load fast, and article content is one click away.

Given its satisfactory growth, editorial renewal, and strong support from the category theory community, future prospects for TAC are excellent.

Other free electronic journals that started at about the same time as TAC have had varying, but successful, experiences: some rapidly began to publish many dozens of articles; others, like TAC, grew more slowly. Over the last decade and a half, a number of other free electronic journals have started up across several mathematics subject areas. Some have even begun to provide low-cost print editions.

The AMS noted the growth of electronic journals through articles published in the *Notices*. Further, the AMS began to disseminate statistics from electronic journals, giving median refereeing and acceptance-to-publication times. Most important was the support for free electronic journals shown by Math Reviews and Zentralblatt in reviewing journals like TAC. It made them visibly respectable. Initiatives like MathSciNet, the online Zentralblatt, and EMIS have been essential to a change in culture among mathematicians: we now look first to high-quality electronic sources for information. Getting the new free journals noticed by commercial abstracting services is considerably harder. For example, TAC began to be covered by Thomson's Web of Science only recently and after considerable pressure. In my opinion such delays arise because free journals help to expose the lack of value added by those companies.

What does experience with TAC suggest about the future of the mathematical literature? If asked in 1995, I would have predicted that by now there would be several hundred electronic math journals and many fewer commercial journals. Instead, there are a few dozen free electronic journals and some others that provide variations of "open access". Despite occasional outrage from the mathematical and scientific community, there is little sign of commercial journals disappearing. Unfortunately, the publishers harvest more money than ever from taxpayer support for science. Nevertheless, progress over the last decade has led me from pessimism about the slowness of change to cautious optimism that the free journals are making a difference.

So, if your subject area of mathematics doesn't yet have a free electronic journal, *it's time to start!* All that's needed is a strong editorial board and *very little* time from a small team. Colleagues in my field often suppose that managing a subject area electronic journal is a heroic endeavor. The truth is very different. In 1995 a few hours were required to publish the average article in TAC. That was almost entirely due to TeX submissions that varied wildly in quality. Today, authors produce much better source code, and each article published in TAC requires well under an hour of the managing editor's time. Some articles do still go to our TeX experts for improvement, but we seldom see really awful code. Nevertheless, TAC has a visual standard that matches the best of the print journals. TAC's simple publication system might not scale to many hundreds of articles yearly, but most mathematics journals are not of that size. Moreover, the scripting required to automate article handling for much larger numbers is not difficult. Several free journals do it that way.

There is nothing to stop the mathematical community from freeing its literature electronically. *Let's just do it.*

Mathematics People

Chudnovsky and Spielman Selected as MacArthur Fellows

MARIA CHUDNOVSKY of Columbia University and DANIEL SPIELMAN of Yale University have been named MacArthur Fellows for 2012. Each Fellow receives a grant of US$500,000 in support, with no strings attached, over the next five years.

Chudnovsky was honored for her work on the classifications and properties of graphs. The prize citation reads, "Maria Chudnovsky is a mathematician who investigates the fundamental principles of graph theory. In mathematics, a graph is an abstraction that represents a set of similar things and the connections between them—e.g., cities and the roads connecting them, networks of friendship among people, websites and their links to other sites. When used to solve real-world problems, like efficient scheduling for an airline or package delivery service, graphs are usually so complex that it is not possible to determine whether testing all the possibilities individually will find the best solution in a practical time period. Chudnovsky explores classifications and properties of graphs that can serve as shortcuts to brute-force methods; showing that a specific graph belongs to a certain class often implies that it can be calculated relatively quickly. In an early breakthrough, Chudnovsky and colleagues proved a conjecture offered in the early 1960s, known as the 'Strong Perfect Graph Theorem', that identifies specific criteria required for a graph to fall into the 'perfect' class. Any perfect graph can be colored efficiently (i.e., no node is connected to another node of the same color), and graph coloring bears a direct relation to finding efficient solutions to problems such as allocating noninterfering radio frequencies in communication networks. Since this landmark accomplishment, Chudnovsky has continued to generate a series of important results in graph theory. Although her research is highly abstract, she is laying the conceptual foundations for deepening the connections between graph theory and other major branches of mathematics, such as linear programming, geometry, and complexity theory."

Spielman was honored for his work connecting theoretical and applied computing to resolve issues in code optimization theory, with implications for how we measure, predict, and regulate our environment and behavior. The prize citation reads, "Daniel Spielman is a theoretical computer scientist studying abstract questions that nonetheless affect the essential aspects of daily life in modern society—how we communicate and how we measure, predict, and regulate our environment and our behavior. Spielman's early research pursued aspects of coding theory, the mathematical basis for ensuring the reliability of electronic communications. When transferring digital information, sometimes even one incorrect bit can destroy the integrity of the data stream; adding verification 'codes' to the data helps to test its accuracy. Spielman and collaborators developed several families of codes that optimize speed and reliability, some approaching the theoretical limit of performance. One, an enhanced version of low-density parity checking, is now used to broadcast and decode high-definition television transmissions. In a separate line of research, Spielman resolved an enduring mystery of computer science: why a venerable algorithm for optimization (e.g., to compute the fastest route to the airport, picking up three friends along the way) usually works better in practice than theory would predict. He and a collaborator proved that small amounts of randomness convert worst-case conditions into problems that can be solved much faster. This finding holds enormous practical implications for a myriad of calculations in science, engineering, social science, business, and statistics that depend on the simplex algorithm or its derivatives. Most recently, Spielman has championed the application of linear algebra to solve optimization problems in graph theory. He and his colleagues have offered a new approach to maximizing the flow through unidirectional graphs, promising significant improvements in the speed of a wide range of applications, such as scheduling, operating system design, and DNA sequence analysis. Through these projects and fundamental insights into a host of other areas, such as complexity theory and spectral theory, Spielman is connecting theoretical and applied computer science in both intellectually and socially profound ways."

Maria Chudnovsky received her B.A. (1996) and M.Sc. (1999) degrees from Technion, Israel Institute of Technology, and an M.A. (2002) and a Ph.D. (2003) from Princeton University. She was affiliated with Princeton University from 2003 to 2006 and was a research fellow at the Clay Mathematics Institute from 2003 to 2008. At Columbia she is currently an associate professor in the Department of Industrial Engineering and Operations Research, with a courtesy appointment in the Department of Mathematics.

Daniel Spielman received his B.A. (1992) from Yale University and his Ph.D. (1995) from the Massachusetts Institute of Technology. He was affiliated with the Massachusetts Institute of Technology from 1996 to 2005. At Yale University he is currently Henry Ford II Professor of Computer Science, Mathematics, and Applied Mathematics in the Department of Computer Science.

The MacArthur Fellows Program awards unrestricted fellowships to talented individuals who have shown extraordinary originality and dedication in their creative pursuits and a marked capacity for self-direction. There are three criteria for selection of Fellows: exceptional creativity, promise for important future advances based on a track record of significant accomplishment, and potential for the fellowship to facilitate subsequent creative work.

—From a MacArthur Foundation announcement

Levin Awarded Knuth Prize

Leonid Levin of Boston University has been awarded the 2012 Knuth Prize "for his visionary research in complexity, cryptography, and information theory, including the discovery of NP-completeness." Working in the Soviet Union at the same time as Stephen Cook in the United States, Levin made his discovery of NP-completeness, the core concept of computational complexity. He also developed the theory of "average-case NP-completeness" for problems considered intractable on average. He has also done significant work on the PCP theorem, the cornerstone of the theory of computational hardness of approximation, and in cryptography theory.

The Knuth Prize, named in honor of Donald Knuth of Stanford University, is given every eighteen months by the Association for Computing Machinery (ACM) Special Interest Group on Algorithms and Computation Theory (SIGACT) and the Institute of Electrical and Electronics Engineers (IEEE) Technical Committee on the Mathematical Foundations of Computing. It carries a cash award of US$5,000.

—From an ACM announcement

Parimala Awarded Noether Lectureship

Raman Parimala of Emory University has been named the 2013 Noether Lecturer by the Association for Women in Mathematics (AWM). She was honored "for her fundamental work in algebra and algebraic geometry with significant contributions to the study of quadratic forms, hermitian forms, linear algebraic groups and Galois cohomology." She will deliver the lecture at the 2013 Joint Mathematics Meetings. The Noether Lectureship honors women who have made fundamental and sustained contributions to the mathematical sciences.

—From an AWM announcement

Australian Mathematical Society Prizes

The Australian Mathematical Society has awarded several major prizes. Anthony Henderson of the University of Sydney and Stephen Keith of the Macquarie Group have been awarded the Australian Mathematical Society Medal for 2012. The medal is given to a Society member or members under the age of forty for distinguished research in the mathematical sciences.

Ross Street of Macquarie University and Neil Trudinger of the Australian National University were awarded the 2012 George Szekeres Medal for outstanding contributions to the mathematical sciences in the fifteen years prior to the year of the award.

Akshay Venkatesh of Stanford University has been awarded the Mahler Lectureship for 2013. The lectureship is awarded every two years to a distinguished mathematician who preferably works in an area of mathematics associated with the work of Kurt Mahler.

Imam Tashdid ul Alam of the Australian National University received the Bernhard Neumann Prize for an outstanding talk presented by a student at the annual meeting of the Australian Mathematical Society.

The Australia and New Zealand Industrial and Applied Mathematics Division (ANZIAM) has awarded its ANZIAM Medal to Robert McKibbin of Massey University for outstanding achievement in applied and industrial mathematics in Australia, and Matthew Simpson of Queensland University received the ANZIAM J. H. Michell Medal for outstanding new researchers.

—From Australian Mathematical Society announcements

AAAS Elects New Members

The American Academy of Arts and Sciences (AAAS) has chosen 202 new fellows and 17 foreign honorary members for 2012. Following are the names and affiliations of the new members who work in the mathematical sciences or whose work involves considerable mathematics: Bonnie Berger, Massachusetts Institute of Technology; Joan S. L. Birman, Barnard College, Columbia University; Russel E. Caflisch, University of California Los Angeles; Thomas M. Liggett, University of California Los Angeles; Bao Châu Ngô, University of Chicago; Judea Pearl, University of California Los Angeles; Bjorn M. Poonen, Massachusetts Institute of Technology; Steven H. Strogatz, Cornell University; and Richard L. Taylor, Institute for Advanced Study. Elected as a foreign honorary member was Louis Boutet de Monvel, Université Pierre et Marie Curie, Paris, France.

—From an AAAS announcement

Royal Society of Canada Elections

The Royal Society of Canada has elected two new fellows who work in the mathematical sciences. Anne Condon of the University of British Columbia is a researcher in computational complexity theory and algorithms. Jeffrey S. Rosenthal of the University of Toronto has made "profound and deep contributions to probability and statistics, including highly original and influential results on the mathematical analysis of Markov chain Monte Carlo methods." He is the author of the popular book *Struck by Lightning*.

—From a Royal Society announcement

Mathematics Opportunities

AMS-Simons Travel Grants Program

Starting February 1, 2013, the AMS will begin accepting applications for the AMS-Simons Travel Grants program, with support from the Simons Foundation. Each grant provides an early-career mathematician with US$2,000 per year for two years to reimburse travel expenses related to research. Sixty new awards will be made in 2013.

Applications will be accepted starting February 1 through `www.mathprograms.org`. The deadline for 2013 applications is **March 31, 2013**.

To apply, applicants must be located in the United States or be U.S. citizens. For complete details of eligibility and application instructions, visit: `www.ams.org/programs/travel-grants/AMS-SimonsTG` or contact Steven Ferrucci, email: `ams-simons@ams.org`, telephone: 800-321-4267, ext. 4113.

—AMS announcement

Proposal Due Dates at the DMS

The Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) has a number of programs in support of mathematical sciences research and education. Listed below are some of the programs and their proposal due dates for the year 2013. Please refer to the program announcement or contact the program director for more information.

January 7, 2013 (full proposal): Innovation Corps Sites Program

January 10, 2013 (full proposal): Algorithms for Threat Detection (ATD)

January 24, 2013 (full proposal): Major Research Instrumentation Program

January 30, 2013 (full proposal): Secure and Trustworthy Cyberspace

January 31, 2013 (full proposal): Expeditions in Training, Research, and Education for Mathematics and Statistics through Quantitative Explorations of Data

February 4, 2013 (full proposal): Research Coordination Networks

March 15, 2013 (full proposal): Innovation Corps Teams Program

May 24, 2013 (full proposal): Research Experiences for Undergraduates (REU), Antarctica Program

June 4, 2013 (full proposal): Mentoring through Critical Transition Points in the Mathematical Sciences (MCTP)

June 4, 2013 (full proposal): Research Training Groups in the Mathematical Sciences

June 15, 2013 (full proposal): Workforce Program in the Mathematical Sciences

June 17, 2013 (full proposal): Innovation Corps Teams Program

July 1, 2013 (full proposal): Innovation Corps Sites Program

July 24, 2013 (full proposal): Faculty Early Career Development (CAREER) Program

August 20, 2013 (full proposal): International Research Experiences for Students (IRES)

August 28, 2013 (full proposal): Research Experiences for Undergraduates (REU)

September 16, 2012 (full proposal): Innovation Corps Teams Program

September 30, 2013 (full proposal): Secure and Trustworthy Cyberspace

October 1, 2013 (full proposal): Analysis; Combinatorics; Foundations

October 4, 2013 (letter of intent): ADVANCE: Increasing the Participation and Advancement of Women in Academic Science and Engineering Careers

October 11, 2013 (full proposal): Algebra and Number Theory

October 16, 2013 (full proposal): Mathematical Sciences Postdoctoral Research Fellowships

For further information see the website `http://www.nsf.gov/funding/pgm_list.jsp?org=DMS&ord=date`. The mailing address is Division of Mathematical Sciences, National Science Foundation, Room 1025, 4201 Wilson Boulevard, Arlington, VA 22230. The telephone number is 703-292-5111.

—From the DMS website

NSF Major Research Instrumentation Program

The National Science Foundation (NSF) Major Research Instrumentation (MRI) program seeks to increase access to shared scientific and engineering instruments for research and research training in institutions of higher education, museums, science centers, and not-for-profit organizations in the United States. This program especially seeks to improve the quality and expand the scope of research and research training in science and engineering by providing shared instrumentation that fosters the integration of research and education in research-intensive learning environments. Proposals must be for either acquisition or development of a single instrument or for equipment that, when combined, serves as an integrated research instrument (physical or virtual).

Proposals may be submitted only by institutions of higher education in the United States or its territories or possessions or by nonprofit organizations such as museums, science centers, observatories, research laboratories, professional societies, and similar organizations involved in research or educational activities. The deadline for full proposals is **January 24, 2013**. For more information see `http://www.nsf.gov/pubs/2011/nsf11503/nsf11503.htm`.

—From an NSF announcement

NSF Algorithms for Threat Detection

The Division of Mathematical Sciences (DMS) at the National Science Foundation (NSF) has formed a partnership with the Defense Threat Reduction Agency (DTRA) to develop the next generation of mathematical and statistical algorithms for the detection of chemical agents, biological threats, and threats inferred from geospatial information. Proposals are solicited from the mathematical sciences community in two main areas: mathematical and statistical techniques for genomics and mathematical and statistical techniques for the analysis of data from sensor systems. The deadline for full proposals is **January 10, 2013**. For more details, see `http://www.nsf.gov/pubs/2012/nsf12502/nsf12502.htm`.

—From an NSF announcement

National Academies Research Associateship Programs

The Policy and Global Affairs Division of the National Academies is sponsoring the 2013 Postdoctoral and Senior Research Associateship Programs. The programs are meant to provide opportunities for Ph.D., Sc.D., or M.D. scientists and engineers of unusual promise and ability to perform research at more than one hundred research laboratories throughout the United States and overseas.

Full-time associateships will be awarded for research in the fields of mathematics, chemistry, earth and atmospheric sciences, engineering, applied sciences, life sciences, space sciences, and physics. Most of the laboratories are open to both U.S. and non-U.S. nationals and to both recent doctoral recipients and senior investigators. Amounts of stipends depend on the sponsoring laboratory. Support is also provided for allowable relocation expenses and for limited professional travel during the period of the award.

Awards will be made four times during the year, in February, May, August, and November. The deadline for application materials to be postmarked or for electronic submissions for the February 2013 review is **February 1, 2013**. Materials for the May review are due **May 1, 2013**; for the August review, **August 1, 2013**; and for the November review, **November 1, 2013**. Note that not all sponsors participate in all four reviews. Applicants should refer to the specific information for the laboratory to which they are applying.

For further information and application materials, see the National Academies website at `http://sites.nationalacademies.org/PGA/RAP/PGA_050491` or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email `rap@nas.edu`.

—From an NRC announcement

CAIMS/PIMS Early Career Award

The Canadian Applied and Industrial Mathematics Society (CAIMS) and the Pacific Institute for Mathematical Sciences (PIMS) sponsor the Early Career Award in Applied Mathematics to recognize exceptional research in any branch of applied mathematics, interpreted broadly. The nominee's research should have been conducted primarily in Canada or in affiliation with a Canadian university. The prize is to be awarded every year to a researcher less than ten years past the date of Ph.D. at the time of nomination.

The award consists of a cash prize of C$1,000 and a commemorative plaque presented at the CAIMS Annual Meeting. The recipient will be invited to deliver a plenary lecture at the CAIMS annual meeting in the year of the award. A travel allowance will be provided. The deadline for nominations is **January 31, 2013**. For more information see `http://www.pims.math.ca/pims-glance/prizes-awards`.

—From a PIMS announcement

News from the CRM

The Centre de Recerca Matemàtica (CRM) in Barcelona, Spain, announces several activities for the spring of 2013.

Research Programs

The Mathematics of Planet Earth: through May 2013. See `www.crm.cat/2012/RPMPE2013`

Conformal Geometry and Geometric PDEs: May–July 2013. See `www.crm.cat/2013/RPConformalGeometry`

Conferences

New Trends in Regular Theory and Methods for Geomathematical Problems: May 6–10, 2013. See `www.crm.cat/2013/CGeomathicalProblems`

Geometrical Analysis: July 1–5, 2013. See `www.crm.cat/2013/CGeometricalAnalysis`

Advanced Courses

Imperial College–CRM International School and Research Workshop on Complex Systems: April 8–13, 2013. See `www.crm.cat/2013/ACComplexSystems`

Compactifying Moduli Spaces: May 27–June 1, 2013. See `www.crm.cat/2013/ACCompactifying`

FESS 2013: June 18–21, 2013. See `www.crm.cat/FESS2013`

Topics in Conformal Geometry and Geometry Analysis: June 25–28, 2013. See `www.crm.cat/2013/ACConformalGeometry`

Workshop

Slow-Fast Dynamics: Theory, Numerics, Application to Life and Earth Sciences: June 3–7, 2013. See `www.crm.cat/2013/WKEarthSciences`

—From a CRM announcement

For Your Information

News from PIMS

Alejandro Adem has been appointed to a second five-year term as director of the Pacific Institute for the Mathematical Sciences (PIMS), beginning July 1, 2013.

Adem is a professor of mathematics at the University of British Columbia and holds the Canada Research Chair in Algebraic Topology. His mathematical interests span a variety of topics in algebraic topology, group cohomology, and related areas. He has published more than sixty research articles, as well as two books, and has delivered more than two hundred fifty invited talks throughout the world. In 1992 he received the U.S. National Science Foundation Young Investigator Award. He has held visiting positions at the Institute for Advanced Study in Princeton, the ETH-Zurich, the Max Planck Institute in Bonn, the University of Paris VII and XIII, and Princeton University.

Adem has a wealth of international scientific and administrative experience. He has served as chair of the Department of Mathematics at the University of Wisconsin-Madison and as cochair of the Scientific Advisory Committee for the Mathematical Sciences Research Institute in Berkeley. He sits on the board of directors for the Atlantic Association for Research in the Mathematical Sciences and the Banff International Research Station, as well as on the AMS Council. He has served on the steering committees for the Long-Range Plan for the Mathematical and Statistical Sciences in Canada, for the 2011 ICIAM Conference, and for the 2013 Mathematical Congress of the Americas. He played a leading role in organizing the first joint meetings between the Canadian and Mexican Mathematical Societies and has been instrumental in the development of the Pacific Rim Mathematical Association (PRIMA). He has also fostered academic connections between Canada and France by serving as director of the Unité Mixte Internationale, etablished at PIMS in 2007 through the sponsorship of the Centre National de la Recherche Scientifique.

— From a PIMS news release

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Conformal Geometry and Geometric PDEs: May–July 2013. See www.crm.cat/2013/RPConformalGeometry

Conferences

New Trends in Regular Theory and Methods for Geomathematical Problems: May 6–10, 2013. See www.crm.cat/2013/CGeomathicalProblems

Geometrical Analysis: July 1–5, 2013. See www.crm.cat/2013/CGeometricalAnalysis

Advanced Courses

Imperial College–CRM International School and Research Workshop on Complex Systems: April 8–13, 2013. See www.crm.cat/2013/ACComplexSystems

Compactifying Moduli Spaces: May 27–June 1, 2013. See www.crm.cat/2013/ACCompactifying

FESS 2013: June 18–21, 2013. See www.crm.cat/FESS2013

Topics in Conformal Geometry and Geometry Analysis: June 25–28, 2013. See www.crm.cat/2013/ACConformalGeometry

Workshop

Slow-Fast Dynamics: Theory, Numerics, Application to Life and Earth Sciences: June 3–7, 2013. See www.crm.cat/2013/WKEarthSciences

—From a CRM announcement

For Your Information

News from PIMS

Alejandro Adem has been appointed to a second five-year term as director of the Pacific Institute for the Mathematical Sciences (PIMS), beginning July 1, 2013.

Adem is a professor of mathematics at the University of British Columbia and holds the Canada Research Chair in Algebraic Topology. His mathematical interests span a variety of topics in algebraic topology, group cohomology, and related areas. He has published more than sixty research articles, as well as two books, and has delivered more than two hundred fifty invited talks throughout the world. In 1992 he received the U.S. National Science Foundation Young Investigator Award. He has held visiting positions at the Institute for Advanced Study in Princeton, the ETH-Zurich, the Max Planck Institute in Bonn, the University of Paris VII and XIII, and Princeton University.

Adem has a wealth of international scientific and administrative experience. He has served as chair of the Department of Mathematics at the University of Wisconsin-Madison and as cochair of the Scientific Advisory Committee for the Mathematical Sciences Research Institute in Berkeley. He sits on the board of directors for the Atlantic Association for Research in the Mathematical Sciences and the Banff International Research Station, as well as on the AMS Council. He has served on the steering committees for the Long-Range Plan for the Mathematical and Statistical Sciences in Canada, for the 2011 ICIAM Conference, and for the 2013 Mathematical Congress of the Americas. He played a leading role in organizing the first joint meetings between the Canadian and Mexican Mathematical Societies and has been instrumental in the development of the Pacific Rim Mathematical Association (PRIMA). He has also fostered academic connections between Canada and France by serving as director of the Unité Mixte Internationale, etablished at PIMS in 2007 through the sponsorship of the Centre National de la Recherche Scientifique.

— From a PIMS news release

Inside the AMS

Erdős Memorial Lecture

The Erdős Memorial Lecture is an annual invited address named for the prolific mathematician Paul Erdős (1913–1996). The lectures are supported by a fund created by Andrew Beal, a Dallas banker and mathematics enthusiast. The Beal Prize Fund, now US$100,000, is being held by the AMS until it is awarded for a correct solution to the Beal Conjecture (see `www.math.unt.edu/~mauldin/beal.html`). At Beal's request, the interest from the fund is used to support the Erdős Memorial Lecture.

The Erdős Memorial Lecturers for 2012 were Avi Wigderson of the Institute for Advanced Study and Vaughan Jones of Vanderbilt University. Wigderson gave a lecture titled "The power and weakness of randomness (when you are short on time)", and Jones gave a talk titled "Why Flatland is a great place to do algebra". The lecture was held jointly with the International Conference on Mathematics and Statistics (ICOMAS) at the University of Memphis May 17–18, 2012.

—AMS announcement

From the AMS Public Awareness Office

2012 contest winner Shayam Narayanan.

National Who Wants to Be a Mathematician. The contest will be held at the 2013 Joint Mathematics Meetings on Thursday, January 10, from 9:30 a.m. to 11:00 a.m. in Room 6F of the San Diego Convention Center. Come see—and cheer on—ten of the nation's best high school math contestants as they compete for the top prize of $10,000. Find out more about the contestants at `http://www.ams.org/programs/students/wwtbam/contestants-2013`.

Celebrating 125 Years. In 2013 the American Mathematical Society celebrates its 125th anniversary. Founded in 1888 to further the interests of mathematics research and scholarship, the AMS continues to serve the national and international community through its meetings, publications, advocacy, and other programs. The celebrations will launch at the 2013 Joint Mathematics Meetings, with the AMS sponsorship of a WiFi hotspot, special sales and cake in the AMS exhibit area in aisle 800, and a 125th Anniversary Gala. See more information at `http://www.ams.org/about-us/ams-125th`.

AMS and Mathematics of Planet Earth 2013. The American Mathematical Society is partnering with Mathematics of Planet Earth 2013, a program of research institutes, scientific societies, universities, and foundations around the globe. See resources on mathematics and the environment to inform and inspire (Mathematical Moments, sessions at the Joint Mathematics Meetings, AMS books, articles in *Notices of the AMS*, Mathematics Awareness Month themes, Feature Column essays, and more) at `http://www.ams.org/samplings/mpe-2013`.

AMS at the 2012 SACNAS National Conference. The AMS hosted an exhibit and a version of the Who Wants to Be a Mathematician game for undergraduate students at the 2012 SACNAS National Conference hold in Seattle, WA, October 11–14. See a report by the Public Awareness Office on all the mathematics at the conference, as well as photographs, at `http://www.ams.org/meetings/sacnas2012-mtg`.

Annette Emerson and Mike Breen
Public Awareness Officers
`paoffice@ams.org`

Deaths of AMS Members

Irving Adler, of Bennington, Vermont, died on September 22, 2012. Born on April 27, 1913, he was a member of the Society for 53 years.

David Chillag, professor, Technion-Israel Institute of Technology, died on July 29, 2012. Born on Ocotber 22, 1946, he was a member of the Society for 38 years.

Edward Cline, professor, University of Oklahoma, died on March 23, 2012. Born on May 30, 1940, he was a member of the Society for 48 years.

Alfred Descloux, of Ocala, Florida, died on September 7, 2012. Born on April 27, 1923, he was a member of the Society for 55 years.

Earnest J. Eckert, of Aalborg, Denmark, died on August 27, 2012. Born on May 9, 1926, he was a member of the Society for 55 years.

John Killeen, of Berkeley, California, died on August 15, 2012. Born on July 28, 1925, he was a member of the Society for 55 years.

Ernesto A. Lacomba, professor, University Autónoma Metropolitana, died on June 26, 2012. Born on December 2, 1945, he was a member of the Society for 41 years.

Jacob R. Matijevic, of Los Angeles, California, died on August 20, 2012. Born on November 3, 1947, he was a member of the Society for 40 years.

Daniel Orloff, of San Francisco, California, died on June 20, 2012. Born on May 31, 1921, he was a member of the Society for 65 years.

Benjamin Rickman, of Wembley, United Kingdom, died on October 16, 2012. Born on June 16, 1950, he was a member of the Society for 35 years.

Electronic Backfile Collections from the AMS

Sergei Gelfand

If you ask university librarians what is the most precious resource their library needs, most would answer “funding”, but perhaps many would also say “shelf space”. With more and more journals and books being published every year, fitting them all in becomes a real challenge for libraries. To help librarians, many publishers have started to supplement their new publications with electronic versions that can be accessed from and read on personal computers, tablets, e-readers, and other such devices.

But what about older books and journals published before the concept of digital media (or computers for that matter) was even conceived? Even if a library possesses paper copies of important books and journals, having them in electronic format is still quite useful, e.g., in cases where a publication is lost or damaged or a library wants to start a new collection or a new library is organized. Another important advantage of electronic publications is the ability to easily search the content both inside one article or book and throughout the entire collection.

Realizing this need, the AMS started an initiative to retrodigitize its older publications (i.e., convert them to digital format). Over the past few years, the AMS has digitized all issues of its primary journals (*Journal of the AMS, Mathematics of Computation, Proceedings of the AMS*, and *Transactions of the AMS*), and, in addition, all of the issues of these journals that are more than five years old are now freely accessible on the AMS website.

Following the digitization of its primary journals, AMS next looked to do the same for its books. In 2011 the AMS retrodigitized all 560 volumes of its largest book series, *Contemporary Mathematics*, and in early 2012 released the collection of all past and current volumes in digital format for purchase by libraries. Throughout the year, the AMS has continued to prepare collections in two more of its proceedings series, Proceedings of *Symposia in Pure Mathematics* (86 volumes, approximately 49,000 pages) and *Proceedings of Symposia in Applied Mathematics* (71 volumes, approximately 17,000 pages). These sets will be offered to libraries by the end of 2012. In early 2013 the AMS will release the full digital collection of *Memoirs of the AMS* (946 volumes, approximately 107,000 pages). Next in our plans is to retrodigitize volumes in leading monograph series, such as *Mathematical Surveys and Monographs*, *Graduate Studies in Mathematics*, and *Student Mathematical Library*. We anticipate that the digitization of these series will be completed in the first half of 2013 and the back-volume sets will be available to libraries by the end of that year.

It should be mentioned that the retrodigitization of back volumes is a complicated process that involves substantial personnel and financial resources. The AMS has been publishing books since 1894, and the majority of the books in its history were published before the dawn of the digital age, so the technical and legal issues that can be encountered are nontrivial, time consuming, and often require the joint efforts of staff from several AMS groups and departments. Some of the challenges of retrodigitalization are the extensive research efforts and staff commitment required to clarify rights and permissions issues associated with electronic rights; taking the appropriate measures to ensure the best possible digital quality; efforts related to scanning, reformatting, and verification of huge amounts of data; and incorporating searchability and discovery options in the prepared electronic files.

We all hope that the final product will turn out to be useful and beneficial to librarians and to the research community in general.

More information about existing and forthcoming back-volume products available from the AMS can be found on the AMS website, in particular, at `http://www.ams.org/publications/ebooks/ebooks`.

Sergei Gelfand is the AMS publisher. His email address is `sxg@ams.org`.

DOI: http://dx.doi.org/10.1090/noti947

Reference and Book List

The ***Reference*** *section of the* Notices *is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.*

Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are `notices@math.wustl.edu` in the case of the editor and `smf@ams.org` in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Information for *Notices* Authors

The *Notices* welcomes unsolicited articles for consideration for publication, as well as proposals for such articles. The following provides general guidelines for writing *Notices* articles and preparing them for submission. Contact information for *Notices* editors and staff may be found on the *Notices* website, `http://www.ams.org/notices`.

Upcoming Deadlines

December 14, 2012: Applications for NDSEG Fellowship Program. See `http://ndseg.asee.org/`.

December 15, 2012: Applications for National Science Foundation STaR (Service, Teaching and Research) program. See the website `http://matheddb.missouri.edu/star/` or contact Robert Reys, Curators' Professor of Mathematics Education, University of Missouri, email: `reysr@missouri.edu`.

December 15, 2012: Applications for AMS Epsilon Fund grants. See `http://www.ams.org/programs/edu-support/epsilon/emp-epsilon` or contact the AMS Membership and Programs Department, email: `prof-serv@ams.org`; telephone: 800-321-4267, ext. 4170.

December 15, 2012: Applications for Fields Institute postdoctoral fellowships for the Thematic Program on Calabi-Yau Varieties: Arithmetic, Geometry, and Physics. See `http://www.fields.utoronto.ca/honours/postdoc.html`.

Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

December 31, 2012: Nominations for IMU Prizes: Leelavati Prize, Fields Medals, Nevanlinna Prize, Gauss Prize, Chern Medal, and Noether Lectureship. See http://www.mathunion.org/general/prizes/leelavati/details/ for the Leelavati Prize; http://www.mathunion.org/general/prizes/nomination-guidelines/ for other prizes; and http://www.mathunion.org/general/prizes/prize-committee-chairs/2014/ for committee chairs.

December 31, 2012: Nominations for Otto Neugebauer Prize of the EMS. See the website http://www.euro-math-soc.eu/otto_neugebauer_prize.html.

January 2, 2013: Applications for priority consideration for Los Angeles, New York City, Utah, and Washington, D.C., fellowships of Math for America. See http://www.mathforamerica.org/.

January 7, 2013: Applications for Institut Mittag-Leffler postdoctoral fellowship grants for 2013–2014. See http://www.mittag-leffler.se.

January 10, 2013: Full proposals for NSF Algorithms for Threat Detection. See "Mathematics Opportunities" in this issue.

January 31, 2013: Nominations for CAIMS/PIMS Early Career Award. See "Mathematics Opportunities" in this issue.

January 13, 2013: Applications for Jefferson Science Fellows Program. See email jsf@nas.edu, telephone 202-334-2643, or see the website http://sites.nationalacademies.org/PGA/Jefferson/PGA_046612.

January 24, 2013: Full proposals for NSF Major Research Instrumentation Program. See "Mathematics Opportunities" in this issue.

January 31, 2013: Entries for AWM Essay Contest. Contact the contest organizer, Heather Lewis, at hlewis5@naz.edu.

February 1, 2013: Applications for February review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

February 1, 2013: Applications for AWM Travel Grants, Mathematics Education Research Travel Grants, Mathematics Mentoring Travel Grants, and Mathematics Education Research Mentoring Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; or telephone: 703-934-0163; e-mail: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

February 4, 2013: Proposals for programs in mathematical sciences for Institut Mittag-Leffler for academic year 2015–2016. See http://www.mittag-leffler.se.

February 5, 2013: Applications for Boston, Los Angeles, New York City, Utah, and Washington, D.C., fellowships of Math for America. See http://www.mathforamerica.org/.

February 12, 2013: Applications for IPAM undergraduate program Research in Industrial Projects for Students (RIPS). See www.ipam.ucla.edu.

February 15, 2013: Nominations for AWM-Microsoft Research Prize in Algebra and Number Theory and AWM-Sadosky Research Prize in Analysis. See http://www.awm-math.org.

February 15, 2013: Applications for AMS Congressional Fellowship. See http://www.ams.org/programs/ams-fellowships/ams-aaas/ams-aaas-congressional-fellowship, or contact the AMS Washington Office at 202-588-1100, email: amsdc@ams.org.

February 25, 2013: Applications for EDGE Summer Program. See http://www.edgeforwomen.org/.

March 31, 2013: Applications for AMS-Simons Travel Grants. See "Mathematics Opportunities" in this issue.

March 31, 2013: Applications for IPAM graduate summer school on computer vision. See www.ipam.ucla.edu.

April 1, 2013: Letters of intent for proposals for one-semester programs at the Bernoulli Center (CIB). See http://cib.epfl.ch/.

April 15, 2013: Applications for fall 2013 semester of Math in Moscow. See http://www.mccme.ru/mathinmoscow, or write to: Math in Moscow, P.O. Box 524, Wynnewood, PA 19096; fax: +7095-291-65-01; e-mail: mim@mccme.ru. Information and application forms for the AMS scholarships are available on the AMS website at http://www.ams.org/programs/travel-grants/mimoscow, or by writing to: Math in Moscow Program, Membership and Programs Department, American Mathematical Society, 201 Charles Street, Providence RI 02904-2294; email student-serv@ams.org.

May 1, 2013: Applications for May review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

May 1, 2013: Applications for AWM Travel Grants and Mathematics Education Research Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; or telephone: 703-934-0163; email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

August 1, 2013: Applications for August review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

October 1, 2013: Applications for AWM Travel Grants and Mathematics Education Research Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; or telephone: 703-934-0163; email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

October 4, 2013: Letters of intent for NSF Program ADVANCE Institutional Transformation and Institutional Transformation Catalyst awards. See http://www.nsf.gov/pubs/2012/nsf12584/nsf12584.htm?WT.mc_id=USNSF_36&WT.mc_ev=click.

November 1, 2013: Applications for November review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

November 12, 2013: Full proposals for NSF Program ADVANCE Institutional Transformation and Institutional Transformation Catalyst awards. See http://www.nsf.gov/pubs/2012/nsf12584/nsf12584.htm?WT.mc_id=USNSF_36&WT.mc_ev=click .

National Science Board

The National Science Board is the policymaking body of the National Science Foundation. Listed below are the current members of the NSB. For further information, visit the website http://www.nsf.gov/nsb/.

Dan E. Arvizu (Chair)
Director and Chief Executive
National Renewable Energy Laboratory

Bonnie Bassler
Howard Hughes Medical Institute Investigator
Squibb Professor of Molecular Biology
Princeton University

Ray M. Bowen
President Emeritus
Texas A&M University

France A. Córdova
President Emeritus
Purdue University

Kelvin K. Droegemeier
(Vice Chair)
Vice President for Research University of Oklahoma

Esin Gulari
Dean of Engineering and Science
Clemson University

Alan Leshner
Chief Executive Officer and Executive Publisher, *Science*
American Association for the Advancement of Science

W. Carl Lineberger
E. U. Condon Distinguished Professor of Chemistry
University of Colorado

G. P. Peterson
President
Georgia Institute of Technology

Douglas D. Randall
Professor of Biochemistry and Thomas Jefferson Fellow Director, Interdisciplinary Plant Group
University of Missouri, Columbia

Anneila I. Sargent
Benjamin M. Rosen Professor of Astronomy
Vice President for Student Affairs
California Institute of Technology

Diane L. Souvaine
Professor, Computer Science
Tufts University

Arnold F. Stancell
Emeritus Professor and Turner Leadership Chair
School of Chemical and Biomolecular Engineering
Georgia Institute of Technology

Claude M. Steele
Dean, School of Education
Stanford University

Robert J. Zimmer
President
University of Chicago

The contact information for the Board is: National Science Board, 4201 Wilson Boulevard, Room 1225N, Arlington, VA 22230; telephone 703-292-7000; fax 703-292-9008; email NationalScienceBrd@nsf.gov; World Wide Web http://www.nsf.gov/nsb/.

Book List

The Book List highlights recent books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. Suggestions for books to include on the list may be sent to notices-booklist@ams.org.

*Added to "Book List" since the list's last appearance.

Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys, by David Joyner. Johns Hopkins University Press (second edition), December 2008. ISBN-13: 978-08018-9013-0.

Algorithmic Puzzles, by Anany Levitin and Maria Levitin. Oxford University Press, October 2011. ISBN-13: 978-01997-404-44.

American Mathematicians as Educators, 1893–1923: Historical Roots of the "Math Wars", by David Lindsay Roberts. Docent Press, July 2012, ISBN-13: 978-09837-004-49.

The Annotated Turing: A Guided Tour Through Alan Turing's Historic Paper on Computability and the Turing Machine, by Charles Petzold. Wiley, June 2008. ISBN-13: 978-04702-290-57. (Reviewed September 2011.)

The Beginning of Infinity: Explanations That Transform the World, by David Deutsch. Viking Adult, July 2011. ISBN-13: 978-06700-227-55. (Reviewed April 2012.)

The Best Writing on Mathematics 2012, edited by Mircea Pitici. Princeton University Press, November 2012. ISBN-13: 978-06911-565-52.

Bibliography of Raymond Clare Archibald, by Scott Guthery. Docent Press, April 2012. ISBN-13: 978-0983700425.

The Big Questions: Mathematics, by Tony Crilly. Quercus, April 2011. ISBN: 978-18491-624-01. (Reviewed October 2012.)

Calculating Curves: The Mathematics, History, and Aesthetic Appeal of T. H. Gronwall's Nomographic Work, by Thomas Hakon Gronwall, with contributions by Ron Doerfler and Alan Gluchoff, translation by Paul Hamburg, and bibliography by Scott Guthery. Docent Press, April 2012. ISBN-13: 978-09837-004-32.

The Calculus of Selfishness, by Karl Sigmund. Princeton University Press, January 2010. ISBN-13: 978-06911-427-53. (Reviewed January 2012.)

Chasing Shadows: Mathematics, Astronomy, and the Early History of Eclipse Reckoning, by Clemency Montelle. Johns Hopkins University Press, April 2011. ISBN-13: 978-08018-969-10. (Reviewed March 2012.)

Classic Problems of Probability, by Prakash Gorroochurn. Wiley, May 2012. ISBN: 978-1-1180-6325-5.

The Crest of the Peacock: Non-European Roots of Mathematics, by George Gheverghese Joseph. Third edition. Princeton University Press, October 2010. ISBN-13: 978-0-691-13526-7.

The Crossing of Heaven: Memoirs of a Mathematician, by Karl

Gustafson. Springer, January 2012. ISBN-13: 978-36422-255-74.

Elegance with Substance, by Thomas Colignatus. Dutch University Press, 2009. ISBN-13: 978-90361-013-87.

Elliptic Tales: Curves, Counting, and Number Theory, by Avner Ash and Robert Gross. Princeton University Press, March 2012. ISBN-13: 978-06911-511-99.

Emmy Noether's Wonderful Theorem, by Dwight E. Neuenschwander. Johns Hopkins University Press, November 2010. ISBN-13: 978-08018-969-41.

Excursions in the History of Mathematics, by Israel Kleiner. Birkhäuser, 2012. ISBN-13: 978-08176-826-75.

Experimental and Computational Mathematics: Selected Writings, by Jonathan Borwein and Peter Borwein. PSIpress, 2011. ISBN-13: 978-19356-380-56.

Flatland, by Edwin A. Abbott, with notes and commentary by William F. Lindgren and Thomas F. Banchoff. Cambridge University Press, November 2009. ISBN-13: 978-05217-599-46.

The Foundations of Geometry And Religion From An Abstract Standpoint, by Salilesh Mukhopadhyay. Outskirts Press, July 2012. ISBN: 978-1-4327-9424-8.

Galileo's Muse: Renaissance Mathematics and the Arts, by Mark Austin-Peterson. Harvard University Press, October 2011. ISBN-13: 978-06740-597-26. (Reviewed November 2012.)

Game Theory and the Humanities: Bridging Two Worlds, by Steven J. Brams. MIT Press, September 2012. ISBN-13: 978-02625-182-53.

Gösta Mittag-Leffler: A Man of Conviction, by Arild Stubhaug (translated by Tiina Nunnally). Springer, November 2010. ISBN-13: 978-36421-167-11.

Guesstimation 2.0: Solving Today's Problems on the Back of a Napkin, by Lawrence Weinstein. Princeton University Press, September 2012. ISBN: 978-06911-508-02.

In Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation, by William J. Cook. Princeton University Press, December 2011. ISBN-13: 978-06911-527-07.

In Pursuit of the Unknown: 17 Equations That Changed the World, by Ian Stewart. Basic Books, March 2012. ISBN-13: 978-04650-297-30. (Reviewed December 2012.)

In Service to Mathematics: The Life and Work of Mina Rees, by Amy Shell-Gellasch. Docent Press, December 2010. ISBN-13: 978-0-9837004-1-8.

Infinity: New Research Frontiers, edited by Michael Heller and W. Hugh Woodin. Cambridge University Press, February 2011. ISBN-13: 978-11070-038-73.

The Infinity Puzzle: Quantum Field Theory and the Hunt for an Orderly Universe, by Frank Close. Basic Books, November 2011. ISBN-13: 978-04650-214-44. (Reviewed September 2012.)

The Information: A History, a Theory, a Flood, by James Gleick. Pantheon, March 2011. ISBN-13: 978-03754-237-27.

Introduction to Mathematical Thinking, by Keith Devlin. Keith Devlin, July 2012. ISBN-13: 978-06156-536-31.

The Irrationals: A Story of the Numbers You Can't Count On, by Julian Havil. Princeton University Press, June 2012. ISBN-13: 978-0691143422.

Knots Unravelled: From String to Mathematics, by Meike Akveld and Andrew Jobbings. Arbelos, October 2011. ISBN-13: 978-09555-477-20.

The Joy of x: A Guided Tour of Math, from One to Infinity, by Steven Strogatz. Eamon Dolan/Houghton Mifflin Harcourt, October 2012. ISBN-13: 978-05475-176-50.

Late Style—Yuri I. Manin Looking Back on a Life in Mathematics. A DVD documentary by Agnes Handwerk and Harrie Willems. Springer, March 2012. ISBN NTSC: 978-3-642-24482-7; ISBN PAL: 978-3-642-24522-0. (Reviewed in this issue.)

Lemmata: A Short Mathematical Thriller, by Sam Peng. CreateSpace, December 2011. ISBN-13: 978-14681-442-39.

The Logician and the Engineer: How George Boole and Claude Shannon Created the Information Age, by Paul J. Nahin, Princeton University Press, October 2012. ISBN: 978-06911-510-07.

Lost in a Cave: Applying Graph Theory to Cave Exploration, by Richard L. Breisch. National Speleological Society, January 2012. ISBN-13: 978-1-879961-43-2.

The Lost Millennium: History's Timetables Under Siege, by Florin Diacu. Johns Hopkins University Press (second edition), November 2011. ISBN-13: 978-14214-028-88.

Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks, by Persi Diaconis and Ron Graham. Princeton University Press, November 2011. ISBN-13: 978-06911-516-49. (Reviewed August 2012.)

The Man of Numbers: Fibonacci's Arithmetic Revolution, by Keith Devlin. Walker and Company, July 2011. ISBN-13: 978-08027-781-23. (Reviewed May 2012.)

Math Girls, by Hiroshi Yuki (translated from the Japanese by Tony Gonzalez). Bento Books, November 2011. ISBN-13: 978-09839-513-15. (Reviewed August 2012.)

Math Goes to the Movies, by Burkard Polster and Marty Ross. Johns Hopkins University Press, July 2012. ISBN-13: 978-14214-048-44.

Math is Murder, by Robert C. Brigham and James B. Reed. Universe, March 2012. ISBN-13: 978-14697-972-81.

The Mathematical Writings of Évariste Galois, edited by Peter M. Neumann. European Mathematical Society, October 2011. ISBN-13: 978-3-03719-104-0. (Reviewed December 2012.)

Mathematical Excursions to the World's Great Buildings, by Alexander J. Hahn. Princeton University Press, July 2012. ISBN-13: 978-06911-452-04

A Mathematician Comes of Age, by Steven G. Krantz. Mathematical Association of America, December 2011. ISBN-13: 978-08838-557-82.

Mathematics in Popular Culture: Essays on Appearances in Film, Fiction, Games, Television and Other Media, edited by Jessica K. Sklar and Elizabeth S. Sklar. McFarland, February 2012. ISBN-13: 978-07864-497-81.

Mathematics in Victorian Britain, by Raymond Flood, Adrian Rice, and Robin Wilson. Oxford University Press, October 2011. ISBN-13: 978-019-960139-4.

Meaning in Mathematics, edited by John Polkinghorne. Oxford University Press, July 2011. ISBN-13: 978-01996-050-57.

Measurement, by Paul Lockhart. Belknap Press of Harvard University Press, September 2012. ISBN-13: 978-06740-575-55.

Newton and the Counterfeiter: The Unknown Detective Career of the World's Greatest Scientist, by Thomas Levenson. Houghton Mifflin Harcourt, June 2009. ISBN-13: 978-01510-127-87.

The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century, by Yvette Kosmann-Schwarzbach. Springer, December 2010. ISBN-13: 978-03878-786-76.

Nine Algorithms That Changed the Future: The Ingenious Ideas That Drive Today's Computers, by John MacCormick. Princeton University Press, December 2011. ISBN-13: 978-06911-471-47.

Number-Crunching: Taming Unruly Computational Problems from Mathematical Physics to Science Fiction, by Paul J. Nahin. Princeton University Press, August 2011. ISBN: 978-06911-442-52.

Numbers: A Very Short Introduction, by Peter M. Higgins. Oxford University Press, February 2011. ISBN-13: 978-0-19-958405-5. (Reviewed January 2012.)

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Proof and Other Dilemmas: Mathematics and Philosophy, edited by Bonnie Gold and Roger A. Simons. Mathematical Association of America, July 2008. ISBN-13: 978-08838-556-76. (Reviewed December 2011.)

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Scientific Reflections: Selected Multidisciplinary Works, by Richard Crandall. PSIpress, 2011. ISBN-13: 978-19356-380-87.

Secrets of Triangles: A Mathematical Journey, by Alfred S. Posamentier and Ingmar Lehman. Prometheus Books, August 2012. ISBN-13: 978-16161-458-73.

Seduced by Logic: Emilie Du Châelet, Mary Somerville and the Newtonian Revolution, by Robyn Arianrhod. Oxford University Press, September 2012. ISBN: 978-01999-316-13.

Simon: The Genius in My Basement, by Alexander Masters. Delacorte Press, February 2012. ISBN-13: 978-03853-410-80.

Six Gems of Geometry, by Thomas Reale. PSIpress, 2010. ISBN-13: 978-19356-380-25.

Sources in the Development of Mathematics: Series and Products from the Fifteenth to the Twenty-first Century, by Ranjan Roy. Cambridge University Press, June 2011. ISBN-13: 978-05211-147-07.

A Strange Wilderness: The Lives of the Great Mathematicians, by Amir D. Aczel. Sterling, October 2011.ISBN-13: 978-14027-858-49.

Taking Sudoku Seriously: The Math behind the World's Most Popular Pencil Puzzle, by Jason Rosenhouse and Laura Taalman. Oxford University Press, January 2012. ISBN-13: 978-01997-565-68.

The Theory That Would Not Die: How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy, by Sharon Bertsch McGrayne. Yale University Press, April 2011. ISBN-13: 978-03001-696-90. (Reviewed May 2012.)

Top Secret Rosies: The Female Computers of World War II. Video documentary, produced and directed by LeAnn Erickson. September 2010. Website: `http://www.topsecretrosies.com`. (Reviewed February 2012.)

Transcending Tradition: Jewish Mathematicians in German Speaking Academic Culture, edited by Birgit Bergmann, Moritz Epple, and Ruti Ungar. Springer, January 2012. ISBN: 978-3642224638.

Turbulent Times in Mathematics: The Life of J.C. Fields and the History of the Fields Medal, by Elaine McKinnon Riehm and Frances Hoffman. AMS, November 2011. ISBN-13: 978-08218-691-47.

Uneducated Guesses: Using Evidence to Uncover Misguided Education Policies, by Howard Wainer. Princeton University Press, August 2011. ISBN-13: 978-06911-492-88. (Reviewed June/July 2012.)

The Universe in Zero Words: The Story of Mathematics as Told through Equations, by Dana Mackenzie. Princeton University Press, April 2012. ISBN-13: 978-06911-528-20. (Reviewed in this issue.)

Vilim Feller, istaknuti hrvatsko-americki matematicar/William Feller, Distinguished Croatian-American Mathematician, by Darko Zubrinic. Bilingual Croatian-English edition, Graphis, 2011. ISBN-13: 978-953-279-016-0.

A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing, edited by Benjamin Wardhaugh. Princeton University Press, April 2012. ISBN-13: 978-06911-477-58.

What's Luck Got to Do with It? The History, Mathematics and Psychology of the Gambler's Illusion, by Joseph Mazur. Princeton University Press, July 2010. ISBN-13: 978-0-691-13890-9. (Reviewed February 2012.)

Who's #1?: The Science of Rating and Ranking, by Amy N. Langville and Carl D. Meyer. Princeton University Press, February 2012. ISBN-13: 978-06911-542-20. (Reviewed in this issue.)

Why Beliefs Matter: Reflections on the Nature of Science, by E. Brian Davies. Oxford University Press, June 2010. ISBN-13: 978-01995-862-02. (Reviewed April 2012.)

Why Cats Land on Their Feet (and 76 Other Physical Paradoxes and Puzzles), by Mark Levi. Princeton University Press, May 2012. ISBN-13: 978-0691148540.

Mathematics Calendar

Please submit conference information for the Mathematics Calendar through the Mathematics Calendar submission form at `http://www.ams.org/cgi-bin/mathcal-submit.pl`. The most comprehensive and up-to-date Mathematics Calendar information is available on the AMS website at `http://www.ams.org/mathcal/`.

January 2013

* 7–March 8 **Algorithmic Game Theory and Computational Social Choice**, Institute for Mathematical Sciences, National University of Singapore, Singapore.
Description: The program will include two workshops and one winter school. For the winter school, several world-class experts will give tutorials on a number of active research directions in Algorithmic Game Theory and Computational Social Choice.
Workshops: There will be two workshops, one right before and one right after the winter school: Workshop 1: Algorithmic Game Theory Workshop 2: Computational Social Choice. The workshops will include presentations by world-class researchers and provide ample opportunities for new research collaborations.
Information: `http://www2.ims.nus.edu.sg/Programs/013game/index.php`.

February 2013

* 19–23 **Recent trends in Algebraic Analysis**, Math Department, University of Padova, Padova, Italy.
Description: The aim of the conference is to bring together some of the leading experts in the field of Algebraic Analysis, to report on some recent and exciting developments. Established researchers will be joined by younger colleagues with the aim of promoting the exchange of ideas and seeking new partnerships. Among the topics covered we would like to point out: (not necessarily regular) holonomic systems, integral transforms, deformation-quantization in the complex field, microlocal analysis of sheaves and symplectic geometry.
Information: `http://events.math.unipd.it/aa2013/`.

* 21–23 **Combinatorial Probability and Statistical Mechanics Workshop**, Queen Mary, University of London, London, England.
Description: The workshop is the first in the new series of meetings co-organized by the School of Mathematical Sciences at QMUL and the Mathematics Institute of the University of Warwick.
Aim: Of the workshop is to bring together researchers who study various aspects and applications of random combinatorial structures, in particular from the viewpoints of stochastic processes, asymptotic enumeration, analytic combinatorics and the analysis of algorithms. If you are coming to the workshop, it would be helpful if you email: `D.S.Stark@qmul.ac.uk`, so that we have some idea of the number of attendees.
Information: `http://www.maths.qmul.ac.uk/content/combinatorial-probability-and-statistical-mechanics-workshop`.

* 25–March 1 **SIAM Conference on Computational Science & Engineering (CSE13)**, The Westin Boston Waterfront, Boston, Massachusetts.
Description: The SIAM CS&E conference seeks to enable in-depth technical discussions on a wide variety of major computational efforts on large problems in science and engineering, foster the

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.
An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.
In general, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to `notices@ams.org` or `mathcal@ams.org`.
In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.
The complete listing of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.
The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: `http://www.ams.org/`.

interdisciplinary culture required to meet these large-scale challenges, and promote the training of the next generation of computational scientists.
Information: http://www.siam.org/meetings/cse13/.

March 2013

* 12–15 **Interactions Between Analysis and Geometry Tutorials**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.
Description: The long program "Interactions between Geometry and Analysis" will start with two tutorials on the "Geometry of negatively curved spaces" by Juan Souto, University of British Columbia, and on "Analysis on metric spaces" by Nages Shanmugalingam, University of Cincinnati.
Lectures: Each tutorial will consist of four lectures and will introduce non-experts to these subjects. We hope that this will provide some common ground for communications between geometers and analysts during our program.
Registration: For tutorials is free, to encourage broad participation from researchers who are not part of the long program.
Information: http://www.ipam.ucla.edu/programs/iagtut/.

* 18–22 **Analysis on Metric Spaces**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.
Description: The last few decades have seen the trend to extend the scope of classical analytic and geometric theories from the familiar Euclidean space or manifold setting to more general metric spaces, often of a non-smooth or fractal nature. Recently there has been spectacular progress on the development of a theory of general metric spaces resembling manifolds with Ricci curvature bounds by the work of Lott, Villani, and Sturm. Their approach is based on convexity properties of an entropy functional in optimal transport. Other approaches include the use of Bochner type formulas by Baudoin and Garofalo and the use of heat kernels by Koskela, Rajala and Shanmugalingam and by Ambrosio, Gigli and Savare. However, many of the underlying analytic and geometric questions are still poorly understood. This workshop will further explore these approaches and related challenging issues, and foster new collaborations among various groups of researchers.
Information: http://www.ipam.ucla.edu/programs/iagws1/.

* 25–29 **4th International conference "Function spaces. Differential operators. General topology. Problems of mathematical education"**, Peoples' Friendship University of Russia (PFUR), Moscow, Russia.
Description: The conference is dedicated to corresponding member of RAS, member of the European Academy of Sciences, Professor L.D. Kudryavtsev in occasion of his 90th anniversary. Conference sections: theory of functions and function spaces; differential operators and their applications; nonlinear differential equations of mathematical physics; general topology and its applications; inverse and ill-posed problems; problems of mathematical education; education and morality; history of mathematics and science; information technologies in education.
Organizers: Steklov Mathematical Institute (MIRAS) (Russia), Lomonosov Moscow State University (MSU) (Russia), Peoples' Friendship University of Russia (PFUR) (Russia), Moscow Institute of Physics and Technology (MIPT) (Russia) and Methodological Council on Mathematical Education (MCME) (Russia).
Information: http://LDKudryavtsev90.rudn.ru.

* 25–29 **Master class on geometry and Teichmüller theory**, Erwin Schrödinger Institute, Vienna, Austria.
Description: The master class consists of a series of five intensive courses on geometry around Teichmüller theory, given by: Norbert A'Campo (Basel), Samuel Lelièvre (Paris), Julien Marché (Paris), Hugo Parlier (Fribourg), Vlad Sergiescu (Grenoble). The courses are especially addressed to young researchers, and in particular to Ph.D. students and new post-docs. The master class is part of a 3-month program on Teichmüller theory at the Erwin Schrödinger Institute, organized by L. Funar (Grenoble), Y. Neretin (Vienna and Moscow), R. C. Penner (UCLA and Aarhus) and A. Papadopoulos (Strasbourg). For more information and for participation to the master-class, contact papadop@math.unistra.fr.
Information: http://modgroup.math.cnrs.fr/teichmuller.html.

April 2013

* 1–5 **AIM Workshop: Gromov-Witten invariants and number theory**, American Institute of Mathematics, Palo Alto, California.
Description: This workshop, sponsored by AIM and the NSF, will focus on the current interactions of Gromov-Witten invariants and number theory.
Information: http://www.aimath.org/ARCC/workshops/gromwitnumthry.html.

* 1–26 **Modular Representation Theory of Finite and p-adic Groups**, Institute for Mathematical Sciences, National University of Singapore, Singapore.
Description: This month-long program is devoted to the representation theory over a field of nonzero characteristic of finite and p-adic groups, as well as related algebras, especially those arising naturally in Lie theory.
Topics: The two main topics of the program are: (i) Modular Representation Theory of Finite Groups and Related Algebras (ii) Modular Representation Theory in the Langlands Programme. We will have 6 tutorials (instructional lecture series of 3 or 4 talks) during the programme. In the last two weeks of the programme, there will be a research conference with about 2 or 3 talks a day.
Information: http://www2.ims.nus.edu.sg/Programs/013mod/index.php.

* 3–6 **Recreational Mathematics Colloquium III**, University of Azores, Ponta Delgada, Portugal.
Description: The Gathering for Gardner (G4G) happens on even number years in Atlanta, GA, USA. On odd numbered years there is a similar event in Europe: the Recreational Mathematics Colloquium. This will be the third in the series.
Information: http://ludicum.org/ev/rm/13.

* 19–21 **Underrepresented Students in Topology and Algebra Research Symposium**, Purdue University, West Lafayette, Indiana.
Description: The Underrepresented Students in Topology and Algebra Research Symposium (USTARS) is in its third year and is a project proposed by a group of underrepresented young mathematicians. After successful meetings in April 2011 and 2012, the committee is planning the third meeting for April 2013 to be held at the Purdue University, known for its commitment to math education, faculty development and underrepresented student issues. The symposium is structured over two days where underrepresented speakers will give 30-minute parallel research talks. Two distinguished graduate students and one invited faculty member give 1-hour presentations. A poster session featuring invited undergraduates is also planned. While the committee welcomes postdoctoral and faculty speakers to give 30 minute talks, graduate students will give at least 75%.
Information: http://www.ustars.org.

* 24–28 **Algebra, Combinatorics and Representation Theory: a conference in honor of the 60th birthday of Andrei Zelevinsky**, Northeastern University, Boston, Massachusetts.
Description: The topics of the conference cover a broad swathe of combinatorics, commutative algebra, representation theory, and algebraic geometry. It will feature talks by: Arkady Berenstein, Joseph Bernstein, Alexander Braverman, Harm Derksen, Sergey Fomin, Alexander Goncharov, Mikhail Kapranov, Allen Knutson, Bernard Leclerc,

Robert Marsh, Ezra Miller, Tomoki Nakanishi, Vladimir Retakh, Claus Michael Ringel, Vera Serganova, Michael Shapiro, Catharina Stroppel, Bernd Sturmfels, Lauren Williams, Jerzy Weyman
Information: `http://www.math.neu.edu/~bwebster/ACRT/`.

* 29–May 3 **Non-Smooth Geometry**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.
Description: Many contemporary investigations in geometry lead to analytic questions on non-smooth and fractal spaces different from the usual Euclidean setting. In this workshop we intend to pursue some of these directions with an emphasis on more geometric aspects (another workshop in this program on "Analysis on Metric Spaces" has a more analytic bias). Topics will include analytic problems that arise in geometric group theory or for expanding dynamical systems, differentiability properties of Lipschitz functions, currents and isoperimetric problems on metric spaces, quasiconformal geometry of fractals, and sub-Riemannian geometry.
Information: `http://www.ipam.ucla.edu/programs/iagws3/`.

May 2013

* 3–5 **Atkin Memorial Lecture and Workshop: Cohen-Lenstra Heuristics**, University of Illinois at Chicago, Chicago, Illinois.
Description: This year Akshay Venkatesh (Stanford) is scheduled to give the Atkin Lecture on May 3rd, 2013.
Confirmed speakers: Thomas Church (Stanford), Jordan Ellenberg (University of Wisconsin), Wei Ho (Columbia), Andy Putman (Rice), Arul Shankar (Institute for Advanced Study, Princeton), Frank Thorne (South Carolina), Jacob Tsimerman (Harvard), Akshay Venkatesh (Stanford), Craig Westerland (Melbourne), and Melanie Matchett Wood (University of Wisconsin).
Funding: We may have some funding available for participants. Please see the workshop website for information.
Information: `http://www.math.uic.edu/~rtakloo/atkin2013.html`.

* 7–11 **Analysis, Complex Geometry, and Mathematical Physics: A Conference in Honor of Duong H. Phong**, Columbia University, New York, New York.
Description: This conference, to be held at Columbia University, will focus on recent developments in analysis, complex geometry and mathematical physics. The conference will also serve as an occasion to celebrate the achievements and impacts of the mathematician D.H. Phong. Phong has worked on a wide range of fields which span many topics within analysis, complex geometry and mathematical physics. In particular he has made fundamental contributions in areas including: the $\overline{\partial}$-Neumann problem, singular oscillatory integrals, pseudodifferential operators, the complex Monge-Ampère equation, Kähler-Einstein geometry, the Ricci flow, as well as important work in areas of mathematical physics such as string theory, conformal field theory, integrable systems, and Seiberg-Witten theory.
Information: `http://math.columbia.edu/phong2013`.

* 20–24 **Quasiconformal Geometry and Elliptic PDEs**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.
Description: The theories of quasiconformal mappings and elliptic partial differential equations have classical connections dating back the work of Vekua, Bers, Bojarski, and others. During the last ten years these connections have been revitalized through new methods and breakthroughs and surprising applications that merge geometric and analytic methods. These include the solution of Calderon's problem of impedance tomography in the plane by Astala and Paivarinta. Current research suggests that the methods of geometric analysis will also be applicable to problems in materials sciences, such as in elasticity and stochastic homogenization. Another development is the extension of the theory to degenerate elliptic equations, through the work of David, Iwaniec, Koskela, Martin, and many others. Here the geometric counterpart is the theory of mappings of finite distortion, which played a vital role in recent work on random geometry. The workshop will also study the conformal geometry related open problem.
Information: `http://www.ipam.ucla.edu/programs/iagws4/`.

* 21–24 **Questions, Algorithms, and Computations in Abstract Group Theory**, University of Braunschweig, Germany.
Description: Our aim is to combine researchers from the areas of group theory, computer science and algebraic geometry to obtain new advances in the algorithmic theory of abstract groups.
Speakers: Jack Button (Cambridge), Bob Gilman (New York), Derek Holt (Warwick), Jim Howie (Edinburgh), Olga Kharlampovich (Montreal), Martin Kreuzer (Passau), Markus Lohrey (Leibzig), Sarah Rees (Newcastle), Saul Schleimer (Warwick), Ben Steinberg (Ottawa and CUNY).
Information: `http://www.icm.tu-bs.de/~beick/conf/qac.html`.

* 26–June 27 **Topological, Symplectic and Contact Spring in Toulouse**, Université Paul Sabatier, Toulouse, France.
Description: The following mathematical events will take place in and around Toulouse from the end of May to the end of June. They are focused on topology and symplectic geometry. (1) Introductory week (May 27 to May 31), (2) GESTA workshop (June 3 to June 7), break(1 week), (3) Summer school on Donaldson hypersurfaces (June 17 to June 21), (4) Toulouse Topology Conference on the occasion of Michel Boileau's 60th birthday (June 24 to June 28).
Information: `http://www.math.univ-toulouse.fr/~barraud/Juin2013/`.

* 27–31 **Control, index, traces and determinants: The journey of a probabilist. A conference related to the work of Jean-Michel Bismut**, Amphithéâtre Lehman Bâtiment 200 Université Paris-Sud F-91405, Orsay, France.
Description: The conference is on the occasion of Jean-Michel Bismut's 65th birthday.
Information: `http://www.math.u-psud.fr/~repsurf/ERC/Bismutfest/Bismutfest.html`.

* 27–31 **Introductory week to the Topological, Symplectic and Contact Spring in Toulouse**, Université Paul Sabatier, Toulouse, France.
Description: The introductory week is targeted at Ph.D. students and nonspecialist mathematicians, but is of course open to everyone. It consists of 4 lecture series introducing to subjects in low dimensional topology, and symplectic geometry: Felix Schlenk (Université de Neuchâatel, symplectic packing); Michèle Audin (Université de Strasbourg, introduction to symplectic geometry, examples); Steve Boyer (UQAM, Montréal, low dimensional topology); Juan Souto (University of British Columbia, Vancouver, TBA).
Information: `http://www.math.univ-toulouse.fr/~barraud/Juin2013`.

June 2013

* 2–8 **8th Spring School on Analysis**, Paseky nad Jizerou, Czech Republic.
Invited speakers and titles: Lars Diening, Lipschitz truncation, Discrete Sobolev spaces; Javier Duoandikoetxea, Forty years of Muckenhoupt weights; Carlos Perez, Singular integrals and weights: a modern introduction; Vladimir Stepanov, Integral operators on the semiaxis: boundedness, compactness and estimates of the characteristic numbers.
Organizers: Jaroslav Lukes, Lubos Pick, Petr Posta.
Deadlines: For a reduced fee or support: February 1, 2013; email: `paseky@karlin.mff.cuni.cz`.
Information: `http://www.karlin.mff.cuni.cz/katedry/kma/ss/jun13/`.

* 3–7 **GESTA 2013 (Topological, Symplectic and Contact Spring in Toulouse)**, Université Paul Sabatier, Toulouse, France.

Description: GESTA is an acronym for Geometría Simpléctica con Técnicas algebraicas (Symplectic Geometry with Algebraic Techniques). It is the name of a Spanish group of mathematicians interested in symplectic geometry, algebraic geometry and mathematical physics. The group organizes a workshop every year. The workshop 2013 will be held in Toulouse, and is of course open to everyone.
Courses: Four minicourses will be taught during this workshop on topics covering the different aspects of symplectic geometry in a wide sense. Joel Fine (Université Libre de Bruxelles); Viktor Ginzburg (University of California, Santa Cruz); Emmy Murphy (Massachusetts Institute of Technology); Nguyen Tien Zung (Université de Toulouse III).
Language: All courses will be taught in English.
Funding: Limited funding for graduate students will be available.
Information: `http://www.math.univ-toulouse.fr/~barraud/Juin2013`.

* 17–21 **Summer school on Donaldson hypersurfaces (in Topological, Symplectic and Contact Spring in Toulouse)**, La Llagonne, France.
Description: The aim of this summer school is the study of Donaldson's construction of symplectic hypersurfaces and Lefschetz pencils. During the morning sessions, the sections of Donaldson's original papers will be explained by participants. The afternoons are dedicated to ramifications of these techniques in contact geometry (open books techniques) and symplectic geometry (through the study of Lagrangian submanifolds from the Lagrangian cobordism point of view): Emmanuel Giroux (ENS de Lyon) will give lectures about applications to contact geometry. Octav Cornea (Université de Montréal) will present Lagrangian cobordisms and Fukaya category.
Information: `http://www.math.univ-toulouse.fr/~barraud/Juin2013`.

* 17–28 **Algebraic Graph Theory**, University of Wyoming, Laramie, Wyoming.
Description: The purpose of this summer program is to introduce participants to topics in algebraic graph theory, including spectral graph theory, association schemes, and graph polynomials, with applications to quantum walks, error correcting codes, designs, the Potts model, and DNA self-assembly. Each of the three main speakers will give a total of ten lectures. No specialized prior knowledge will be assumed. In the afternoons there will be contributed talks by participants on their research.
Sponsors: Rocky Mountain Mathematics Consortium.
Funding: Additional funding is pending.
Aims: To prepare participants to pursue open problems in the subject, and build ties between young researchers and graduate students.
Speakers: Chris Godsil (University of Waterloo); William J. Martin (Worcester Polytechnic Institute); Joanna Ellis-Monaghan (St. Michael's College).
Deadline: For applications/abstracts of talks: April 5, 2013.
Contact: Jason Williford; `jwillif1@uwyo.edu`.
Information: `http://www.uwyo.edu/jwilliford/rmmc2013/rmmc_2013.html`.

* 17–August 9 **SUMMERICERM: 2013 Undergraduate Summer Research Program Geometry and Dynamics**, ICERM, Providence, Rhode Island.
Description: The 2013 SummerICERM program is designed for a select group of 10-12 undergraduate scholars. Students will work in small groups of two or three, supervised by a faculty advisor and aided by a teaching assistant. A variety of activities around various research themes will allow participants to engage in collaborative research, communicate and examine their findings in formal and informal settings, and report-out their findings with a finished product.
Information: `http://icerm.brown.edu/summerug_2013`.

* 19–21 **Nonlinear Elliptic and Parabolic Partial Differential Equations**, Dipartimento di Matematica, Politecnico di Milano, Italy.
Description: The purpose of the workshop is to bring together specialists in the field of Elliptic and Parabolic Nonlinear Partial Differential Equations in order to emphasize future theoretical trends and their applications. Among other topics, the workshop will focus on qualitative properties of solutions to elliptic equations, regularity of solutions, asymptotic behavior of solutions to parabolic equations, new equations from mathematical physics and biology. The workshop also aims to be a good state of art information for young researchers.
Invited speakers: Lucio Boccardo (Roma), José Antonio Carrillo (London), Jean Dolbeault (Paris), Alessio Figalli (Austin)-to be confirmed, Marek Fila (Bratislava), Ugo Gianazza (Pavia), Hans-Christoph Grunau (Magdeburg), Giuseppe Mingione (Parma), Enzo Mitidieri (Trieste), Filomena Pacella (Roma), Patrizia Pucci (Perugia), Bernhard Ruf (Milano), Sandro Salsa (Milano), Philippe Souplet (Paris), Guido Sweers (Köln), Susanna Terracini (Torino), Juan Luis Vazquez (Madrid).
Information: `http://www.mate.polimi.it/nep2de/`.

* 19–22 **4th International conference "Nonlinear Dynamics-2013"**, Sevastopol, Ukraine.
Description: The objective of the Conference is to bring together scientists and engineers to present and discuss recent developments on the different problems of nonlinear dynamics.
Main topics: Analytical and numerical methods in nonlinear dynamics resonances, stability analysis and bifurcations in nonlinear systems, nonlinear normal modes, transient, localization of energy chaotic dynamics, nonlinear dynamics of continuous systems, in particular, plates and shells vibro-impact systems and other nonsmooth systems, nonlinear dynamics of structures and machines and other problems of engineering applications, mini-symposium creep and plasticity at cyclic loading will also take place in the framework of the conference.
Organizer: By NTU "KhPI" (Ukraine) and McGill University (Canada).
Information: `http://sites.google.com/site/ndkhpi2013/home`.

* 24–26 **Numerical Analysis and Scientific Comlputation with Applications (NASCA13)**, University of Littoral Cote d'Opale, Calais, France.
Description: The conference is organized by the LMPA "Laboratory of Pure and Applied Mathematics" and the Engineering School EIL Côte d'Opale.
Aim: To bring together researchers working in numerical analysis, scientific computation and applications. Participants will present and discuss their latest results in this area.
Topics: Large linear systems and eigenvalue problems with preconditioning, linear algebra and control, model reduction, ill-posed problems, regularisations, numerical mthods for PDEs, approximation theory, radial basis functions, meshless approximation, optimization, applications to image and signal processing, environment, energy minimization, internet search engines.
Information: `http://www-lmpa.univ-littoral.fr/NASCA13/`.

* 24–28 **Low-dimensional Topology and Geometry in Toulouse**, Université Paul Sabatier, Institut de Mathématiques de Toulouse, Toulouse, France.
Description: On the occasion of Michel Boileau's 60th birthday. This area has shown rapid and spectacular progress over the last few years (the Poincaré conjecture, Thurston's geometrisation conjecture, the classification of hyperbolic manifolds, Marden's conjecture, infinite virtual Betti numbers, the virtual fibration conjecture, Heegaard-Floer homology, quantum invariants, curve complexes...). Among them, the conference will address low-dimensional topology (especially dimension 3), hyperbolic and differential geometry, knot theory, applications of geometric group theory and representation

theory. This conference is the last week of a thematic month dedicated to topology and symplectic/contact geometry.
Information: http://www.math.univ-toulouse.fr/top-geom-conf-2013/en/ldtg-mb/.

July 2013

* 8–12 **2013 SIAM Annual Meeting (AN13)**, Town and Country Resort & Convention Center, San Diego, California
Description: SIAM's Annual Meeting provides a broad view of the state of the art in applied mathematics, computational science, and their applications through invited presentation, prize lectures, minisymposia, and contributed papers and posters.
Information: http://www.siam.org/meetings/an13/.

* 8–12 **Topics in Numerical Analysis for Differential Equations**, Instituto de Ciencias Matemáticas-ICMAT, campus de Cantoblanco, Madrid, Spain.
Description: The meeting aims at bringing together leading experts working in numerical analysis of differential equations and related areas. It is composed of five sessions. Each session of the workshop is organized around a specific topic which has attracted a great deal of current research. The background and current status of each field is introduced in a plenary talk by a leading expert.
Topics: Includes the followings fields and plenary speakers: Geometric Numerical Integration, Jes Sanz-Serna (Valladolid); Highly Oscillatory Problems, Arieh Iserles (Cambridge); Discrete Mechanics and Control Theory, David Martín de Diego (ICMAT-CSIC); Splitting Methods and Time Integration of PDEs, Christian Lubich (Tübingen); Algebraic Structures in Numerical Integration, Hans Munthe-Kaas (Bergen).
Information: http://www.icmat.es/congresos/tnade2013/index.html.

* 14–19 **The Sixth International Congress of Chinese Mathematicians (ICCM)**, Opening ceremony on July 14 in the Big Hall of the Grand Hotel, Taipei, Taiwan. Lectures and invited talks from July 15–19 on the campus of National Taiwan University, Taipei, Taiwan.
Description: ICCM is a triennial event that brings together Chinese and overseas mathematicians to discuss the latest research developments in pure and applied mathematics. During the opening of the ICCM 2013, you will witness the presentation of the Morningside Medals, the most prestigious awards for Chinese mathematicians. During the conference, there will be at least fifteen plenary talks, several Morningside talks by international speakers, and over one hundred 45-minute talks. The talks cover a full range of subjects in mathematical sciences, from number theory, geometry, differential equations to statistics and bio-mathematics. The first Congress was held in December 1998 in Beijing. The second Congress took place in Taipei 2001, then in Hong Kong, in Hangzhou, and 2010 in Beijing again. After twelve years, the ICCM 2013 will return to Taipei. We expect that there will be about 1,000 participants and nearly 200 presentations on a broad spectrum of mathematical sciences.
Information: http://iccm.tims.ntu.edu.tw/.

* 15–19 **ICERM IdeaLab 2013: Weeklong Program for Postdoctoral Researchers**, ICERM, Providence, Rhode Island.
Description: The Idea-Lab invites 20 postdoctoral researchers to the institute for a week during the summer. The program will start with brief participant presentations on their research interests in order to build a common understanding of the breadth and depth of expertise. Throughout the week, two or more leading senior researchers will give comprehensive overviews of their research topics. Organizers will create smaller teams of participants who will discuss, in depth, these research questions, obstacles, and possible solutions. At the end of the week, the teams will prepare presentations on the problems at hand and solution ideas. These will be shared with a broad audience including invited program officers from funding agencies.
Information: http://icerm.brown.edu/idealab_2013.

* 31–August 10 **Workshop and Conference on the Topology and Invariants of Smooth 4-Manifolds (First Announcement)**, University of Minnesota, Twin Cities, Minnesota.
Description: The main focus of this workshop will be on the topology and invariants of smooth 4-manifolds. The workshop is a part of the FRG: Collaborative Research: Topology and Invariants of Smooth 4-Manifolds. It is funded by NSF Focused Research Grant DMS-1065955. There will be one week of mini-courses, primarily for graduate students, followed by a conference during the second week.
Information: http://www.math.umn.edu/~akhmedov/UMN2013.

August 2013

* 26–30 **Geometric Function Theory and Applications 2013**, Isik University, Campus of Sile, Istanbul, Turkey.
Description: Dedicated to Professor S. Kahramaner on the occasion of her 100th birthday! The term Complex Function Theory signifies a vast literature with several correspondences among a number of different areas in mathematics. Of these, Univalent Function Theory occupies a special place having a long tradition.
Purpose: The main purpose of the symposium is to bring together leading experts as well as young researchers working on topics mainly related to Univalent and Geometric Function Theory and to present their recent work to the mathematical community.
Main Topics: Univalent function theory, differential subordination, quasiconformal mappings, fractional calculus.
Language: The official language of the symposium is English.
Information: http://gfta.isikun.edu.tr.

* 28–September 2 **Gelfand Centennial Conference: A View of 21st Century Mathematics**, Massachusetts Institute of Technology, Cambridge, Massachusetts.
Description: This conference will bring together mathematicians working in many fields, to review some of the most significant advances in mathematics in the 21st century, and commemorate I. M. Gelfand, a giant of modern mathematics, on the occasion of his Centennial.
Speakers: R. Bezrukavnikov (MIT), M. Bhargava (Princeton), V. Drinfeld* (University of Chicago), M. Emerton* (University of Chicago), D. Gaitsgory (Harvard), A. Goncharov (Yale), M. Hopkins (Harvard), M. Kontsevich (IHES), J. Lurie (Harvard), G. Margulis (Yale), D. McDuff (Columbia), C. Moeglin (Jussieu), N. Nekrasov (IHES and Simons Institute), A. Okounkov (Columbia), D. Orlov (Steklov Institute), R. Rouquier (Oxford), P. Sarnak (Princeton), P. Seidel (MIT), S. Smirnov* (University of Geneva), Y.-T. Siu (Harvard), E. Urban (Columbia), A. Venkatesh (Stanford), C. Voisin (Jussieu), E. Witten* (IAS). * tentatively scheduled.
Information: http://math.mit.edu/conferences/Gelfand/index.php.

September 2013

* 15–20 **ICERM Workshop: Exotic Geometric Structures**, ICERM, Providence, Rhode Island.
Description: This workshop will focus on recent advances in the study of geometric structures and their associated group representations. As well as featuring hyperbolic structures, the workshop will also consider more exotic structures, such as projective structures, complex hyperbolic and spherical CR-structures and locally homogeneous space-times. A related focus includes aspects of coarse or non-positively curved geometry such as Gromov hyperbolic spaces and CAT(0) complexes. We will explore the interaction between experimental evidence and rigorous proof.
Information: http://icerm.brown.edu/sp-f13-w1.

October 2013

* 15–19 **VII Moscow International Conference on Operations Research (ORM2013)**, Dorodnicyn Computing Center of RAS (CC of RAS) and Lomonosov Moscow State University (MSU), Moscow, Russian Federation.
Description: The conference will bring together scientists from all over the world to discuss theoretical aspects and various applications of operations research.
Language: Working language of the conference is English. Some sections might be in Russian.
Information: `http://io.cs.msu.su/`.

* 21–25 **ICERM Workshop: Topology, Geometry and Group Theory, Informed by Experiment**, ICERM, Providence, Rhode Island.
Description: The mathematical focus of this workshop will include all aspects of the topology and geometry of low-dimensional manifolds and geometric group theory. It has been understood for over a century that these subjects are tightly connected, but the connections have become even deeper as the subjects have matured. Recent advances have given dramatic evidence of this. The workshop aims to further extend the interplay between these subjects.
Information: `http://icerm.brown.edu/sp-f13-w2`.

November 2013

* 18–22 **ICERM Workshop: Geometric Structures in Low-Dimensional Dynamics**, ICERM, Providence, Rhode Island.
Description: This workshop will present topics in low-dimensional dynamics such as billiards, flows on flat surfaces, dynamics on moduli spaces, and piecewise isometric maps. One theme in the workshop will be the appearance of geometric structures such as hyperbolic space and Teichmüller space in connection with dynamical systems which are basically defined in terms of the Euclidean plane. Computer experiments are common in these areas, and will be discussed, but the emphasis will be on the mathematics that comes out of the experiments.
Information: `http://icerm.brown.edu/sp-f13-w3`.

The following new announcements will not be repeated until the criteria in the next to the last paragraph at the bottom of the first page of this section are met.

February 2014

* 3–May 9 **ICERM Semester Program on "Network Science and Graph Algorithms"**, ICERM, Providence, Rhode Island.
Description: The study of computational problems on graphs has long been a central area of research in computer science. However, recent years have seen qualitative changes in both the problems to be solved and the tools available to do so. Application areas such as computational biology, the Web, social networks, and machine learning give rise to large graphs and complex statistical questions that demand new algorithmic ideas and computational models. At the same time, techniques such as semidefinite programming and combinatorial preconditioners have been emerging for addressing these challenges. There will be four international conferences associated with this program, including an applications-oriented opening event.
Information: `http://icerm.brown.edu/sp-s14`.

July 2014

* 14–August 8 **Theory of Water Waves**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom.
Description: Water waves impact every aspect of life on the planet. At smaller length scales the ripples driven by surface tension affect remote sensing. At intermediate length scales waves in the mid-ocean affect shipping and near the shoreline they control the coastal morphology and the ability to navigate along shore. At larger length scales waves such as tsunamis and hurricane-generated waves can cause devastation on a global scale. Across all length scales an exchange of momentum and thermal energy between ocean and atmosphere occurs affecting the global weather system and the climate. From a mathematical viewpoint water waves pose rich challenges. New methodologies are emerging and computational approaches are becoming much more sophisticated.
Themes: Covered in this conference include: The initial-value problem (IVP); Existence and classification of waves; Linear and nonlinear stability of waves; Dynamical systems and geometric techniques; Beyond irrotational flow.
Information: `http://www.newton.ac.uk/programmes/TWW/`.

* 21–August 15 **Quantum Control Engineering: Mathematical Principles and Applications**, Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom.
Description: We are currently entering a new technological era in which we are able to build systems whose performance is limited by quantum physical effects and in which it may be possible to exploit non-classical phenomena in novel ways. To this end, there has been considerable recent interest in engineering quantum systems and at the heart of this is the development of a quantum control theory dedicated to extending classical control to the quantum domain. Examples already utilizing control of one sort or another include quantum electromechanical systems, quantum dots, cooper-pair boxes, superconducting interference devices, ion traps, as well as a large selection of optical devices. It is clear that a mathematical framework is essential for the future development of quantum control as an engineering discipline. The aim of the programme is to bring together experimentalists and theoreticians working in quantum engineering to identify the core mathematical issues and challenges ahead.
Information: `http://www.newton.ac.uk/programmes/QCE/`.

September 2014

* 2–5 **Introductory Workshop: Geometric Representation Theory**, Mathematical Sciences Research Institute, Berkeley, California.
Description: Geometric Representation Theory is a very active field, at the center of recent advances in Number Theory and Theoretical Physics. The principal goal of the Introductory Workshop will be to provide a gateway for graduate students and new post-docs to the rich and exciting, but potentially daunting, world of geometric representation theory. The aim is to explore some of the fundamental tools and ideas needed to work in the subject, helping build a cohort of young researchers versed in the geometric and physical sides of the Langlands philosophy.
Information: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9804`.

January 2015

* 12–May 22 **Dynamics on Moduli Spaces of Geometric Structures Program**, Mathematical Sciences Research Institute, Berkeley, California.
Description: The program will focus on the deformation theory of geometric structures on manifolds, and the resulting geometry and dynamics. This subject is formally a subfield of differential geometry and topology, with a heavy infusion of Lie theory. Its richness stems from close relations to dynamical systems, algebraic geometry, representation theory, Lie theory, partial differential equations, number theory, and complex analysis.
Information: `http://www.msri.org/web/msri/scientific/programs/show/-/event/Pm9002`.

New Publications Offered by the AMS

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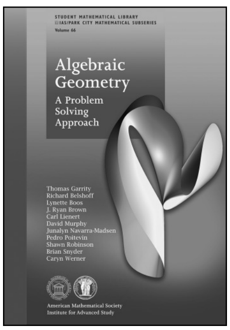
Algebra and Algebraic Geometry



Algebraic Geometry

A Problem Solving Approach

Thomas Garrity, *Williams College, Williamstown, MA*, **Richard Belshoff**, *Missouri State University, Springfield, MO*, **Lynette Boos**, *Providence College, RI*, **J. Ryan Brown**, *Georgia College and State University, Milledgeville, GA*, **Carl Lienert**, *Fort Lewis College, Durango, CO*, **David Murphy**, *Hillsdale College, MI*, **Junalyn Navarra-Madsen**, *Texas Woman's University, Denton, TX*, **Pedro Poitevin**, *Salem State University, MA*, **Shawn Robinson**, *Colorado Mesa University, Grand Junction, CO*, **Brian Snyder**, *Lake Superior State University, Sault Ste. Marie, MI*, and **Caryn Werner**, *Allegheny College, Meadville, PA*

Algebraic Geometry has been at the center of much of mathematics for hundreds of years. It is not an easy field to break into, despite its humble beginnings in the study of circles, ellipses, hyperbolas, and parabolas.

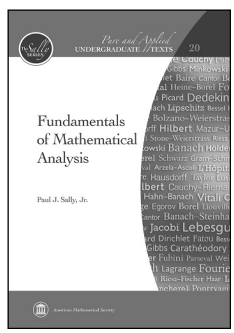
This text consists of a series of exercises, plus some background information and explanations, starting with conics and ending with sheaves and cohomology. The first chapter on conics is appropriate for first-year college students (and many high school students). Chapter 2 leads the reader to an understanding of the basics of cubic curves, while Chapter 3 introduces higher degree curves. Both chapters are appropriate for people who have taken multivariable calculus and linear algebra. Chapters 4 and 5 introduce geometric objects of higher dimension than curves. Abstract algebra now plays a critical role, making a first course in abstract algebra necessary from this point on. The last chapter is on sheaves and cohomology, providing a hint of current work in algebraic geometry.

Contents: Conics; Cubic curves and elliptic curves; Higher degree curves; Affine varieties; Projective varieties; The next steps: Sheaves and cohomology; Bibliography; Index.

Student Mathematical Library, Volume 66

February 2013, approximately 353 pages, Softcover, ISBN: 978-0-8218-9396-8, LC 2012037402, 2010 *Mathematics Subject Classification:* 14-01, **AMS members US$42.40**, List US$53, Order code STML/66

Analysis



Fundamentals of Mathematical Analysis

Paul J. Sally, Jr., *University of Chicago, IL*

This is a textbook for a course in Honors Analysis (for freshman/sophomore undergraduates) or Real Analysis (for junior/senior undergraduates) or Analysis-I (beginning graduates). It is intended for students who completed a course in "AP Calculus", possibly followed by a routine course in multivariable calculus and a computational course in linear algebra.

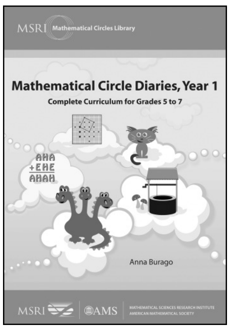
There are three features that distinguish this book from many other books of a similar nature and which are important for the use of this book as a text. The first, and most important, feature is the collection of exercises. These are spread throughout the chapters and should be regarded as an essential component of the student's learning. Some of these exercises comprise a routine follow-up to the material, while others challenge the student's understanding more deeply. The second feature is the set of independent projects presented at the end of each chapter. These projects supplement the content studied in their respective chapters. They can be used to expand the student's knowledge and understanding or as an opportunity to conduct a seminar in Inquiry Based Learning in which the students present the material to their class. The third really important feature is a series of challenge problems that increase in impossibility as the chapters progress.

Contents: The construction of real and complex numbers; Metric and Euclidean spaces; Complete metric spaces; Normed linear spaces; Differentiation; Integration; Fourier analysis on locally compact abelian groups; Sets, functions, and other basic ideas; Linear algebra; Bibliography; Index of terminology; Index of notation definitions.

Pure and Applied Undergraduate Texts, Volume 20

March 2013, approximately 370 pages, Hardcover, ISBN: 978-0-8218-9141-4, 2010 *Mathematics Subject Classification:* 15-01, 22B05, 26-01, 28-01, 42-01, 43-01, 46-01, **AMS members US$59.20**, List US$74, Order code AMSTEXT/20

General Interest



Mathematical Circle Diaries, Year 1

Complete Curriculum for Grades 5 to 7

Anna Burago, *Prime Factor Math Circle, Seattle, WA*

Early middle school is a great time for children to start their mathematical circle education. This time is a period of curiosity and openness to learning. The thinking habits and study skills acquired by children at this age stay with them for a lifetime. Mathematical circles, with their question-driven approach and emphasis on creative problem-solving, have been rapidly gaining popularity in the United States. The circles expose children to the type of mathematics that stimulates development of logical thinking, creativity, analytical abilities and mathematical reasoning. These skills, while scarcely touched upon at school, are in high demand in the modern world.

This book contains everything that is needed to run a successful mathematical circle for a full year. The materials, distributed among 29 weekly lessons, include detailed lectures and discussions, sets of problems with solutions, and contests and games. In addition, the book shares some of the know-how of running a mathematical circle. The curriculum, which is based on the rich and long-standing Russian math circle tradition, has been modified and adapted for teaching in the United States. For the past decade, the author has been actively involved in teaching a number of mathematical circles in the Seattle area. This book is based on her experience and on the compilation of materials from these circles.

The material is intended for students in grades 5 to 7. It can be used by teachers and parents with various levels of expertise who are interested in teaching mathematics with the emphasis on critical thinking. Also, this book will be of interest to mathematically motivated children.

In the interest of fostering a greater awareness and appreciation of mathematics and its connections to other disciplines and everyday life, MSRI and the AMS are publishing books in the Mathematical Circles Library series as a service to young people, their parents and teachers, and the mathematics profession.

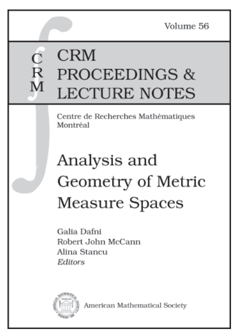
Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

Contents: Preliminaries; *Session plans:* Introduction; How to solve a problem; Knights and liars; How to turn lies into truth; Mathematical auction; Word problems and common sense; More word problems; Odd and even numbers I. Magic paper cups; Odd and even numbers II. Definitions and properties; Halloween math hockey I; Odd and even numbers III. Alternations; Weighings an counterfeit coins; Mathematical olympiad I; Meet the cube. First lesson in 3d geometry; Cross sections. Second lesson in 3d geometry; Mathematical auction; Combinatorics I; Combinatorics II; Mathematical hockey II; Numerical puzzles I. Runaway digits; Numerical puzzles II. Encrypted problems; Mathematical olympiad II; Divisibility I. Definition and properties; Divisibility II. Prime numbers and prime factorization; Mathematical auction; Divisibility III. Divisibility rules; Divisibility IV. Relatively prime numbers; Mathematical games of strategy I; Mathematical games of strategy II; Mathematical olympiad III; *Mathematical contests and competitions:* Mathematical contests; Mathematical auction; Mathematical hockey; Mathematical olympiads; Short entertaining math games; *More teaching advice:* How to be a great math circle teacher; Math circle day-to-day; More questions?; *Solutions:* Solutions; Bibliography.

MSRI Mathematical Circles Library, Volume 11

January 2013, 335 pages, Softcover, ISBN: 978-0-8218-8745-5, 2010 *Mathematics Subject Classification:* 97A20, 97A80, 00A07, 00A08, 00A09, 97D50, **AMS members US$36**, List US$45, Order code MCL/11

Geometry and Topology



Analysis and Geometry of Metric Measure Spaces

Lecture Notes of the 50th Séminaire de Mathématiques Supérieures (SMS), Montréal, 2011

Galia Dafni, *Concordia University, Montreal, QC, Canada*, **Robert John McCann**, *University of Toronto, ON, Canada*, and **Alina Stancu**, *Concordia University, Montreal, QC, Canada*

This book contains lecture notes from most of the courses presented at the 50th anniversary edition of the Séminaire de Mathématiques Supérieure in Montréal. This 2011 summer school was devoted to the analysis and geometry of metric measure spaces, and featured much interplay between this subject and the emergent topic of optimal transportation. In recent decades, metric measure spaces have emerged as a fruitful source of mathematical questions in their own right, and as indispensable tools for addressing classical problems in geometry, topology, dynamical systems, and partial differential equations. The summer school was designed to lead young scientists to the research frontier concerning the analysis and geometry of metric measure spaces, by exposing them to a series of minicourses featuring leading researchers who highlighted both the state-of-the-art and some of the exciting challenges which remain. This volume attempts to capture the excitement of the summer school itself, presenting the reader with glimpses into this active area of research and its connections with other branches of contemporary mathematics.

This item will also be of interest to those working in differential equations, analysis, and probability and statistics.

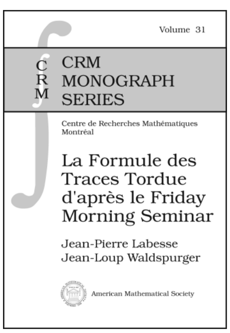
Titles in this series are co-published with the Centre de Recherches Mathématiques.

Contents: L. Ambrosio, An overview on calculus and heat flow in metric measure spaces and spaces with Riemannian curvature bounded from below; **M. T. Barlow**, Analysis on the Sierpinski carpet; **T. Coulhon**, Heat kernel estimates, Sobolev-type inequalities and Riesz transform on noncompact Riemannian manifolds; **G. David**, Regularity of minimal and almost minimal sets and cones: J. Taylor's theorem for beginners; **Y.-H. Kim**, Lectures on Ma-Trudinger-Wang curvature and regularity of optimal transport maps; **R. J. McCann** and **N. Guillen**, Five lectures on optimal transportation: Geometry, regularity and applications; **E. Milman**, A proof of Bobkov's spectral bound for convex domains via Gaussian fitting and free energy estimation; **Y. Ollivier**, A visual introduction to Riemannian curvatures and some discrete generalizations.

CRM Proceedings & Lecture Notes, Volume 56

February 2013, approximately 228 pages, Softcover, ISBN: 978-0-8218-9418-7, 2010 *Mathematics Subject Classification:* 53C23; 35-06, 53-06, 58-06, 60-06, 49Q15, 49N60, **AMS members US$79.20**, List US$99, Order code CRMP/56

Number Theory



La Formule des Traces Tordue d'après le Friday Morning Seminar

Jean-Pierre Labesse, *Institut Mathématique de Luminy, Marseille, France*, and **Jean-Loup Waldspurger**, *Institut Mathématique de Jussieu, Paris, France*

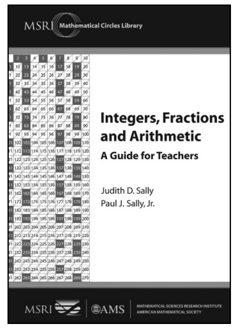
The trace formula for an arbitrary connected reductive group over a number field was developed by James Arthur. The twisted case was the subject of the Friday Morning Seminar at the Institute for Advanced Study in Princeton during the 1983–1984 academic year. During this seminar, lectures were given by Laurent Clozel, Jean-Pierre Labesse and Robert Langlands. Having been written quite hastily, the lecture notes of this seminar were in need of being revisited. The authors' ambition is to give, following these notes, a complete proof of the twisted trace formula in its primitive version, i.e., its noninvariant form. This is a part of the project of the Parisian team led by Laurent Clozel and Jean-Loup Waldspurger. Their aim is to give a complete proof of the stable form of the twisted trace formula, and to provide the background for the forthcoming book by James Arthur on twisted endoscopy for the general linear group with application to symplectic and orthogonal groups.

Titles in this series are co-published with the Centre de Recherches Mathématiques.

Contents: *Géométrie et combinatoire:* Racines et convexes; Espaces tordus; Théorie de la réduction; *Théorie spectrale, troncatures et noyaux:* Lópérateur de troncature; Formes automorphes et produits scalaires; Le noyau intégral; Décomposition spectrale; *La formule des traces grossère:* Formule des traces: état zéro; Développement géométrique; Développement spectral grossier; Formule des traces: propriétés formelles; *Forme explicite des termes spectraux:* Introduction d'une fonction B; Calcul de $A^T(B)$; Formules explicites; Bibliographie; Index des notations.

CRM Monograph Series, Volume 31

February 2013, approximately 248 pages, Hardcover, ISBN: 978-0-8218-9441-5, 2010 *Mathematics Subject Classification:* 11F72; 11R56, 20G35, **AMS members US$79.20**, List US$99, Order code CRMM/31



Integers, Fractions and Arithmetic

A Guide for Teachers

Judith D. Sally, *Northwestern University, Evanston, IL*, and **Paul J. Sally, Jr.**, *University of Chicago, IL*

This book, which consists of twelve interactive seminars, is a comprehensive and careful study of the fundamental topics of K–8 arithmetic. The guide aims to help teachers understand the mathematical foundations of number theory in order to strengthen and enrich their mathematics classes. Five seminars are dedicated to fractions and decimals because of their importance in the classroom curriculum. The standard topics are covered in detail, but are arranged in an order that is slightly different from the usual one. Multiplication is treated first, and with that in hand, common denominators and equivalent fractions are more readily understood and are available for use when discussing addition.

The book is intended for the professional development of teachers. It is appropriate for teacher education programs as well as for enrichment programs such as Mathematical Circles for Teachers. There are numerous activities in each seminar that teachers can bring into their classrooms.

In the interest of fostering a greater awareness and appreciation of mathematics and its connections to other disciplines and everyday life, MSRI and the AMS are publishing books in the Mathematical Circles Library series as a service to young people, their parents and teachers, and the mathematics profession.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

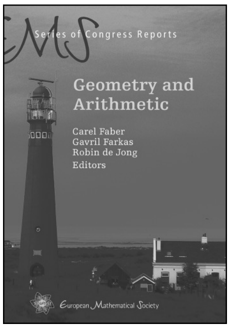
Contents: Number systems; Divisibility and order in the integers; GCD's and the division algorithm; Prime numbers and factorization into primes; Applications of prime power factorization; Modular arithmetic and divisibility tests; More modular arithmetic; The arithmetic of fractions; Properties of multiplication of fractions; Addition of fractions; The decimal expansion of a fraction; Order and the number line; The mysterious long division algorithm; The pigeonhole principle; Index.

MSRI Mathematical Circles Library, Volume 10

December 2012, 208 pages, Softcover, ISBN: 978-0-8218-8798-1, 2010 *Mathematics Subject Classification:* 11Axx, 97B50, 11-01, **AMS members US$31.20**, List US$39, Order code MCL/10

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Geometry and Arithmetic

Carel Faber, *Royal Institute of Technology, Stockholm, Sweden*, **Gavril Farkas**, *Humboldt University of Berlin, Germany*, and **Robin de Jong**, *University of Leiden, The Netherlands*, Editors

This volume contains 21 articles written by leading experts in the fields of algebraic and arithmetic geometry. The treated topics range over a variety of themes, including moduli spaces of curves and abelian varieties, algebraic cycles, vector bundles and coherent sheaves, curves over finite fields, and algebraic surfaces, among others.

The volume originates from the conference "Geometry and Arithmetic," which was held on the island of Schiermonnikoog in The Netherlands in September 2010.

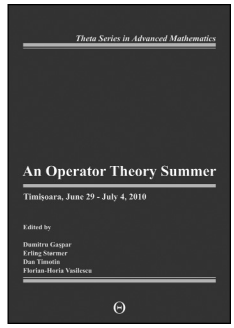
This item will also be of interest to those working in number theory.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: V. Alexeev and **D. Swinarski**, Nef divisors on $\bar{M}_{0,n}$ from GIT; **I. Bauer** and **F. Catanese**, Inoue type manifolds and Inoue surfaces: A connected component of the moduli space of surfaces with $K^2 = 7, p_g = 0$; **A. Beauville**, Non-rationality of the symmetric sextic Fano threefold; **C. Ciliberto** and **F. Flamini**, Brill-Noether loci of stable rank-two vector bundles on a general curve; **J. I. Cogolludo-Agustín** and **R. Kloosterman**, Mordell-Weil groups and Zariski triples; **J.-M. Couveignes** and **B. Edixhoven**, Approximate computations with modular curves; **F. Eusen** and **F.-O. Schreyer**, A remark on a conjecture of Paranjape and Ramanan; **A. Gibney**, On extensions of the Torelli map; **B. H. Gross**, The classes of singular moduli in the generalized Jacobian; **G. Harder**, The Eisenstein motive for the cohomology of $\mathrm{GSp}_2(\mathbb{Z})$; **J. Heinloth**, Cohomology of the moduli stack of coherent sheaves on a curve; **E. W. Howe** and **K. E. Lauter**, New methods for bounding the number of points on curves over finite fields; **H. Ito** and **S. Schröer**, Wildly ramified actions and surfaces of general type arising from Artin-Schreier curves; **T. Katsura** and **S. Kondō**, A note on a supersingular K3 surface in characteristic 2; **S. J. Kovács**, The intuitive definition of Du Bois singularities; **H. Lange** and **P. E. Newstead**, Bundles of rank 2 with small Clifford index on algebraic curves; **R. Pandharipande** and **A. Pixton**, Descendents on local curves: Stationary theory; **D. Petersen**, A remark on Getzler's semi-classical approximation; **R. Schoof**, On the modular curve $X_0(23)$; **C. Voisin**, Degree 4 unramified cohomology with finite coefficients and torsion codimension 3 cycles; **Y. G. Zarhin**, Poincaré duality and unimodularity.

EMS Series of Congress Reports, Volume 7

October 2012, 384 pages, Hardcover, ISBN: 978-3-03719-119-4, 2010 *Mathematics Subject Classification:* 14-XX, 11-XX, **AMS members US$78.40**, List US$98, Order code EMSSCR/7



An Operator Theory Summer

Timişoara, June 29–July 4, 2010

Dumitru Gaşpar, *West University of Timişoara, Romania*, **Erling Størmer**, *University of Oslo, Norway*, **Dan Timotin**, *Romanian Academy, Bucharest, Romania*, and **Florian-Horia Vasilescu**, *University of Lille I, Villeneuve d'Ascq, France*, Editors

This volume contains the proceedings of the 23rd International Conference on Operator Theory, held in Timişoara, Romania, from June 29 to July 4, 2010. It includes three survey articles on traces on a C^*-algebra, amalgamated products of C^*-bundles, and compactness of composition operators, as well as ten papers containing original research on several topics: single operator theory, C^*-algebras, moment problems, differential and integral operators, and complex function theory.

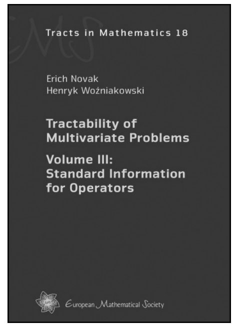
A publication of the Theta Foundation. Distributed worldwide, except in Romania, by the AMS.

Contents: A. M. Bikchentaev, Characterization of traces on C^*-algebras: A survey; **E. Blanchard**, Amalgamated products of C^*-bundles; **E. Boasso**, The Drazin spectrum in Banach algebras; **B. E. Breckner** and **C. Varga**, On problems involving the weak Laplacian operator on the Sierpinski gasket; **S. Grigorian** and **A. Kuznetsova**, On a category of nuclear C^*-algebras; **L. Lemnete-Ninulescu**, Truncated trigonometric and Hausdorff moment problems for operators; **N. Lupa** and **M. Megan**, Rolewicz type theorems for nonuniform exponential stability of evolution operators on the half-line; **M. Megan**, **T. Ceauşu**, and **L. Biriş**, On some concepts of stability and instability for cocycles of linear operators in Banach spaces; **H. Queffélec**, A tentative comparison of compactness of composition operators on Hardy-Orlicz and Bergman-Orlicz spaces; **F. Rădulescu**, A universal, non-commutative C^*-algebra associated to the Hecke algebra of double cosets; **C. Stoica** and **M. Megan**, On nonuniform exponential dichotomy for linear skew-evolution semiflows in Banach spaces; **L. Suciu** and **N. Suciu**, On the asymptotic limit of a bicontraction; **A. Totoi**, Integral operators applied on classes of meromorphic functions defined by subordination and superordination.

International Book Series of Mathematical Texts

June 2012, 148 pages, Hardcover, ISBN: 978-973-87899-8-2, 2010 *Mathematics Subject Classification:* 00B25, 46-06, 47-06, **AMS members US$28.80**, List US$36, Order code THETA/16

Applications



Tractability of Multivariate Problems

Volume III: Standard Information for Operators

Erich Novak, *University of Jena, Germany*, and **Henryk Woźniakowski**, *Columbia University, New York, NY*

This is the third volume of a three-volume set comprising a comprehensive study of the tractability of multivariate problems. The third volume deals with algorithms using standard information consisting of function values. Linear and selected nonlinear *operators* are studied.

The most important example studied in volume III is the approximation of multivariate functions. Many other linear and some nonlinear problems are closely related to the approximation of multivariate functions. While the lower bounds obtained in volume I for the class of linear information also yield lower bounds for the standard class of function values, new techniques for upper bounds are presented in volume III. One of the main issues here is to verify when the power of standard information is nearly the same as the power of linear information. In particular, for the approximation problem defined over Hilbert spaces, the power of standard and linear information is the same in the randomized and average case (with Gaussian measures) settings, whereas in the worst case setting this is not true.

The book is of interest to researchers working in computational mathematics, especially in approximation of high-dimensional problems. It may be well suited for graduate courses and seminars. The text contains 58 open problems for future research in tractability.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Contents: Examples of multivariate approximation; Randomized setting: Multivariate approximation; Randomized setting: Linear problems; Average case setting: Multivariate approximation; Average case setting: Linear problems; Worse case setting: Multivariate approximation; Worse case setting: Linear problems; Nonlinear problems; Power of function values for multivariate approximation; List of open problems; Errata for volumes I and II; Bibliography; Index.

EMS Tracts in Mathematics, Volume 18

October 2012, 604 pages, Hardcover, ISBN: 978-3-03719-116-3, 2010 *Mathematics Subject Classification:* 65-02, 65Y20, 68Q17, 68Q25, 41A63, 46E22, 65N99, 65R20, 28C20, 46E30, **AMS members US$102.40**, List US$128, Order code EMSTM/18

Classified Advertisements

Positions available, items for sale, services available, and more

CALIFORNIA

UNIVERSITY OF CALIFORNIA, LOS ANGELES
Institute for Pure and Applied Mathematics

The Institute for Pure and Applied Mathematics (IPAM) at UCLA is seeking an Associate Director (AD), to begin a two-year appointment on August 1, 2013. The AD is expected to be an active and established research mathematician or scientist in a related field, with experience in conference organization. The primary responsibility of the AD will be running individual programs in coordination with the organizing committees. More information on IPAM's programs can be found at: `http://www.ipam.ucla.edu`. The selected candidate will be encouraged to continue his or her personal research program within the context of the responsibilities to the institute. For a detailed job description and application instructions, go to: `http://www.ipam.ucla.edu/jobopenings/assocdirector.aspx`. Applications will receive fullest consideration if received by February 1, 2013, but we will accept applications as long as the position remains open. UCLA is an Equal Opportunity/Affirmative Action Employer.

000001

UNIVERSITY OF SOUTHERN CALIFORNIA
Department of Mathematics

The Department of Mathematics in the Dana and David Dornsife College of Letters, Arts and Sciences of the University of Southern California seeks to fill positions of Assistant Professor (NTT) of Mathematics with an anticipated start date of August 2013. These are non-tenure-track positions with terms of up to 3 years.

Candidates in all fields of mathematics will be considered. Candidates should demonstrate great promise in research and evidence of strong teaching, and will be required to teach three semester courses per year. Several positions are likely to be available. Applicants should have a doctoral degree in an appropriate field of study.

To apply, please submit the following materials: letter of application and curriculum vitae, including your e-mail address, telephone and fax numbers, preferably with the standardized AMS Cover Sheet. Candidates should also arrange for at least three letters of recommendation to be sent, at least one of which addresses teaching skills. Please submit applications electronically through MathJobs at `www.mathjobs.org`. As an alternative and only if necessary, materials can be mailed to:

Search Committee,
Department of Mathematics,
Dornsife College of Letters, Arts and Sciences
University of Southern California,
3620 Vermont Avenue, KAP 104,
Los Angeles, CA 90089-2532

In order to be considered for this position, applicants are also required to submit an electronic USC application; follow this job link or paste in a browser: `https://jobs.usc.edu/applicants/Central?quickFind=66668`.

Review of applications will begin December 15, 2012, and will continue until the positions are filled. Additional information about the USC Dornsife's Department of Mathematics can be found at our website: `http://dornsife.usc.edu/mathematics/`.

USC strongly values diversity and is committed to Equal Opportunity in employment. Women and men, and members of all racial and ethnic groups are encouraged to apply.

000008

GEORGIA

GEORGIA STATE UNIVERSITY
Mathematics and Statistics

Georgia State University Mathematics and Statistics Department, located in Atlanta, GA, invites applications for an assistant professor and a lecturer requiring a Ph.D. Submit applications to `http://www.mathjobs.org`. An offer of employment will be conditional upon background verification. Georgia State University is a Research University of the University System of Georgia and an EEO/AAA institution.

000009

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

The 2012 rate is \$3.50 per word with a minimum two-line headline. No discounts for multiple ads or the same ad in consecutive issues. For an additional \$10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: February 2013 issue–November 28, 2012; March 2013 issue–January 2, 2013; April 2013 issue–January 30, 2013; May 2013–February 28, 2013; June/July 2013 issue–April 26, 2013; August 2013 issue–May 29, 2013.

U.S. laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the U.S. cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to U.S. laws. Details and specific wording may be found on page 1373 (vol. 44).

Situations wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the U.S. and Canada or 401-455-4084 worldwide for further information.

Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or send email to `classads@ams.org`. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 20904. Advertisers will be billed upon publication.

MAINE

UNIVERSITY OF MAINE
Department of Mathematics & Statistics

The Department of Mathematics & Statistics invites applications for one or more tenure-track, academic-year Assistant Professorships in pure mathematics, effective September 1, 2013. Exceptionally well-qualified candidates whose research interests complement those of the department are strongly encouraged to apply.

A Ph.D. in mathematics is required by date of hire. Candidates are required to have strong commitment to and experience in teaching and research as well as excellent written and oral communication skills.

The University of Maine is the primary graduate institution in the state of Maine. The department offers BA and MA degrees.

More detailed information about the duties and responsibilities of this faculty career opportunity, required application materials, and deadline for application can be found at: `http://www.math.umaine.edu`.

Review of applications will begin December 1, 2012, and continue until the position is filled. UMaine is committed to diversity in our workforce.

The University of Maine is an Equal Employment Opportunity/Affirmative Action Employer.

000002

MASSACHUSETTS

BOSTON UNIVERSITY
Department of Mathematics and Statistics
Postdoctoral Position–Number Theory

The Department of Mathematics and Statistics, at Boston University, invites applications for a three-year postdoctoral position in Number Theory, starting July 2013 (pending budgetary approval). Strong commitment to research and teaching is essential. Submit AMS cover sheet, CV, research statement, teaching statement and at least four letters of recommendation, one of which addresses teaching, to `mathjobs.org`. Alternatively, send all material to Number Theory Postdoctoral Search Committee, Department of Mathematics and Statistics, Boston University, 111 Cummington Mall, Boston, MA 02215. Application deadline: January 15, 2013. Boston University is an Affirmative Action/Equal Opportunity Employer.

000007

MISSOURI

UNIVERSITY OF MISSOURI-KANSAS CITY (UMKC)
Department of Mathematics and Statistics

The Department of Mathematics and Statistics at the University of Missouri-Kansas City (UMKC) seeks applicants for three nine-month faculty positions to start on August 15, 2013. Two of these positions are tenure-track, both at the level of Assistant Professor, with one in statistics (00054406) and one in mathematics (00056606). The third one is a non-tenure-track full-time Teaching Assistant Professor position (00056607). For position 00054406, candidates must have a Ph.D. in statistics or in mathematics with specialization in Statistics by August 2013. For position 00056606, candidates must have a Ph.D. in mathematics with specialization in numerical analysis of PDEs by August 2013. For position 00056607, candidates must have a Ph.D. in mathematics by August 2013 and must demonstrate strong potential to succeed in teaching. Please visit: `http://www.umkc.edu/jobs` for detailed information regarding these positions. UMKC accepts online applications only. It is the fundamental policy of UMKC to provide equal opportunity regardless of race, creed, color, sex, sexual orientation, national origin, age, veteran status, or disability status in all education, employment, and contracted activities. All final candidates will be required to successfully pass a criminal background check prior to beginning employment.

000003

NEW JERSEY

PRINCETON UNIVERSITY
Program in Applied and Computational Mathematics
and the Department of Mechanical and Aerospace Engineering
School of Engineering and Applied Science

The Program in Applied and Computational Mathematics (PACM) and the Department of Mechanical and Aerospace Engineering at Princeton University invite applications for a tenure-track assistant professor position in the general field of applied mathematics in areas relevant to mechanical and aerospace engineering. We are especially interested candidates in nonlinear mechanics, but all areas of expertise relevant to mechanical and aerospace engineering will be considered. Applicants must hold a Ph.D. in mathematics, engineering, physics, or a related field, and have a demonstrated record of excellence in research with evidence of an ability to establish an independent research program. We seek faculty members who will create a climate that embraces excellence and diversity, with a strong commitment to teaching and mentoring that will enhance the work of the department and attract and retain students of all races, nationalities and genders. We strongly encourage applications from members of all underrepresented groups.

The Program in Applied and Computational Mathematics is an interdisciplinary and interdepartmental program that provides a home for people from many fields and directions, who share a passion for mathematics and its applications. Our core faculty is presently constituted by 11 faculty members, all of whom hold a joint appointment between their home department and PACM; these core faculty also function as an executive committee for PACM in all its important decisions. Drawing from a wide range of departments, from the physical sciences and engineering to the biological sciences, PACM counts an additional 48 Princeton faculty members among its associate faculty; they provide mentoring and advising to our students interested in their fields of expertise. More information can be found at: `http://www.pacm.princeton.edu`.

To ensure full consideration, applications should be received by December 1, 2012, however, we will continue to accept applications until the position is filled. Applicants should submit a curriculum vitae, including a list of publications and presentations, a summary of research accomplishments and future plans, a teaching statement, and contact information for at least three references online at `http://jobs.princeton.edu`. Personal statements that summarize teaching experience and contributions to diversity are encouraged.

Princeton University is an Equal Opportunity Employer and complies with applicable EEO and affirmative action regulations.

000004

HONG KONG

THE UNIVERSITY OF HONG KONG
MR Post-doctoral Fellow in the Department of Mathematics
(Ref.: 201200919)

Applications are invited for appointment as Postdoctoral Fellow (2-3 posts) in the Department of Mathematics, from September 1, 2013, or as soon as possible thereafter, on a two-year term and with the possibility of renewal. An earlier start date can be discussed if a successful candidate finds it desirable.

Applicants should possess a Ph.D. degree. Preference will be given to those who have obtained their Ph.D. degrees within the last 6 years and whose areas of expertise fall within the following areas of concentration: algebra and number theory; geometry and lie groups; optimization, theoretical computer science and information theory; several complex variables and complex geometry; and probability theory, stochastics and mathematical finance. They are expected to show strong potential in mathematical research. The appointees will be associated with the Institute of Mathematical Research (IMR, `http://www0.hku.hk/math/`) and will be primarily engaged in mathematical research. They will also take active roles in the activities of IMR and contribute to the academic activities as assigned by the Head of the Department of Mathematics.

A highly competitive salary commensurate with qualifications and experience will be offered. Annual leave and medical benefits will be provided.

Applicants should send a completed application form, together with a C.V. containing information on educational experience, including the title of the thesis and the name of the thesis supervisor for the Ph.D. degree, professional experience, a list of publications, a survey of past research experience, a description of current research interests and a research plan for the next few years by e-mail to `imrpostdoc@maths.hku.hk`. Please indicate clearly **Ref.: 201200919** and (Post-doctoral Fellow (2-3 posts) in the Department of Mathematics) in the subject of the email. Application forms (341/1111) can be obtained at `http://www.hku.hk/apptunit/form-ext.doc`. Further particulars can be obtained at `http://jobs.hku.hk/`. Closes January 31, 2013.

The University thanks applicants for their interest, but advises that only shortlisted applicants will be notified of the application result.

The University is an Equal Opportunity Employer and is committed to a No-Smoking Policy.

000100

SINGAPORE

NATIONAL UNIVERSITY OF SINGAPORE (NUS)
Dean, Faculty of Science

The National University of Singapore (NUS) invites applications for the position of Dean, Faculty of Science. NUS is a multi-campus university of global standing, with distinctive strengths in education and research and an entrepreneurial dimension. It has an enrollment of 26,700 undergraduate and more than 10,500 graduate students from 100 countries. NUS has three Research Centres of Excellence and 22 university-level research institutes and centres, and also enjoys a close association with 16 national-level research institutes and centres. In addition, a spirit of entrepreneurship and innovation promotes creative enterprise university-wide. NUS plays an active role in international academic networks such as the International Alliance of Research Universities (IARU) and Association of Pacific Rim Universities (APRU). It is ranked among the best universities in the world, and is well-regarded for research and teaching in multiple disciplines. For more information, please visit: `http://www.nus.edu.sg`. The NUS Faculty of Science continues to be a leading hub of research and world-class education. Established in 1929 as a single department, the Faculty has evolved to become one of the largest faculties in NUS with some 4,600 undergraduates, 1,590 graduate students and over 270 research-active academic staff. As a significant member in the global community of leading scientists engaged in fundamental as well as applied research, the Faculty hosts two of Singapore's five Research Centres of Excellence and constantly develops seamless transitions from research discovery to tomorrow's technology. The Faculty of Science embraces talent and fosters the spirit of enterprise. In our vigorous curriculum that spans across six departments, namely biological sciences, chemistry, mathematics, pharmacy, physics and statistics and applied probability, Science graduates have been known to perform exceedingly well in diverse fields and are competent drivers of Singapore's economy. For more information, please visit: `http://www.science.nus.edu.sg`. The university seeks to appoint a scholar, with distinguished academic achievements and outstanding administrative leadership quality as Dean of the Faculty of Science. The dean will lead the Faculty in all administrative and academic matters, including strategic planning, research and program development, funding and implementation of academic review process. The successful candidate is expected to be a dynamic leader with strong management and communication skills. Please submit your curriculum vitae (CV) in confidence to the Dean Search Committee at the following address, including a full list of publications, and a vision statement as well as full contacts of at least five referees, including those who can comment on your scholarship as well as leadership and management qualities. We will approach the nominated referees of shortlisted candidates for confidential reports. The application deadline is February 15, 2013. Shortlisted candidates will be invited for a campus visit.

Science Dean Search Committee
c/o Ms. Maria Chia
Faculty of Science Dean's Office
National University of Singapore
Blk S16 Level 9
6 Science Drive 2, Singapore 117546;
fax: (65) 6777-4279. Attn. to Ms. Maria Chia;
email: `scicpym@nus.edu.sg`.

000006

TAIWAN

NATIONAL CHENGCHI UNIVERSITY, TAIWAN
Department of Mathematical Sciences

The information of our teaching position is as follows: Tenure-Track Faculty financial mathematics/mathematical sciences

Requirement:

The successful candidate must hold a doctoral degree and be able to communicate effectively in Chinese.

Interested candidates should send a cover letter and indicate when he/she is available to start the job, together with the following documents.

1. CV

2. A photocopy of diploma and a photocopy of previous job contracts or appointment letters

3. A doctoral dissertation or research publications in the past five years

4. Two letters of recommendation

5. A list of the graduate/undergraduate courses (with syllabi) that the candidate can teach

6. Graduate transcripts (those who have already gotten a university teaching certificate in Taiwan may only need to submit a photocopy of the certificate in lieu of the graduate transcripts)

Screening and hiring process will begin upon receipt of applications and will continue until filled or the absolute deadline of Feburary 15, 2013.

Submit to:

Faculty Search Committee
Department of Mathematical Sciences
National Chengchi University
Wen-Shan, Taipei 11605
Taiwan;
fax: 886-2-2938-7905;
email: `math@nccu.edu.tw`.

All application materials will NOT be returned.

000005

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `http://www.ams.org/meetings/`. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 9–12, 2013

Wednesday – Saturday

Meeting #1086

Joint Mathematics Meetings, including the 119th Annual Meeting of the AMS, 96th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2012
Program first available on AMS website: November 1, 2012
Program issue of electronic *Notices*: January 2012
Issue of *Abstracts*: Volume 34, Issue 1

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: Expired

Oxford, Mississippi

University of Mississippi

March 1–3, 2013

Friday – Sunday

Meeting #1087

Southeastern Section
Associate secretary: Robert J. Daverman
Announcement issue of *Notices*: December 2012
Program first available on AMS website: December 13, 2012
Program issue of electronic *Notices*: March 2013
Issue of *Abstracts*: Volume 34, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Patricia Hersh, North Carolina State University, *An interplay of combinatorics with topology*.

Daniel Krashen, University of Georgia, *Topology, arithmetic, and the structure of algebraic groups*.

Washington Mio, Florida State University, *Taming shapes and understanding their variation*.

Slawomir Solecki, University of Illinois at Urbana-Champaign, *An abstract approach to Ramsey theory with applications*.

Special Sessions

Algebraic Combinatorics, **Patricia Hersh**, North Carolina State University, and **Dennis Stanton**, University of Minnesota.

Approximation Theory and Orthogonal Polynomials, **David Benko**, University of South Alabama, **Erwin Mina-Diaz**, University of Mississippi, and **Edward Saff**, Vanderbilt University.

Banach Spaces and Operators on Them, **Qingying Bu** and **Gerard Buskes**, University of Mississippi, and **William B. Johnson**, Texas A&M University.

Commutative Algebra, **Sean Sather-Wagstaff**, North Dakota State University, and **Sandra M. Spiroff**, University of Mississippi.

Connections between Matroids, Graphs, and Geometry, **Stan Dziobiak**, **Talmage James Reid**, and **Haidong Wu**, University of Mississippi.

Dynamical Systems, **Alexander Grigo**, University of Oklahoma, and **Saša Kocić**, University of Mississippi.

Fractal Geometry and Ergodic Theory, **Manav Das**, University of Louisville, and **Mrinal Kanti Roychowdhury**, University of Texas-Pan American.

Graph Theory, **Laura Sheppardson** and **Bing Wei**, University of Mississippi, and **Hehui Wu**, McGill University.

Modern Methods in Analytic Number Theory, **Nathan Jones** and **Micah B. Milinovich**, University of Mississippi, and **Frank Thorne**, University of South Carolina.

Set Theory and Its Applications, **Christian Rosendal**, University of Illinois at Chicago, and **Slawomir Solecki**, University of Illinois at Urbana-Champaign.

Chestnut Hill, Massachusetts

Boston College

April 6–7, 2013

Saturday – Sunday

Meeting #1088

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2013
Program first available on AMS website: February 21, 2013
Program issue of electronic *Notices*: April 2013
Issue of *Abstracts*: Volume 34, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: December 18, 2012
For abstracts: February 12, 2013

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Roman Bezrukavnikov, Massachusetts Institute of Technology, *Canonical bases and geometry*.

Marston Conder, University of Auckland, *Discrete objects with maximum possible symmetry*.

Alice Guionnet, Ecole Normale Supérieure de Lyon, *Title to be announced*.

Yanir Rubinstein, University of Maryland, *Geometry: (very) local meets global*.

Special Sessions

Algebraic and Geometric Structures of 3-manifolds (Code: SS 3A), **Ian Biringer**, **Tao Li**, and **Robert Meyerhoff**, Boston College.

Algorithmic Problems of Group Theory and Applications to Information Security (Code: SS 14A), **Delaram Kahrobaei**, City University of New York Graduate Center and New York College of Technology, and **Vladimir Shpilrain**, City College of New York and City University of New York Graduate Center.

Arithmetic Dynamics and Galois Theory (Code: SS 6A), **John Cullinan**, Bard College, and **Farshid Hajir** and **Siman Wong**, University of Massachusetts, Amherst.

Combinatorics and Classical Integrability (Code: SS 16A), **Amanda Redlich** and **Shabnam Beheshti**, Rutgers University.

Commuting Matrices and the Hilbert Scheme (Code: SS 11A), **Anthony Iarrobino**, Northeastern University, and **Leila Khatami**, Union College.

Complex Geometry and Microlocal Analysis (Code: SS 2A), **Victor W. Guillemin** and **Richard B. Melrose**, Massachusetts Institute of Technology, and **Yanir A. Rubinstein**, Stanford University.

Counting and Equidistribution on Symmetric Spaces (Code: SS 5A), **Dubi Kelmer**, Boston College, and **Alex Kontorovich**, Yale University.

Discrete Geometry of Polytopes (Code: SS 10A), **Barry Monson**, University of New Brunswick, and **Egon Schulte**, Northeastern University.

Financial Mathematics (Code: SS 18A), **Hasanjan Sayit** and **Stephan Sturm**, Worcester Polytechnic Institute.

History and Philosophy of Mathematics (Code: SS 7A), **James J. Tattersall**, Providence College, and **V. Frederick Rickey**, United States Military Academy.

Homological Invariants in Low-dimensional Topology. (Code: SS 1A), **John Baldwin**, **Joshua Greene**, and **Eli Grigsby**, Boston College.

Homology and Cohomology of Arithmetic Groups (Code: SS 13A), **Avner Ash**, Boston College, **Darrin Doud**, Brigham Young University, and **David Pollack**, Wesleyan University.

Hopf Algebras and their Applications (Code: SS 9A), **Timothy Kohl**, Boston University, and **Robert Underwood**, Auburn University Montgomery.

Moduli Spaces in Algebraic Geometry (Code: SS 4A), **Dawei Chen** and **Maksym Fedorchuk**, Boston College, and **Joe Harris** and **Yu-Jong Tzeng**, Harvard University.

Real and Complex Dynamics of Difference Equations with Applications (Code: SS 12A), **Ann Brett**, Johnson and

Wales University, and **M. R. S. Kulenovic**, University of Rhode Island.

Recursion and Definability (Code: SS 15A), **Rachel Epstein**, Harvard University, **Karen Lange**, Wellesley College, and **Russell Miller**, Queens College and City University of New York Graduate Center.

Research by Undergraduates and Students in Post-Baccalaureate Programs (Code: SS 8A), **Chi-Keung Cheung**, Boston College, **David Damiano**, College of the Holy Cross, **Steven J. Miller**, Williams College, and **Suzanne L. Weekes**, Worcester Polytechnic Institute.

Topology and Generalized Cohomologies in Modern Condensed Matter Physics (Code: SS 17A), **Claudio Chamon** and **Robert Kotiuga**, Boston University.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include hotel tax. Participants must state that they are with the **American Mathematical Society (AMS) Meeting at Boston College** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

The Inn at Longwood Medical 342 Longwood Ave., Boston, MA, 02115, 617-731-4700; `www.innatlongwood.com`. Rates are US$179 per night for single/double occupancy. Please note that hotel tax in Boston is 14.45%. Amenities include complimentary in-room coffee, free local calls, high speed DSL internet, and wireless Internet. Affordable covered and secured parking is located in the garage adjacent to the hotel lobby. The Inn is also connected to the Longwood Galleria, which contains over 20 shops and eateries. This property is located approximately 4 miles from the campus by car and approximately 1 hour by public transportation ("T"). Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 3:00 p.m.; checkout time: 12:00 p.m. **The deadline for reservations at this rate is March 15, 2013.**

Holiday Inn Boston-Brookline, 1200 Beacon St., Brookline, MA, 02134, 617-277-1200; `www.holidayinn.com/bos-bookline`. Rates are US$149 per night for single/double occupancy. Please note that hotel tax in Brookline is 11.7%. Amenities include complimentary wireless Intenet access, indoor heated saltwater pool and hot tub, 24-hour business center, fitness center, full service restaurant and lounge, climate-controlled parking garage, indoor putting green, and giant chess board. This property is located approximately 2 miles from the campus by car and approximately 1 hour by public transportation ("T"). Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 4:00 p.m. and checkout time is 11:00 a.m. **Reservations can be made at this rate at anytime until the dates of meeting, subject to room availablity.**

Hotel Indigo, 339 Grove Street, Newton, MA, 02462, 617-969-5300; `www.newtonboutiquehotel.com`. Rates are US$129 per night for single/double occupancy. Please note that hotel tax in Newton is 11.7%. Amenities include complimentary parking, fitness center, complimentary business center, complimentary shuttle service within a five-mile radius of the hotel, and free wired and wireless Internet in guest rooms. This property is located approximately one mile from the campus by car and approximately 50 minutes by by public transportation ("T"). Cancellation and early checkout policies vary; be sure to check when you make your reservation. **The deadline for reservations at this rate is February 18, 2013.**

Crowne Plaza, 320 Washington Street, Newton, MA, 02458, 617-969-3010; `www.crowneplaza.com/hotels/us/en/newton`. Rates are US$159 per night for single occupancy. Please note that hotel tax in Newton is 11.7%. Amenities include complimentary in-room coffee, business center, indoor pool, fitness room, and Internet in guest rooms for a fee. Parking is available at the Gateway Parking Garage adjacent to the hotel—cash only payment at US$4 per hour, not to exceed US$28 in a day. Overnight registered parking is US$9 per night and can be billed to your guest room. There is a restaurant on property. This property is located approximately 3 miles from the campus by car and approximately one hour by public transportation ("T"). Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 3:00 p.m. and checkout time is 12:00 p.m. **The deadline for reservations at this rate is March 5, 2013.**

The Boston Park Plaza Hotel and Towers, 50 Park Plaza at Arlington St., Boston, MA, 02116, 800-225-2008 (toll free); `www.BostonParkPlaza.com`. Rates are US$195 per night for single/double occupancy. Amenities include concierge service, room service, business center, valet parking (available for US$46), self parking (available for US$31 per day), and Internet is available in guest rooms for a fee. This property is pet friendly; there is an additional US$50 per stay cleaning fee assessed for guests with pets. This property is located approximately 7 miles from the campus by car and approximately 45 minutes by public transportation ("T"). Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is at 3:00 p.m.; checkout time is 12:00 p.m. The deadline for reservations at this rate is **March 6, 2013.**

Food Services

On Campus: There are two options on campus that will be open to meeting participants. On the Middle Campus located in McElroy Hall are *Carney's,* serving meals and snacks between 8:00 a.m.–8:00 p.m., and *On the Fly,* a mini mart offering beverages, snack foods, microwavable entrees, ice cream, and grab-n-go items (credit cards only), location open between 10:00 a.m.–10:00 p.m.

Off Campus: There are many choices for dining convenient to campus in Newton and Brookline via a short walk, car, or cab ride from campus.

Dining close to the BC "T" Stop:

Flat Breads Cafe/Sweet Basil Pizza and Pasta Company, 11 Commonwealth Ave, Newton, 617-964-3516; offering a wide selection of gourmet wraps and Italian cuisine made to order.

Appetito's Restaurant, 761 Beacon St., Newton Centre, 617-244-9881; offering Italian cuisine.
Crazy Dough, 2201 Commonwealth Ave., Brighton, 617-202-5769; offering award-winning pizza.
El Pelon, 2197 Commonwealth Ave., Brighton, 617-779-9090; offering Mexican cuisine.
Eagle's Deli, 1918 Beacon St., Brighton, 617-731-3232; deli featured on the *Travel Channel,* specializing in sandwiches and burgers.
Dining close to the Chestnut Hill "T" **Stop:**
Legal Sea Foods, 43 Boylston Street, Chestnut Hill, 617-277-7300; offering fresh seafood
Aquitaine Chestnut Hill, 11 Boylston Street, Chestnut Hill, 617-734-8400; offering French cuisine and brunch on Sundays.
Metropolitan Club, 1210 Boylston St, Chestnut Hill, 617-731-0600; offering contemporary American cuisine at brunch, lunch, and dinner.
Oishii, 612 Hammond Street, Chestnut Hill, 617-277-7888; serving sushi and Japanese cuisine.
Bernard's Restaurant, 199 Boylston Street, Chestnut Hill, 617-969-3388; Chinese cuisine. Located in the Chestnut Hill Mall.
Cheesecake Factory, 300 Boylston St, Newton, 617-964-3001; offering a large, varied menu and located in the Chestnut Hill Mall.
Charley's, 199 Boylston Street, Chestnut Hill, 617- 964-1200; featuring American cuisine and located in the Chestnut Hill Mall.
Forty Carrots, 225 Boylston Street, Chestnut Hill, 617-630-6640; health food, smoothies/juice bar, salads, and frozen yogurt, located in Bloomingdale's in the Chestnut Hill Mall.
The Cottage, 47 Boylston Street, Chestnut Hill, 617-916-5413; featuring an eclectic menu.
Please visit `www.bostonusa.com/`, for more casual and fine dining options in Boston.

Registration and Meeting Information

Registration and the AMS book exhibit will be located in the Lyons Hall Welch Dining Room. Special Sessions will be held in Fulton Hall and all Invited Addresses will be in Devlin Hall. Please refer to the campus map for the Chestnut Hill campus at `www.bc.edu/content/bc/a-z/maps/s-chestnuthill.html` for specific locations. The registration desk will be open on Saturday, April 6, 7:30 a.m.–4:00 p.m. and Sunday, April 7, 8:00 a.m.–12:00 p.m. Fees are US$53 for AMS members, US$74 for nonmembers; and US$5 for students, unemployed mathematicians, and emeritus members. Fees are payable on-site via cash, check, or credit card; advance registration is not available.

Other Activities

Book Sales: Stop by the on-site AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

The Boston College Mathematics Department will host a **reception** for participants on Saturday evening in the Lyons Welch Dining Room. Please watch for more details on this event on the AMS website and at the registration desk on site at the meeting. The AMS thanks our hosts for their gracious hospitality.

Local Information and Maps

This meeting will take place on the Chestnut Hill Campus of Boston College. A campus map for all of Boston College can be found at `www.bc.edu/a-z/maps.html.` Information about the Boston College Department of Mathematics may be found at `www.bc.edu/content/bc/schools/cas/math.html` Please watch the website available at `www.ams.org/meetings/sectional/sectional.html` for additional information on this meeting. Please visit the Boston College website at `www.bc.edu` for additional information on the campus.

Parking

Visitor parking is only available in the garages on campus. All participants should park in either the Beacon Street Garage on level three or four or in the Commonwealth Garage between levels three and six. The rates at the time of publication were as follows: weekend parking costs US$5 per exit, every 24 hours in the garage. No permit is required. All vehicles must exit the garage by 2:00 a.m. Monday morning. Vehicles left in the garage after that time are subject to ticketing/towing. Additional information on visitor parking can be found at `www.bc.edu/content/bc/offices/transportation/visitor.html.`

Travel

Boston College is located in the Chestnut Hill section of Newton, Massachusetts. The campus is approximately six miles west of downtown Boston. Logan International Airport is the closest airport to Boston College. The most common types of transportation used from the airport, are taxis, hotel shuttle, and the public transportation system called the "T".

By Air: Logan International Airport (BOS) is the closest airport to Boston College. The most common types of transportation used from the airport, are taxis, hotel shuttle, and the public transportation system called the "T".

Taxis can be found at the ground transportation area outside each terminal. Fare to downtown Boston will be approximately US$35 and to Boston College about $US50 depending on traffic and the time of day.

The campus can also be reached via mass transit on the MBTA "T" system. Shuttle bus #33 picks up passengers in front of the airline terminals and brings them to the airport subway stop. Take the Blue Line subway inbound to Government Center. At Government Center change (go upstairs) to the Boston College Green Line outbound train, i.e., Line "B" to Boston College. Stay on the train to the end of the line. It will cost US$2.50 for a one-way trip on the "T".

Shuttle service is another option to arrive on campus from the airport. Contact your hotel to see if they provide a free shuttle or if there is a shuttle company that they recommend.Some hotels have established service with reputable providers. For more information on alternate ground transportation please visit, `www.massport.com/massport/gtu/Pages/default.aspx`.

By Train: The Boston region is served by Amtrak: reservations can be made at `www.amtrak.com`. If traveling by rail, you will arrive at South Station, located at Summer Street & Atlantic Avenue, or North Station, located at 135 Causeway Street. From South Station you can you can board the MBTA "T" Red Line and transfer to the Green Line "B" to travel to the campus. From North Station you can board the Green Line directly, look for a "B" train. Both stations also offer taxis and buses.

By Bus: The Boston region is served by Greyhound Lines; reservations can be made at `www.greyhound.com`. Peter Pan also provides service to the area; reservations can be made at `www.peterpanbus.com`. You will arrive at South Station. For ground transportation options please see the "by train", listing above.

By Car: If you are driving to campus and using a GPS, please note that Boston Colllege's GPS address is 140 Commonwealth Avenue, Chestnut Hill, MA. Once you arrive on campus, you will be instructed to park in the Beacon Street Parking Garage. Please note that the Commonwealth Avenue garage is reserved for Boston College faculty and staff.

FROM POINTS NORTH AND SOUTH: Take Interstate 95 (Route 128) to Exit 24 (Route 30). Proceed east on Route 30, also known as Commonwealth Avenue, and follow for about five miles to Boston College.

FROM POINTS WEST: Take the Massachusetts Turnpike (Route 90) to Exit 17. At the first set of lights after the exit ramp, take a right onto Centre Street. Follow Centre Street to the fourth set of lights, and turn left onto Commonwealth Avenue. Follow Commonwealth Avenue 1-1/2 miles to Boston College.

FROM DOWNTOWN BOSTON: Take the Massachusetts Turnpike (Route 90) to Exit 17. Take a left over the bridge after passing the Crowne Plaza Hotel. Take the first right onto Centre Street. Follow above directions from Centre Street.

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at `www.hertz.com`, click on the box "I have a discount", and type in our convention number (CV): 04N30003. You can also call Hertz directly at 800-654-2240 (U.S. and Canada) or 405-749-4434 (other countries). At the time this announcement was prepared, rates were US$24 to US$74 per day on the weekend. At the time of your reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

Local Transportation

Taxi Service: Licensed, metered taxis are available throughout Chestnut Hill and Boston.

Bus and Subway Service: The Massachusetts Bay Transportation Authority (MBTA, known as the "T" in Boston), offers access to the BC campus. The meeting will be held on the upper part of the Chestnut Hill campus. A one-way ticket costs US$2.50. The three stops closest to campus include the Boston College stop which is approximately a 10-minute walk from the campus, the Chestnut Hill station which is approximately a 14-minute walk from campus, and Cleveland Circle which is approximately a 28-minute walk from campus. To access the campus via the Boston College stop utilize the branch of the MBTA's "Green Line" known as the "B" line that ends at the Boston-Newton boundary on Commonwealth Avenue. Exit the train at the Boston College stop, cross the street, walk by St. Ignatius Church, and follow the perimeter road around to campus entrances. There is also a campus shuttle from Cleveland Circle to campus that runs regularly; please inquire at registration for more details.

For more information on the "B" line please visit `www.mbta.com`.

Weather

The average high temperature for April is approximately 58 degrees Fahrenheit and the average low is approximately 40 degrees Fahrenheit. Rain is common in Boston for this time of year. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `sites.nationalacademies.org/pga/biso/visas/` and `travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `mac@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Boulder, Colorado

University of Colorado Boulder

April 13–14, 2013

Saturday – Sunday

Meeting #1089

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: January 2013
Program first available on AMS website: February 28, 2013
Program issue of electronic *Notices*: April 2013
Issue of *Abstracts*: Volume 34, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: December 26, 2012
For abstracts: February 19, 2013

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Gunnar Carlsson, Stanford University, *Title to be announced*.

Jesus A. De Loera, University of California, Davis, *Title to be announced*.

Brendan Hassett, Rice University, *Title to be announced*.

Raphael Rouquier, University of California Los Angeles, *Title to be announced*.

Special Sessions

Advances in Mathematical Biology (Code: SS 16A), **Liming Wang**, California State University, Los Angeles, and **Jiangguo Liu**, Colorado State University.

Algebraic Geometry (Code: SS 14A), **Sebastian Casalaina-Martin**, University of Colorado, **Renzo Cavalieri**, Colorado State University, **Brendan Hassett**, Rice University, and **Jonathan Wise**, University of Colorado.

Algebras, Lattices and Varieties (Code: SS 5A), **Keith A. Kearnes** and **Ágnes Szendrei**, University of Colorado, Boulder.

Analysis of Dynamics of the Incompressible Fluids (Code: SS 18A), **Mimi Dai** and **Congming Li**, University of Colorado, Boulder.

Arithmetic Statistics and Big Monodromy (Code: SS 23A), **Jeff Achter**, Colorado State University, and **Chris Hall**, University of Wyoming.

Associative Rings and Their Modules (Code: SS 1A), **Greg Oman** and **Zak Mesyan**, University of Colorado, Colorado Springs.

Cluster Algebras and Related Combinatorics (Code: SS 6A), **Gregg Musiker**, University of Minnesota, **Kyungyong Lee**, Wayne State University, and **Li Li**, Oakland University.

Combinatorial Avenues in Representation Theory (Code: SS 21A), **Richard Green**, University of Colorado Boulder, **Anne Shepler**, University of North Texas, and **Nathaniel Thiem**, University of Colorado Boulder.

Combinatorial and Computational Commutative Algebra and Algebraic Geometry (Code: SS 7A), **Hirotachi Abo**, University of Idaho, **Zach Teitler**, Boise State University, and **Alexander Woo**, University of Idaho.

Diophantine Approximation on Manifolds and Fractals: Dynamics, Measure Theory and Schmidt Games. (Code: SS 15A), **Wolfgang Schmidt**, University of Colorado at Boulder, and **Lior Fishman**, University of North Texas.

Dynamical Systems: Thermodynamic Formalism and Connections with Geometry (Code: SS 10A), **Keith Burns**, Northwestern University, and **Dan Thompson**, The Ohio State University.

Dynamics and Arithmetic Geometry (Code: SS 2A), **Suion Ih**, University of Colorado at Boulder, and **Thomas J. Tucker**, University of Rochester.

Elliptic Systems and Their Applications (Code: SS 20A), **Wenxiong Chen**, Yeshiva University, and **Congming Li**, University of Colorado at Boulder.

Extremal Graph Theory (Code: SS 3A), **Michael Ferrara**, University of Colorado Denver, **Stephen Hartke**, University of Nebraska-Lincoln, and **Michael Jacobson**, University of Colorado Denver.

Foundations of Computational Mathematics (Code: SS 13A), **Susan Margulies**, Pennsylvania State University, and **Jesus De Loera**, University of California, Davis.

Geometric Methods in the Representation Theory of Reductive Groups (Code: SS 19A), **J. Matthew Douglass**, University of North Texas, **Gerhard Roehrle**, Ruhr-Universitaet Bochum, and **Rahbar Virk**, University of Colorado Boulder.

Harmonic Analysis of Frames, Wavelets, and Tilings (Code: SS 12A), **Veronika Furst**, Fort Lewis College, **Keri Kornelson**, University of Oklahoma, and **Eric Weber**, Iowa State University.

Noncommutative Geometry and Geometric Analysis (Code: SS 22A), **Carla Farsi** and **Alexander Gorokhovsky**, University of Colorado, Boulder.

Nonlinear Waves and Integrable Systems (Code: SS 9A), **Christopher W. Curtis**, University of Colorado, Boulder, **Anton Dzhamay**, University of Northern Colorado, **Willy**

Hereman, Colorado School of Mines, and **Barbara Prinari**, University of Colorado, Colorado Springs.

Number Theory with a Focus on Diophantine Equations and Recurrence Sequences (Code: SS 8A), **Patrick Ingram**, Colorado State University, and **Katherine E. Stange**, University of Colorado, Boulder.

Set Theory and Boolean Algebras (Code: SS 17A), **Natasha Dobrinen**, University of Denver, and **Don Monk**, University of Colorado, Boulder.

Singular Spaces in Geometry, Topology, and Algebra (Code: SS 11A), **Greg Friedman**, Texas Christian University, and **Laurentiu Maxim**, University of Wisconsin, Madison.

Themes in Applied Mathematics: From Data Analysis through Fluid Flows and Biology to Topology (Code: SS 4A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, and Enfitek, Inc.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include the state and local taxs (15%). Participants must state that they are with the **American Mathematical Society (AMS) Meeting at the University of Akron** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

Boulder Inn/Best Western Plus, 770 28th Street, Boulder, Colorado 80303; 800-233-8469; www.boulderinn.com. Rates are US$89 per night for a single/double occupancy. Amenities include complimentary daily hot breakfast, heated seasonal pool, 24-hour business center, free bike use, laundry facilities, and complimentary parking. This property is located across from the University of Colorado. Cancellation and early check-out policies vary; be sure to check when you make your reservation. The deadline for reservations at this rate is **March 12, 2013.**

Millennium Harvest House, Boulder, 1345 28th Street, Boulder, Colorado 80302; 303-443-3850; www.millenniumhotels.com/millenniumboulder/. Rates are US$119 per night single/double occupancy, US$129 triple occupancy, and US$139 quad occupancy. Ammenities include complimentary guest room Internet, a business center, fitness center and massage services, and complimentary parking. The University of Colorado is 100 yards from the hotel. Cancellation and early check-out policies vary; be sure to check when you make your reservation. The deadline for reservations at this rate is **March 13, 2013.**

America's Best Value Inn & Suites, 970 28th Street, Boulder, Colorado 80303; 303-443-7800; www.americas-bestvalueinn.com. Rates are US$65 for single/double occupancy. Ammenities include free continental breakfast, free wireless Internet in lobby, guest laundry facilities and complimentary parking. Hotel is located within walking distance of the University of Colorado. Cancellation and early check-out policies vary; be sure to check when you make your reservation. The deadline for reservations at this rate is **March 1, 2013.**

Boulder Outlook Hotel & Suites, 800 28th Street, Boulder, Colorado 80303; 303-443-3322; www.boulder-outlook.com. Rates are US$79.00 for a standard king or double for 1 or 2 people; US$89.00 for a standard double for 3 or 4 people; US$89.00 for a poolside king or double for 1 or 2 people; US$99.00 for a poolside double for 3 or 4 people. Amenities include complimentary continental breakfast, free wireless Internet in guest rooms, and complimentary parking. The hotel is within close walking distance to the university. Cancellation and early check-out policies vary; be sure to check when you make your reservation. The deadline for reservations at this rate is **February 21, 2013.**

Quality Inn, 2020 Arapahoe Avenue, Boulder, Colorado 80302; 888.449.7550; www.qualityinnboulder.com. Rates are US$118 for a single or double. Amenities include free hot breakfast; complimentary wireless Internet; indoor swimming pool, hot tub, and sauna; and complimentary parking. The hotel is a 10-15 minute walk to campus. Cancellation and early check-out policies vary; be sure to check when you make your reservation. The deadline for reservations at this rate is **March 12, 2013.**

Food Services

On Campus: Campus facilities will be available but limited during the weekend of the meeting. There are a number of options at the University Memorial Center including:
Baby Doe's Coffee and Bakery, Saturday 11:00 a.m. – 5:00 p.m., and noon – 5 p.m. on Sunday.
Subway, Saturday, 10:00 a.m. – 3:00 p.m., closed Sunday.
Domino's Pizza, Saturday 11:00 a.m. – 3:00 p.m., closed Sunday.
Jamba Juice, Saturday 10:00 a.m. – 3:00 p.m., closed Sunday.

Off Campus: There are many dining choices for casual dining and "grab and go" options convenient to campus and local hotels.
The Sink, 1165 13th St., Boulder, CO 80302; 303-444-SINK; www.TheSink.com. A historic Boulder landmark and university tradition. Featuring quality American food and beverages in a casual setting. Open daily 11:00 a.m. – 10:00 p.m.
Jax Fish House and Oyster Bar, 928 Pearl St., Boulder, CO 80302; 303-444-1811; www.jaxboulder.com. Consistently voted to the "Best of" Awards of Boulder and Denver, for 17 years Jax has served up the finest and freshest the ocean has to offer to the masses that squeeze into the packed Pearl Street hotspot. Open at 4:00 p.m. daily.
Pizzeria Locale, 1730 Pearl Street, Boulder, CO 80302; 303-442-3003; www.pizzerialocale.com. Contemporary pizzeria inspired by the traditional pizzerias of Napoli, Italy. Open Friday-Saturday 11:30a.m. – 10:30p.m.; Sunday 11:30 a.m. – 9:00 p.m.
Mountain Sun Pub and Brewery, 1535 Pearl St., Boulder, CO 80302; 303-546-0886; www.mountainsunpub.com. Cuisines: American, Hamburgers, Mexican, Vegetarian. Open Monday through Sunday 11:00 a.m. – 1:00 a.m.

Tangier Moroccan Cuisine, 3070 28th St., Boulder, CO; 303-443-3676; Cuisines: Mediterranean, Middle Eastern, Moroccan, International. Open Monday – Saturday, 5:00 p.m. – 9:00 p.m.

Aji Restaurant, 1601 Pearl St., Boulder, CO 80302; 303-442-3464; www.ajirestaurant.com. Aji Latin American Restaurant brings you the bold, colorful flavors of Latin America. We feature dishes from South and Central America, as well as some Mexican fare. Open for brunch Saturday & Sunday 10:00 a.m. – 3:00 p.m. and dinner starting at 5:00 p.m.

Registration and Meeting Information

Registration and the AMS book exhibit will be located in the lobby of the Eaton Humanities building. Special Sessions will be held in Hellems Arts and Sciences and Eaton Humanities Building. Please refer to the campus map at http://www.colorado.edu/campusmap/map.pdf for specific locations. The registration desk will be open on Saturday, April 13th, 7:30 a.m. – 4:00 p.m. and Sunday, April 14, 8:00 a.m. – 12:00 p.m. Fees are US$53 for AMS members, US$74 for nonmembers; and US$5 for students, unemployed mathematicians, and emeritus members. Fees are payable on-site via cash, check, or credit card; advance registration is not available.

Other Activities

Reception: The Departments of Mathematics and Applied Mathematics of the University of Colorado, Boulder, will host a reception at the Koenig Alumni Center, on Saturday April 13, 2013, from 6:30 p.m. to 8:00 p.m. The AMS thanks our hosts for their gracious hospitality.

Book Sales: Stop by the on-site AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Local Information and Maps

This meeting will take place on the campus of the University of Colorado. A campus map can be found at www.colorado.edu/campusmap or you can print a pdf version of the map here: http://www.colorado.edu/campusmap/map.pdf. Information about the Department of Mathematics, University of Colorado, Boulder: http://math.colorado.edu. Please watch the AMS website available at http://www.ams.org/meetings/sectional/sectional.html for additional information on this meeting. Visit the University of Colorado, Boulder website: www.colorado.edu for additional information on the campus.

Parking

The Euclid Avenue Auto-park is closest to the buildings where the meeting rooms will be. The following University of Colorado website has more details about parking costs: www.colorado.edu/about/visiting-campus/parking-transportation.

Travel

Driving time between Denver International Airport (DIA) and Boulder is approximately 60 to 75 minutes. You can find directions from the aiport to the university here: www.colorado.edu/about/visiting-campus/directions.

Taxi information: There are taxis available at the airport. The approximate cab fare from DIA (Denver International Airport) to Boulder is US$101.00

Shuttle information: SuperShuttle Boulder, website: www.supershuttle.com/Locations/DENBoulderDIAExpressAirportShuttle.aspx. Rates from the airport to campus are US$27 one-way, and round trip is $54 (prices subject to change)

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at www.hertz.com, click on the box "I have a discount", and type in our convention number (CV): 04N30003. You can also call Hertz directly at 800-654-2240 (U.S. and Canada) or 1-405-749-4434 (other countries). At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

Weather

The weather in Boulder in April can vary. Usually the weather is dry and sunny and with the sun out, the temperature can go into the 60's. However, the nights are cooler and can be chilly. The average high is 63 degrees F and the average low in April is 36 degrees F. However, it is not at all unheard of to have significant snowfall in mid-April, and even though Boulder in general has a dry climate, the month of April is the month with the greatest average precipitation of the year (2.74 inches on average in April).

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at http://sites.nationalacademies.org/pga/biso/visas/ and http://travel.state.gov/visa/visa_1750.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to mac@ams.org.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence

- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;
* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;
* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;
* If travel plans will depend on early approval of the visa application, specify this at the time of the application;
* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date.

Ames, Iowa

Iowa State University

April 27–28, 2013

Saturday – Sunday

Meeting #1090

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: February 2013
Program first available on AMS website: March 14, 2013
Program issue of electronic *Notices*: April 2013
Issue of *Abstracts*: Volume 34, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: January 18, 2013
For abstracts: March 5, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Kevin Costello, Northwestern University, *Title to be announced.*

Marianne Csornyei, University of Chicago, *Title to be announced.*

Vladimir Markovic, California Institute of Technology, *Title to be announced.*

Eitan Tadmor, University of Maryland, *Title to be announced.*

Special Sessions

Algebraic and Geometric Combinatorics (Code: SS 4A), **Sung Y. Song**, Iowa State University, and **Paul Terwilliger**, University of Wisconsin-Madison.

Analysis, Dynamics and Geometry In and Around Teichmuller Spaces (Code: SS 17A), **Alistair Fletcher**, Northern Illinois University, **Vladimir Markovic**, California Institute of Technology, and **Dragomir Saric**, Queens College CUNY.

Commutative Algebra and its Environs (Code: SS 6A), **Olgur Celikbas** and **Greg Piepmeyer**, University of Missouri, Columbia.

Commutative Ring Theory (Code: SS 8A), **Michael Axtell**, University of St. Thomas, and **Joe Stickles**, Millikin University.

Computability and Complexity in Discrete and Continuous Worlds (Code: SS 11A), **Jack Lutz** and **Tim McNicholl**, Iowa State University.

Computational Advances on Special Functions and Tropical Geometry (Code: SS 14A), **Lubjana Beshaj**, Oakland University, and **Emma Previato**, Boston University.

Control Theory and Qualitative Analysis of Partial Differential Equations (Code: SS 16A), **George Avalos**, University of Nebraska-Lincoln, and **Scott Hansen**, Iowa State University.

Discrete Methods and Models in Mathematical Biology (Code: SS 18A), **Dora Matache**, University of Nebraska-Omaha, and **Stephen J. Willson**, Iowa State University.

Extremal Combinatorics (Code: SS 7A), **Steve Butler** and **Ryan Martin**, Iowa State University.

Generalizations of Nonnegative Matrices and Their Sign Patterns (Code: SS 3A), **Minerva Catral**, Xavier University, **Shaun Fallat**, University of Regina, and **Pauline van den Driessche**, University of Victoria.

Geometric Elliptic and Parabolic Partial Differential Equations (Code: SS 13A), **Brett Kotschwar**, Arizona State University, and **Xuan Hien Nguyen**, Iowa State University.

Graphs, Hypergraphs and Counting (Code: SS 26A), **Eva Czabarka** and **Laszlo Szekely**, University of South Carolina.

Kinetic and Hydrodynamic PDE-based Descriptions of Multi-scale Phenomena (Code: SS 25A), **James Evans** and **Hailiang Liu**, Iowa State University, and **Eitan Tadmor**, University of Maryland.

Logic and Algebraic Logic (Code: SS 9A), **Jeremy Alm**, Illinois College, and **Andrew Ylvisaker**, Iowa State University.

Multi-Dimensional Dynamical Systems (Code: SS 15A), **Jayadev Athreya**, University of Illinois, Urbana-Champaign, **Jonathan Chaika**, University of Chicago, and **Joseph Rosenblatt**, University of Illinois at Urbana-Champaign.

Numerical Analysis and Scientific Computing (Code: SS 20A), **Hailiang Liu**, **Songting Luo**, **James Rossmanith**, and **Jue Yan**, Iowa State University.

Numerical Methods for Geometric Partial Differential Equations (Code: SS 22A), **Gerard Awanou**, University of Illinois at Chicago, and **Nicolae Tarfulea**, Purdue University.

Operator Algebras and Topological Dynamics (Code: SS 1A), **Benton L. Duncan**, North Dakota State University, and **Justin R. Peters**, Iowa State University.

Partial Differential Equations (Code: SS 12A), **Gary Lieberman** and **Paul Sacks**, Iowa State University, and **Mahamadi Warma**, University of Puerto Rico at Rio Piedras.

Probabilistic and Multiscale Modeling Approaches in Cell and Systems Biology (Code: SS 19A), **Jasmine Foo**, University of Minnesota, and **Anastasios Matzavinos**, Iowa State University.

Quasigroups, Loops, and Nonassociative Division Algebras (Code: SS 21A), **C. E. Ealy Jr.** and **Annegret Paul**, Western Michigan University, **Benjamin Phillips**, University of Michigan Dearborn, **J. D. Phillips**, Northern Michigan University, and **Petr Vojtechovsky**, University of Denver.

Ring Theory and Noncommutative Algebra (Code: SS 24A), **Victor Camillo**, University of Iowa, and **Miodrag C. Iovanov**, University of Bucharest and University of Iowa.

Stochastic Processes with Applications to Physics and Control (Code: SS 10A), **Jim Evans** and **Arka Ghosh**, Iowa State University, **Jon Peterson**, Purdue University, and **Alexander Roitershtein**, Iowa State University.

Topology of 3-Manifolds (Code: SS 23A), **Marion Campisi** and **Alexander Zupan**, University of Texas at Austin.

Zero Forcing, Maximum Nullity/Minimum Rank, and Colin de Verdiere Graph Parameters (Code: SS 2A), **Leslie Hogben**, Iowa State University and American Institute of Mathematics, and **Bryan Shader**, University of Wyoming.

Alba Iulia, Romania

June 27–30, 2013

Thursday – Sunday

Meeting #1091

First Joint International Meeting of the AMS and the Romanian Mathematical Society, in partnership with the "Simion Stoilow" Institute of Mathematics of the Romanian Academy.

Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2013
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/internmtgs.html`.

Invited Addresses

Viorel Barbu, Universitatea Cuza, *Title to be announced.*

Sergiu Klainerman, Princeton University, *Title to be announced.*

George Lusztig, Massachusetts Institute of Technology, *Title to be announced.*

Stefan Papadima, Institute of Mathematics of the Romanian Academy of Sciences, *Title to be announced.*

Dan Timotin, Institute of Mathematics of the Romanian Academy of Sciences, *Title to be announced.*

Srinivasa Varadhan, New York University, *Title to be announced.*

Special Sessions

Algebraic Geometry, **Marian Aprodu**, Institute of Mathematics of the Romanian Academy, **Mircea Mustata**, University of Michigan, Ann Arbor, and **Mihnea Popa**, University of Illinois, Chicago.

Articulated Systems: Combinatorics, Geometry and Kinematics, **Ciprian S. Borcea**, Rider University, and **Ileana Streinu**, Smith College.

Calculus of Variations and Partial Differential Equations, **Marian Bocea**, Loyola University, Chicago, **Liviu Ignat**, Institute of Mathematics of the Romanian Academy, **Mihai Mihailescu**, University of Craiova, and **Daniel Onofrei**, University of Houston.

Commutative Algebra, **Florian Enescu**, Georgia State University, and **Cristodor Ionescu**, Institute of Mathematics of the Romanian Academy.

Discrete Mathematics and Theoretical Computer Science, **Sebastian Cioaba**, University of Delaware, **Gabriel Istrate**, Universitatea de Vest, Timisoara, **Ioan Tomescu**, University of Bucharest, and **Marius Zimand**, Towson University.

Domain Decomposition Methods and their Applications in Mechanics and Engineering, **Lori Badea**, Institute of Mathematics of the Romanian Academy, and **Marcus Sarkis**, Worcester Polytechnic Institute.

Geometry and Topology of Arrangements of Hypersurfaces, **Daniel Matei**, Institute of Mathematics of the Romanian Academy, and **Alexandru I. Suciu**, Northeastern University.

Harmonic Analysis and Applications, **Ciprian Demeter**, Indiana University, Bloomington, and **Camil Muscalu**, Cornell University.

Hopf Algebras, Coalgebras, and their Categories of Representations, **Miodrag C. Iovanov**, University of Bucharest and University of Iowa, **Susan Montgomery**, University of Southern California, and **Siu-Hung Ng**, Iowa State University.

Local and Nonlocal Models in Wave Propagation and Diffusion, **Anca V. Ion**, Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, and **Petronela Radu**, University of Nebraska, Lincoln.

Mathematical Finance, Stochastic Analysis, and Partial Differential Equations, **Lucian Beznea**, Institute of Mathematics of the Romanian Academy, **Paul Feehan**, Rutgers University, **Victor Nistor**, Pennsylvania State University, **Camelia Pop**, University of Pennsylvania, and **Mihai Sirbu**, University of Texas, Austin.

Mathematical Models in Life and Environment, **Gabriela Marinoschi**, Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, and **Fabio Augusto Milner**, Arizona State University.

Mathematical Models in Materials Science and Engineering, **Marian Bocea**, Loyola University, Chicago, and **Bogdan Vernescu**, Worcester Polytechnic Institute.

Noncommutative Ring Theory and Applications, **Toma Albu**, Institute of Mathematics of the Romanian Academy, and **Lance W. Small**, University of California, San Diego.

Nonlinear Evolution Equations, **Daniel Tataru**, University of California, Berkeley, and **Monica Visan**, University of California, Los Angeles.

Operator Algebra and Noncommutative Geometry, **Marius Dadarlat**, Purdue University, and **Florin Radulescu**, Institute of Mathematics of the Romanian Academy and University of Rome Tor Vergata.

Operator Theory and Function Spaces, **Aurelian Gheondea**, Institute of Mathematics of the Romanian Academy and Bilkent University, **Mihai Putinar**, University of California, Santa Barbara, and **Dan Timotin**, Institute of Mathematics of the Romanian Academy.

Probability and its Relation to Other Fields of Mathematics, **Krzysztof Burdzy**, University of Washington, and **Mihai N. Pascu**, Transilvania University of Braşov.

Random Matrices and Free Probability, **Ioana Dumitriu**, University of Washington, and **Ionel Popescu**, Georgia Institute of Technology and Institute of Mathematics of the Romanian Academy.

Several Complex Variables, Complex Geometry and Dynamics, **Dan Coman**, Syracuse University, and **Cezar Joita**, Institute of Mathematics of the Romanian Academy.

Topics in Geometric and Algebraic Topology, **Stefan Papadima**, Institute of Mathematics of the Romanian Academy, and **Alexandru I. Suciu**, Northeastern University.

This announcement was composed with information taken from the website maintained by the local organizers at `imar.ro/ams-ro2013/description.php`. *Please watch this website for the most up-to-date information.*

Abstract Submissions

Talks in Special Sessions are generally by invitation of the organizers. It is recommended that you contact an organizer before submitting an abstract for a Special Session if you have not been specifically invited to speak in the session. Please watch the website cited above for more information on abstract submission. Abstracts may be submitted for consideration as of January 1. The deadline for volunteers for Special Sessions will be March 1 and the deadline for submissions for invited speakers will be April 15.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. These hotels were reserved in their entirety for participants of the meeting, for the dates of the meeting. Participants must state that they are with the **AMS (American Mathematical Society) Meeting in Alba Iulia** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **The deadline for reservations at a discounted rate, for all properties listed below, is May 1, 2013.**

Hotel Parc**,** Str. Primaverii, 4, Alba Iulia, Romania, (258) 811.723; email `office@hotelparc.ro`; `www.hotelparc.ro`. Rates are US$52 per night for a single room. Located in the Central Park of the city, close to Alba Iulia's Town Hall, the Prefect's Office and the Chambers of Commerce. Amenities include private bathroom with shower, color TV, and free wireless Internet in guest rooms. There is a fitness room, heated pool, two saunas, two jacuzzis, a bar, and a restaurant on site. Cancellation and early checkout policies vary; be sure to check when you make your reservation.

Vila Preciosa**,** 10 Lucian Blaga Str., Alba Iulia, Romania; email `office@preciosa.ro`; `www.preciosa.ro`. Rates are US$55 per night for a single room. This property is located only minutes away from the historical and administrative center of Alba Iulia, close to Alba Iulia Citadel, both Cathedrals, the Union Museum, and the Route of the Three Fortifications. Amenities include complimentary breakfast, complimentary minibar with nonalcoholic beverages, LCD television, air conditioning, private parking, and free wireless Internet access. The restaurant on property also serves lunch and dinner. All guestrooms are non-smoking. Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 2:00 p.m.; checkout time is 12:00 noon.

Pensiunea MaryLou****, Str. Lalelelor; nr.18, Alba Iulia, Romania, 510217; (358) 082.450; email `office@pensiunea-marylou.ro`; `www.pensiunea-marylou.ro`. Rates are US$40 per night for a single room. This property is located near Union Cathedral, Roman Catholic Cathedral, National Union Museum, Union Hall, Batthyaneum Library, Apor Palace, Alba Iulia Citadel, and the Route of the Three Fortifications. Amenities include complimentary breakfast, minibar, air conditioning, private parking, and free wireless Internet access. All guestrooms are non-smoking. Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is between 2:00p.m. and 11:00 p.m.; checkout time is 12 noon.

Villa Elisabeta***, Calea Motilor Nr. 158, Alba Iulia, Romania, (258) 839.117, email `vilaelisabeta@yahoo.com`; `www.vilaelisabeta.ro/start`. Rates are US$45 per night for single occupancy. Vila Elisabeta is 2.5 miles from the train station and 43 miles from Sibiu Airport. Amenities include complimentary parking, air conditioning, cable TV, and wireless Internet in guest rooms. There is an outdoor swimming pool, an indoor pool, a sauna, a massage room, and a fitness center. Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 2:00 p.m.; checkout time is 12:00 noon.

Villa Steaua Nordului***, Republicii 5A, langa Bazinul Olimpic, Alba Iulia, Romania; (258) 810 234, email `vila_steauanordului@yahoo.com`; `www.vilasteauanordului.ro/`. Rates are `www.vilasteauanordului.ro/`. Rates are US$40 per night for a single room. This property is situated in Alba Iulia, near the Olympic Basin. Amenities include air conditioning, TV, Internet access, telephone, private bathroom, hair dryer, and minibar. Guests have access to an indoor pool and sauna and an outdoor pool. There is a restaurant on property. Cancellation and early checkout policies vary; be sure to check when you make your reservation.

Hotel Hermes***, Str. Toporasilor nr. 20, Alba Iulia, Romania, (258) 835.008; e-mail `office@hotelhermes.ro`; `www.hotelhermes.ro/`. Rates are US$46 per night for single room. Amenities include cable TV, minibar, private bathroom, and Internet in guest rooms. There is a restuarant on property. Cancellation and early checkout policies vary; be sure to check when you make your reservation.

Hotel Astoria***, DN1, Km 387, Alba Iulia, Romania; email `office@astoriahotels.ro`; `imar.ro/ams-ro2013/accommodations.php`. or `www.astoriahotels.ro`. Rates are US$45 per night for single room. This property is located five miles from central Alba Iulia. Amenities include air conditioning, private bathrooms, cable TV, free parking, room service, complimentary wireless Internet in public areas, complimentary wired Internet in guest rooms. This property has a restaurant on site. Cancellation and early checkout policies vary; be sure to check when you make your reservation.

Hotel Cetate**, Str. Union, No. 3, Alba Iulia, Romania, (258) 815.833 ; `www.tourismguide.ro/tgr/hotel_cetate_alba_iulia_1730.php` Rates are US$35 per night for a single room. The Cetate Hotel is located a short walking distance from the most important monuments and tourist attractions of Alba Iulia, surrounded by fortified walls with towers. Amenities include non-smoking rooms, an in-room minibar, cable/satellite TV, complimentary parking, and free wireless Internet in guest rooms. This property has a restaurant on site. Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 12:00 noon.; checkout time is 11:00 a.m.

Hotel Transilvania**, Piata Iuliu Maniu, nr. 22, Alba Iulia, Romania, (258) 812.052, email `transilvania.alba@unita-turism.ro`; `www.tourismguide.ro/tgr/hotel_transilvania_alba_iulia_1756.php`. Rates are US$45 per night for a single room. This property is located in the historical center of Alba Iulia. Amenities include cable TV, a refrigerator in the room, private bathroom, and Internet access in guest rooms. A restaurant and parking are available on property. Cancellation and early checkout policies vary; be sure to check when you make your reservation. Checkin time is 2:00 p.m.; checkout time is 11:00 a.m.

University Hostels, economical accommodations may be arranged at the University Hostels, located no more than 20-minutes walking distance from the conference locations. The rate for accommodations in these hostels will be €20 per night, including breakfast. Reserving these rooms in the University Hostels must be arranged through the registration process.

Local Information/ Tourism

Local and area maps can be found at `www.apulum.ro/` which features a map of the city and a map of the historical citadel of Alba Iulia. An interactive map of Alba Iulia is also available online at `www.hartaalbaiulia.ro/`.

For more information about visiting Alba Iulia, please visit the Alba Iulia Tourist Information Centre (Centrul de Informare Turistica) at `www.turismalba.ro`, or `www.stiri.turismalba.ro`. The offce can also be reached by email at `carolinatour@ymail.com`. The Tourist Information Centre provides maps, brochures, and information on accommodations, restaurants, and transportation. Additionally, a very useful website for planning your time in Romania can be found at `www.romaniatourism.com/`.

The local organizers are planning an excursion for participants and guests on Friday, June 28th, in the afternoon. The excursion program includes a visit to Turda Salt Mine (`www.salinaturda.eu`). The cave is now a museum of salt mining in Transylvania. A classical music concert will be offered to the participants in the Amphitheatre of the mine, where the acoustics are optimal for musical performances.

Romania's currency is Leu (the plural is lei) and the abbreviation is RON. At the time of publication of this announcement the exchange rate was US$1 is equal to 3.53 Lei. ATM machines are available at main banks and at airports and shopping centers. There are very few ATMs in remote areas or villages. ATMs that have symbols for international networks such as STAR and PLUS will accept US/Canadian banking cards.

Cash (US or Canadian Dollars) can be easily exchanged at any bank or Currency Exchange Office (Casa de Schimb). Exchange rates offered by the exchange offices at the airport may be 10% to 25% less than the official rate.

Major credit cards are accepted in large hotels, car rental companies, and stores in the main cities. However, credit cards will generally not be accepted in small towns or away from tourist areas. A PIN is usually required to make credit card purchases. Many American banks allow cardholders to establish such a PIN prior to travel, in case one is needed. It is highly reccomended that you notify your bank of your international travel and the potential legitimate use of your card abroad, prior to leaving the U.S.

Romania's electrical current is 230 V; 50 cycles. Partipants from the US planning to use their own electronic devices should bring suitable power converter kits. Sockets take the standard continental European dual round-pronged plugs.

Registration and Meeting Information

Please check the local website for instructions to register online The participation fee is US$65 and should be payed at the registration desk on site at the meeting. It covers the conference materials and the refreshments. Participants will be asked during registration about transfer. All locations for conference events are located in the historical centre of the city, within walking distance. Registration and the AMS book exhibit will be located at Apor Palace, the building of the Rectorate of "1 Decembrie 1918", University of Alba Iulia. It is located at Gabriel Bethlen Street, No.5, Alba Iulia. Special Sessions will be held on the main campus of "1 Decembrie 1918", University of Alba Iulia, buildings B and C. Address: Nicolae Iorga Street, No.11-13, Alba Iulia; `www.uab.ro`. The Invited Addresses will take place in the Hall of Culture, a theatre that is about a 10-15 minute walk from the main campus; cssalba.ro.

Opening Reception and Conference Dinner

An opening reception will take place on Thursday, June 27th, in the courtyard of the Medieval Hotel (hotel-medieval.ro/). This hotel is situated in the middle of the star-shaped historical fortress of Alba Iulia, in the neighborhood of the Route of the Three Fortifications. This is one of the most important sites to see in the city, offering the opportunity of a journey through two millennia among the vestiges of three fortifications belonging to three different periods. More information will follow with the details of this reception.

A Conference Dinner will take place on Saturday, June 29. This event will take place in the courtyard of Jidvei Castle (www.jidvei.ro/en/#/ABOUT%20US-THE%20CASTLE), Cetatea de Balta, close to Alba Iulia. The old castle, demolished in the 16th century, was used as a royal residence by the viceroys of Transylvania. The present castle was rebuilt in the 17th century by Count Bethlen and it is today a renowned symbol of Jidvei, a famous vineyard in Transylvania, on Tarnava valley, www.jidvei.ro/en/. Please watch the local website for more details. The AMS thanks our hosts for their gracious hospitality.

Travel

Alba Iulia is located in central Romania. American and Canadian citizens as well as citizens of Australia, New Zealand, and most European countries do not need an entry visa to visit Romania (for stays up to 90 days). No immunizations or unusual health precautions are necessary or required.

By Air: The airport in closest proximity to Alba Iulia is Sibiu Airport (SBZ), www.sibiuairport.ro, and it is approximately 44 miles away. Alternate airports include Cluj-Napoca Airport(CLJ), www.airportcluj.ro, which is located 56 miles away, and Targu Mures Airport (TGM), www.targumuresairport.ro, which is located 82 miles away.

Transfers from Sibiu and Cluj-Napoca to the meeting will be provided by conference shuttles. Participants will be asked during registration if they wish to utilize this transportation. The schedule will be posted on the meeting website at a later date.

By Train: The main train station (Gara Alba Iulia; phone +40 (258) 812.967) is located at Blvd. Ferdinand, about a mile southeast of the Citadel. There are daily trains to and from Budapest (journey time—7 ½ hours). Trains to and from other western European cities run via Budapest. For a list of international trains with service to/from Romania please visit www.RomaniaTourism.com/Transportation.html#ByTrain. To check the latest train schedules for domestic routes please visit the website of the Romanian Railways: www.cfrcalatori.ro.

By Bus: Daily domestic bus servicce is available between many cities in Romania. The Alba Iulia Bus Station is located at Str. Iasilor 94. Information can be found on the web at www.autogari.ro/AlbaIulia or by calling +40 (258) 812.967.

By Car: Documents required by Border Police are the vehicle's registration, proof of insurance, and a valid driver's license. U.S./Canadian/Australian/New Zealand driver's licenses are valid for driving in Romania for 90 days from the date of entry into Romania. Independent travelers entering Romania by car (own or rental) need to obtain a road toll badge, called Rovinieta. Rovinieta is available at any border-crossing point, postal office, and most gas stations at a cost of (the equivalent in Romanian Lei) US$5 to US$8 (valid for up to 7 days) or US$9 to US$15 (valid for 30 days), depending on car type.

Most international auto rental companies and several local companies offer cars in the major cities and airports. Renters must be over 21 and have a valid driver's license and an internationally valid credit card.

Local Transportation

Taxi Service: Several taxi companies offer service in Alba Iulia and at all airports.
Bus Service: Several bus (autobus) routes connect Alba Iulia's main areas and tourist attractions. For the bus routes map in Alba Iulia please visit: http://www.apulum.ro/harta%20transport.html.

Weather

Romania has a temperate climate, similar to the northeastern United States, with four distinct seasons. Summer is quite warm, with extended sunny days. Average temperatures in June are approximately 66 degrees Farenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Louisville, Kentucky

University of Louisville

October 5–6, 2013

Saturday – Sunday

Meeting #1092

Southeastern Section
Associate secretary: Robert J. Daverman
Announcement issue of *Notices*: June 2013
Program first available on AMS website: August 22, 2013
Program issue of electronic *Notices*: October 2013
Issue of *Abstracts*: Volume 33, Issue 3

Deadlines

For organizers: March 5, 2013
For consideration of contributed papers in Special Sessions: June 18, 2013
For abstracts: August 13, 2013

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Michael Hill, University of Virginia, *Title to be announced.*

Suzanne Lenhart, University of Tennessee, *Title to be announced.*

Ralph McKenzie, Vanderbilt University, *Title to be announced.*

Victor Moll, Tulane University, *Title to be announced.*

Special Sessions

Extremal Graph Theory (Code: SS 2A), **Jozsef Balogh**, University of Illinois at Urbana-Champaign, **Louis DeBiasio**, Miami University, Oxford, OH, and **Tao Jiang**, Miami University, Oxford, OH.

Set Theory and Its Applications (Code: SS 1A), **Paul Larson**, Miami University, **Justin Moore**, Cornell University, and **Grigor Sargsyan**, Rutgers University.

Philadelphia, Pennsylvania

Temple University

October 12–13, 2013

Saturday – Sunday

Meeting #1093

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: June 2013
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2013
Issue of *Abstracts*: Volume 33, Issue 3

Deadlines

For organizers: March 12, 2013
For consideration of contributed papers in Special Sessions: June 25, 2013
For abstracts: August 20, 2013

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Patrick Brosnan, University of Maryland, *Title to be announced.*

Xiaojung Huang, Rutgers University, *Title to be announced.*

Barry Mazur, Harvard University, *Title to be announced* (Erdős Memorial Lecture).

Robert Strain, University of Pennsylvania, *Title to be announced.*

Special Sessions

Contact and Symplectic Topology (Code: SS 5A), **Joshua M. Sabloff**, Haverford College, and **Lisa Traynor**, Bryn Mawr College.

Geometric and Spectral Analysis (Code: SS 3A), **Thomas Krainer**, Pennsylvania State Altoona, and **Gerardo A. Mendoza**, Temple University.

Higher Structures in Algebra, Geometry and Physics (Code: SS 2A), **Jonathan Block**, University of Pennsylvania, **Vasily Dolgushev**, Temple University, and **Tony Pantev**, University of Pennsylvania.

History of Mathematics in America (Code: SS 4A), **Thomas L. Bartlow**, Villanova University, **Paul R. Wolfson**, West Chester University, and **David E. Zitarelli**, Temple University.

Recent Advances in Harmonic Analysis and Partial Differential Equations (Code: SS 1A), **Cristian Gutiérrez** and **Irina Mitrea**, Temple University.

Recent Developments in Noncommutative Algebra (Code: SS 6A), **Edward Letzter** and **Martin Lorenz**, Temple University.

Several Complex Variables and CR Geometry (Code: SS 7A), **Andrew Raich**, University of Arkansas, and **Yuan Zhang**, Indiana University-Purdue University Fort Wayne.

St. Louis, Missouri

Washington University

October 18–20, 2013

Friday – Sunday

Meeting #1094

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: August 2013
Program first available on AMS website: September 5, 2013
Program issue of electronic *Notices*: October 2013
Issue of *Abstracts*: Volume 33, Issue 4

Deadlines

For organizers: March 20, 2013
For consideration of contributed papers in Special Sessions: July 2, 2013
For abstracts: August 27, 2013

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Ronny Hadani, University of Texas at Austin, *Title to be announced.*

Effie Kalfagianni, Michigan State University, *Title to be announced.*

Jon Kleinberg, Cornell University, *Title to be announced.*

Vladimir Sverak, University of Minnesota, *Title to be announced.*

Special Sessions

Algebraic and Combinatorial Invariants of Knots (Code: SS 1A), **Heather Dye**, McKendree University, **Allison Henrich**, Seattle University, and **Louis Kauffman**, University of Illinois.

Computability Across Mathematics (Code: SS 2A), **Wesley Calvert**, Southern Illinois University, and **Johanna Franklin**, University of Connecticut.

Interactions between Geometric and Harmonic Analysis (Code: SS 3A), **Leonid Kovalev**, Syracuse University, and **Jeremy Tyson**, University of Illinois, Urbana-Champaign.

Riverside, California

University of California Riverside

November 2–3, 2013

Saturday – Sunday

Meeting #1095

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2013
Program first available on AMS website: September 19, 2013
Program issue of electronic *Notices*: November 2013
Issue of *Abstracts*: Volume 33, Issue 4

Deadlines

For organizers: April 2, 2013
For consideration of contributed papers in Special Sessions: July 15, 2013
For abstracts: September 10, 2013

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Michael Christ, University of California, Berkeley, *Title to be announced.*

Mark Gross, University of California, San Diego, *Title to be announced.*

Matilde Marcolli, California Institute of Technology, *Title to be announced.*

Paul Vojta, California Institute of Technology, *Title to be announced.*

Special Sessions

The Mathematics of Planet Earth (Code: SS 1A), **John Baez**, University of California, Riverside.

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel at Camden Yards

January 15–18, 2014

Wednesday – Saturday

Meeting #1096

Joint Mathematics Meetings, including the 120th Annual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart
Program first available on AMS website: November 1, 2013
Program issue of electronic *Notices*: January 2013
Issue of *Abstracts*: Volume 35, Issue 1

Deadlines

For organizers: April 1, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Knoxville, Tennessee

University of Tennessee, Knoxville

March 21–23, 2014

Friday – Sunday

Meeting #1097

Southeastern Section
Associate secretary: Robert J. Daverman
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 21, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Baltimore, Maryland

University of Maryland, Baltimore County

March 29–30, 2014

Saturday – Sunday

Meeting #1098

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: March 2014
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Albuquerque, New Mexico

University of New Mexico

April 5–6, 2014
Saturday – Sunday

Meeting #1099
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2014
Issue of *Abstracts*: To be announced

Deadlines
For organizers: September 5, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: February 11, 2014

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions
The Inverse Problem and Other Mathematical Methods Applied in Physics and Related Sciences (Code: SS 1A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico and Enfitek Inc.

Lubbock, Texas

Texas Tech University

April 11–13, 2014
Friday – Sunday

Meeting #2000
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: September 18, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions
Topology and Physics (Code: SS 1A), **Razvan Gelca** and **Alastair Hamilton**, Texas Tech University.

Tel Aviv, Israel

Bar-Ilan University, Ramat-Gan and Tel-Aviv University, Ramat-Aviv

June 16–19, 2014
Monday – Thursday
The 2nd Joint International Meeting between the AMS and the Israel Mathematical Union.
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

Special Sessions
Mirror Symmetry and Representation Theory, **David Kazhdan**, Hebrew University, and **Roman Bezrukavnikov**, Massachusetts Institute of Technology.

Nonlinear Analysis and Optimization, **Boris Mordukhovich**, Wayne State University, and **Simeon Reich** and **Alexander Zaslavski**, The Technion-Israel Institute of Technology.

Qualitative and Analytic Theory of ODE's, **Yosef Yomdin**, Weizmann Institute.

Eau Claire, Wisconsin

University of Wisconsin-Eau Claire

September 20–21, 2014
Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: February 20, 2014
For consideration of contributed papers in Special Sessions: To be announced

For abstracts: August 5, 2014

San Francisco, California

San Francisco State University

October 25–26, 2014
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2014
Issue of *Abstracts*: To be announced

Deadlines
For organizers: March 25, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: September 3, 2014

San Antonio, Texas

Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio

January 10–13, 2015
Saturday – Tuesday
Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2015
Issue of *Abstracts*: Volume 36, Issue 1

Deadlines
For organizers: April 1, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Las Vegas, Nevada

University of Nevada, Las Vegas

April 18–19, 2015
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: September 18, 2014
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Porto, Portugal

University of Porto

June 11–14, 2015
Thursday – Sunday
First Joint International Meeting between the AMS and the Sociedade Portuguesa de Matemática.
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Not applicable

Deadlines
For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Chicago, Illinois

Loyola University Chicago

October 3–4, 2015
Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2015
Issue of *Abstracts*: To be announced

Deadlines
For organizers: March 10, 2015
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: `lapidus@math.ucr.edu`; telephone: 951-827-5910.

Central Section: **Georgia Benkart**, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: `benkart@math.wisc.edu`; telephone: 608-263-4283.

Eastern Section: **Steven H. Weintraub**, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: `steve.weintraub@lehigh.edu`; telephone: 610-758-3717.

Southeastern Section: **Robert J. Daverman,** Department of Mathematics, University of Tennessee, Knoxville, TN 37996-6900, e-mail: `daverman@math.utk.edu`; telephone: 865-974-6900.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at** `www.ams.org/meetings/`**.**

Meetings:

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 111 in the the January 2012 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit `http://www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences in Cooperation with the AMS: (see `http://www.ams.org/meetings/` for the most up-to-date information on these conferences.)

July 22–26, 2013: Samuel Eilenberg Centenary Conference (E100), Warsaw, Poland.

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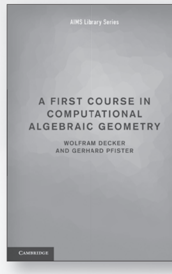

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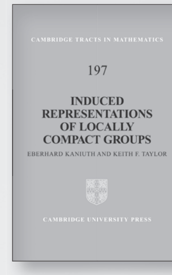

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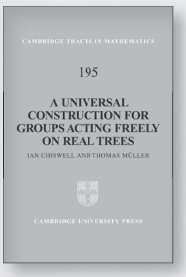

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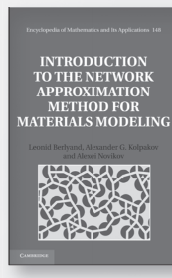

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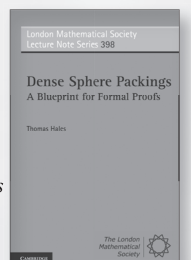

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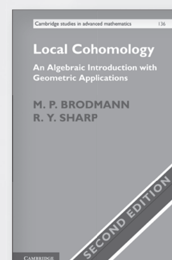

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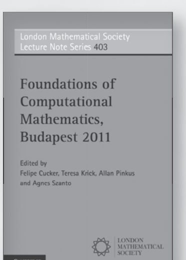

Foundations of Computational Mathematics, Budapest 2011

Edited by Felipe Cucker,
Teresa Krick, Allan Pinkus, *and*
Agnes Szanto

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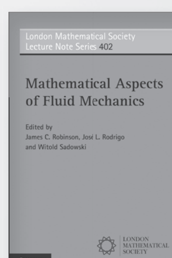

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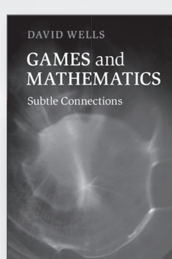

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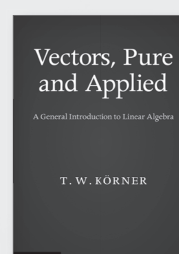

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