Remembering Walter Rudin (1921–2010)

Alexander Nagel and Edgar Lee Stout, Coordinating Editors

Walter Rudin, Vilas Professor Emeritus at the University of Wisconsin-Madison, died on May 20, 2010, at his home in Madison after a long battle with Parkinson's disease. He was born in Vienna on May 2, 1921.

The Rudins were a well-established Jewish family which began its rise to prominence in the first third of the nineteenth century. By the 1830s, Walter's great-grandfather, Aron Pollak, had built a factory to manufacture matches; he also became known for his charitable activities, including the construction of a residence hall where seventy-five needy students at the Technical University in Vienna could live without paying rent. As a result, Aron was knighted by Emperor Franz Joseph in 1869 and took the name Aron Ritter Pollak von Rudin. The Rudin family prospered, and Walter's father, Robert, was a factory owner and electrical engineer, with a particular interest in sound recording and radio technology. He married Walter's mother, Natalie (Natasza) Adlersberg, in 1920. Walter's sister, Vera, was born in 1925.

After the *Anschluss* in 1938, the situation for Austrian Jews became impossible, and the Rudin family left Vienna. Walter served in the British Army and Navy during the Second World War, and rejoined his parents and sister in New York in late 1945. He entered Duke University, obtaining a B.A. in 1947 and a Ph.D. in mathematics in 1949. He was a C. L. E. Moore Instructor at the Massachusetts Institute of Technology and began teaching at the University of Rochester in 1952.

Photograph courtesy of Mary Ellen Rudin.

Walter Rudin and sister, Vera, in Vienna.

While on leave visiting Yale in 1958, Rudin received a call from R. H. Bing at the University of Wisconsin-Madison, asking if he would be interested in teaching summer school. Rudin said that, since he had a Sloan Fellowship, he wasn't interested in summer teaching. Then, as he writes in his autobiography, *As I Remember It*, "my brain slipped out of gear but my tongue kept on talking and I heard it say 'but how about a real job?' " As a result, Walter Rudin joined the Department of Mathematics at UW-Madison in 1959, where he remained until his retirement as Vilas Professor in 1991. He and his wife, the distinguished mathematician Mary Ellen (Estill) Rudin, were popular teachers at both the undergraduate and graduate level and served as mentors for many graduate students. They lived in Madison in a house designed by Frank Lloyd Wright, and its intriguing architecture and two-story-high living room made it a center for social life in the department.

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Photograph courtesy of Mary Ellen Rudin.

Rudin house, Madison.

Walter Rudin was one of the preeminent mathematicians of his generation. He worked in a number of different areas of mathematical analysis, and he made major contributions to each. His early work reflected his classical training and focused on the study of trigonometric series and holomorphic functions of one complex variable. He was also very influenced by the then relatively new study of Banach algebras and function algebras. One of his important results in this area, building on the work of Arne Beurling, is the complete characterization of the closed ideals in the disk algebra in 1956.

Another major area of Walter's interest was the general theory of harmonic analysis on locally compact Abelian groups. In the late 1950s and 1960s this was a very active and popular area of research, and perhaps only partially in jest, Walter suggested that mathematicians introduce a new word, "lgbalcag", to replace the phrase "Let G be a locally compact Abelian group", which is how almost every analysis seminar began in those days. One of Walter's major achievements in this area was his 1959 work with Helson, Kahane, and Katznelson, which characterized the functions that operate on the Fourier transforms of the L^1-algebra. Rudin synthesized this aspect of his mathematical career in his 1962 book, *Fourier Analysis on Groups*.

Walter's interests changed again in the late 1960s, and he began to work on problems in several complex variables. At that time the study of the analytic aspects of complex analysis in several variables was relatively new and unexplored, and it was not even clear what the right several-variable generalization of the one-dimensional unit disk should be. There are at least two candidates: the polydisk and the ball. Walter did important work with both. For example, he showed for the polydisk (1967) and the unit ball (1976) that the zero sets of different H^p classes of functions are all different. His work on the "inner function conjecture" led to a tremendous amount of research, and after the solution by Aleksandrov and Hakim-Sibony-Løw (1981), Walter made additional important contributions to this question. Much of Rudin's work in several complex variables is presented in three of his advanced books. The first, published in 1969, is *Function Theory in Polydiscs*. The second, published in 1980, is *Function Theory in the Unit Ball of* $\mathbb{C}^n$. His work on inner functions was summarized in a series of NSF-CBMS lectures, which were then published in 1986 as *New Constructions of Functions Holomorphic in the Unit Ball of* $\mathbb{C}^n$.

Walter Rudin is also known to generations of undergraduate and graduate students for his three outstanding textbooks: *Principles of Mathematical Analysis* (1953), *Real and Complex Analysis* (1966), and *Functional Analysis* (1973). In 1993 he was awarded the American Mathematical Society's Leroy P. Steele Prize for Mathematical Exposition. He received an honorary degree from the University of Vienna in 2006.

In addition to his widow, Mary Ellen, Walter Rudin is survived by his four children: Catherine Rudin, professor of modern languages and linguistics at Wayne State College, Nebraska; Eleanor Rudin, an engineer working for 3M in St. Paul, Minnesota; Robert Rudin of Madison, Wisconsin; and Charles Rudin, professor of oncology at the Johns Hopkins University in Baltimore. He is also survived by four grandchildren: Adem, Deniz, Sofia, and Natalie.

Jean-Pierre Kahane

Walter Rudin and Harmonic Analysis

The work of Walter Rudin on harmonic analysis is a good part of his life and also of mine. A guide for most of it is the list of his papers on Fourier analysis on groups at the end of his celebrated book. I shall start with a few of them and add some complements. This is nothing but a glance at a large piece of harmonic analysis.

The general inspiration for Walter was to discover questions and results arising from the algebraic structure of some parts of analysis. Wiener and Gelfand had paved the way. It became the main tendency in harmonic analysis in the middle of the twentieth century.

Part of Walter's work deals with the Wiener algebras, that is, the algebras made of Fourier transforms of integrable functions or summable sequences. A related matter is trigonometric series. Another is convolution algebras of Radon measures. A general framework is the Gelfand theory of Banach algebras. Generally speaking, there are several ways to express the same problems and results: classical in the sense of the nineteenth century or abstract and modern in the sense of the twentieth. Moreover, as far as Fourier analysis is concerned, what happens on a group can be expressed on the dual group. Equivalent definitions can be found everywhere, and Walter was an expert in playing this game.

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Let me start with the first paper of his mentioned in *Fourier Analysis on Groups*. The title is "Non analytic functions of absolutely convergent Fourier series", and the year was 1955 [1]. What he discovered was a positive function with an absolutely convergent Fourier series, vanishing at 0, such that its square root does not enjoy the same property. It was the first result of this type, but the question was in the air, if in different forms. The Wiener-Lévy theorem asserts that an analytic function of a function in the Wiener algebra $A(\mathbb{T})$ (that is, its Fourier series converges absolutely) belongs to $A(\mathbb{T})$. In other words, the analytic functions operate on $A(\mathbb{T})$; that is, the convolution algebra $l^1(\mathbb{Z})$ has a symbolic calculus consisting of analytic functions. Can we replace the analytic functions by a wider class? The Wiener-Lévy theorem can be translated into Banach algebras via the theory of Gelfand. The problem can be translated as well: which are the functions that operate in a given Banach algebra? Great progress was made on this question between 1955 and 1958. I proved that absolute values do not operate on the Wiener algebra, then that functions that operate are necessarily infinitely differentiable. Katznelson proved in 1958 the natural conjecture: only analytic functions operate. That occurred in a very informal meeting in Montpellier just before the International Congress in Edinburgh; besides Katznelson and me, Helson, Herz, Rudin, Salem, and others were there, enjoying life and discussing mathematics. Katznelson's theorem immediately had several versions, involving locally compact abelian groups instead of $\mathbb{T}$. Walter was interested in convolution algebras of measures or, what is the same, in multiplicative algebras of Fourier-Stieltjes transforms. What we proved in [2] is that only entire functions operate. It is a way to recover many previous results, going back to the *Wiener-Pitt phenomenon* (the inverse of a Fourier-Stieltjes transform is not necessarily a Fourier-Stieltjes transform, even when it is bounded), through the discoveries of Schreider in 1950 about the algebra of Fourier-Stieltjes transforms. Our results were published in the form of a series of notes in the *Comptes Rendus*, but Walter was incredibly efficient in making them known among mathematicians: the invited report he made at the Cambridge meeting of the AMS in August 1958, "Measure Algebras on Abelian Groups", contained them all. In the general form about locally compact abelian groups they are exposed in *Fourier Analysis on Groups*.

Walter's last contribution to the subject [3] was an extension of Katznelson's theorem, "A strong converse of the Wiener-Lévy theorem" in 1962: if, for each given f in $A(G)$ with values in the interval $(-1,1)$, the composed function $F(f)$ is a Fourier transform of a function which belongs to $L^p(G^*)$ (here G^* is the dual group of G), with $p < 2$ (depending on f), then F is the restriction on $(-1,1)$ of an analytic function in a neighborhood of the closed interval $[-1,1]$.

Photograph courtesy of Mary Ellen Rudin.

Rudin in 1956.

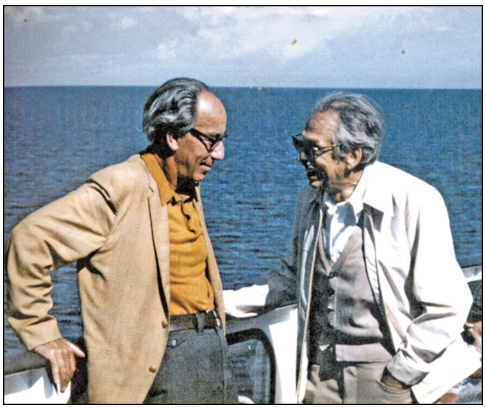
The main event in the domain after 1958 was the theorem of Malliavin on spectral synthesis in 1959. Spectral synthesis can be expressed in many ways, as well as nonspectral synthesis. The contribution of Walter in that subject was to exhibit a function f in $A(G)$ such that the ideals generated by the powers f^n are all different (see [4]).

The question on functions operating on $A(\mathbb{R})$ or $A(\mathbb{T})$ goes back to Paul Lévy (1938) in the paper where the Wiener-Lévy theorem is stated. Paul Lévy asked a second question, on functions operating "below": which are the changes of variables preserving $A(\mathbb{R})$? The obvious example is affine functions.

Actually affine functions are the only ones; it is a theorem of Beurling and Helson, published in 1953. It opened a new and important field, the isomorphisms and endomorphisms of the group algebras. Walter entered the subject in 1956 with his *Acta Mathematica* article [5] on the automorphisms and the endomorphisms of the group algebra of the unit circle. Here is a typical result: a permutation of the integers carries Fourier coefficients into Fourier coefficients if and only if the permutation is equal to an obvious one, up to a finite number of places; an obvious one is a permutation p that satisfies $p(n-g)+p(n+g) = 2p(n)$ for some g. The general result extends both the Beurling-Helson and the Rudin theorems; it deals with homomorphisms of a group algebra into another group algebra and is due to Paul Cohen (1960). Here is a nice particular case, established by Rudin in 1958: the group algebra of a locally compact abelian group G is isomorphic to that of the circle group $\mathbb{T}$ if and only if $G = \mathbb{T} + F$, where F is a finite abelian group.

This question of homomorphism of group algebras is linked with an apparently different question, the characterization of idempotent measures. Here again Helson and Rudin paved the way, and the final result was obtained by Paul Cohen, proving a conjecture of Rudin (Cambridge meeting, 1958): The supports of the Fourier transforms of idempotent measures (these Fourier transforms take values 0 and 1) are the members of the "coset ring" of the group G^*, defined as

Rudin with Jaap Korvaar.



Photos courtesy of Mary Ellen Rudin.

Rudin with Lipman Bers.

generated by all cosets of subgroups of G^* by means of complementation and finite intersection.

There are a number of other results of Rudin in Fourier analysis, about factorization in $L^1(\mathbb{R}^n)$, lacunary sequences, thin sets, weak almost periodic functions, positive definite sequences, and absolutely monotonic functions (another example of *operating functions*). I shall restrict myself to lacunary sequences and thin sets; the name of Rudin is attached to some of them. The name of Rudin appears frequently in relation to automatic sequences and their role in Fourier series; I shall explain the use of Rudin-Shapiro sequences.

Though lacunary sequences and thin sets can be considered in general groups, let me restrict myself to the integers and the circle.

In 1957 Walter defined the Paley sequences as sequences $(n(k))$ such that the coefficients of order $n(k)$ of a function of the Hardy class H^1 (the subspace of $L^1(\mathbb{T})$ generated by the imaginary exponentials with positive frequencies) belong to l^2. The theorem of Paley is that Hadamard sequences (meaning $n(k+1)/n(k) > q > 1$) are Paley. Obviously this extends to finite unions of Hadamard sequences. The theorem of Rudin is that it is a characterization of Paley sequences.

The Sidon sets have very many definitions, and they were studied by Rudin in the important paper of 1960 "Trigonometric series with gaps" [6]. He introduced the $\Lambda(p)$ sets, E, defined by the fact that the L^p norm of a trigonometric polynomial whose frequencies lie in E are dominated by the L^s norms for an $s < p$, up to a constant factor depending on s. He studied the relation between Sidon sets and $\Lambda(p)$ sets, proved an inequality saying that a Sidon set is a $\Lambda(p)$ set for all values of p, and conjectured that this inequality was optimal. This is the case, as proved by Pisier in 1978 using Gaussian processes. Though much is known now about Sidon sets, the main conjecture is still unsolved; it says that Sidon sets are nothing but a finite union of quasi-independent sets, *quasi-independent* meaning that there is no linear relation with coefficients 1, −1, or 0 between its elements. Sidon sets are still a subject of interest, and the subject, including the name of Sidon sets, was introduced by Rudin.

A question was raised by Rudin on the $\Lambda(p)$ sets: There is a natural inclusion between the collections of $\Lambda(p)$ sets, since every $\Lambda(p)$ is $\Lambda(q)$ when $q < p$. Is this inclusion strict? The answer is negative for indices < 2: all $\Lambda(p)$ are the same for $1 < p < 2$. When p is an even integer > 2, Rudin exhibited a $\Lambda(p)$ set that is not $\Lambda(p')$ for any $p' > p$. Only in 1989 was Bourgain able to extend this to all $p > 2$; therefore, the inclusion is strict for $p > 2$. The situation for $p = 2$ is not yet settled.

The Rudin sets on $\mathbb{R}$ or $\mathbb{T}$ are independent sets over the rationals which carry measures whose Fourier transform tends to 0 at infinity. Because of the Kronecker theorem, this cannot happen with countable sets. They are thin, but not too thin. The construction of Rudin is clever (1960). It can be replaced by a random construction. I believe that I am responsible for the name of Rudin sets.

I may be responsible also for the name of the Rudin-Shapiro sequence. The description and the history are well described in the article [7] of Walter Rudin, "Some theorems on Fourier coefficients" of 1959. It is a beautiful and very useful automatic sequence, and I used it as soon as Walter told me about it. It answered a question asked by Raphaël Salem in our informal Montpellier meeting. Salem had overlooked the fact that Harold Shapiro had already answered the question already in 1951. Shapiro deserves recognition, and a few colleagues would prefer to change the name to Shapiro-Rudin. Likely it is too late. There are abuses of that sort in all parts of mathematics, and they usually benefit strong and well-known mathematicians.

The name of Rudin will stay in the history of mathematics by the importance of his contributions to many parts of analysis and by his exceptional talent for exposition, in his articles as well as in his books. Fourier analysis corresponds to part of his life, but only to a part. His whole life as a mathematician and as a human being deserves to be known.

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Jean-Pierre Rosay

I came to Madison for a one-year visit in 1986 with the hope of working with Walter. I admired Walter's work in several complex variables. I especially liked his book *Function Theory on the Unit Ball in* $\mathbb{C}^n$, a stimulating book of supreme elegance where originality is to be found in the least details.

Our collaboration soon began. Working with Walter was pure enjoyment, with daily exchanges. He always came with challenges that gave rise to an immediate desire to work. That cannot surprise readers of his books. Nothing was rushed. Pieces were kept, without any rushing to premature global writing, and at the end the last pleasure was the magic of his elegant writing (not simply cut and paste!).

As soon as I arrived in Madison, I noticed a very special quality of life in the mathematics department. It is obvious that Mary Ellen and Walter Rudin contributed largely to the atmosphere, and they have been wonderful hosts for many. When unexpectedly (the move was never planned) I was invited to stay at Madison, I quickly accepted. Walter was the main reason, but having great colleagues around Walter, such as P. Ahern, A. Nagel, and S. Wainger also played a role. In addition to a collaboration with Walter, a true friendship with Mary Ellen and Walter developed.

Edgar Lee Stout

Walter Rudin and Several Complex Variables. The Beginning

Although Walter Rudin began his mathematical career with work in Euclidean harmonic analysis in a thesis on uniqueness problems concerning Laplace series, from very early on he also pursued investigations in complex analysis. Most of his work in complex analysis until the early 1960s was concerned with one-dimensional theory.

Rudin's main work in several complex variables began in the early 1960s after the publication of his book *Fourier Analysis on Groups*. I was present at this beginning and recall it rather clearly. During the academic year 1963–64 a working seminar dedicated to trying to learn something about the then not-widely-known subject of several complex variables was run in Madison by a group of students and some faculty members, including Walter. It was a question of the blind leading the blind. The volume of Fuks [1] had just appeared in an English translation published by the AMS, so we in the seminar set out to read through it systematically, but before long we recognized that this was not really what was desired. About that time someone found the beautiful Tata lectures [2] of Malgrange which give a concise introduction to the modern theory of higher-dimensional complex analysis, including the important notions of coherent analytic sheaves and the associated fundamental Theorems A and B of Cartan and Serre. The seminar turned to these notes and became a great success.


Photo by Yvonne Nagel.

Mary Ellen and Walter Rudin, 1991.

Walter's principal research efforts soon turned to multivariate complex analysis. Not unnaturally, his efforts in this direction began with some function-theoretic questions on the unit polydisc in $\mathbb{C}^n$, which is the n-fold Cartesian product of the unit disc in the plane with itself. For a classically trained analyst who is approaching multidimensional complex analysis for the first time, it is entirely natural to begin by studying the possible extensions of classical results on the unit disc in the plane to analogous results on the polydisc. One soon realizes that some classical results have direct and often easy analogues on the polydisc and that the analogues of some classical results are simply false. The most interesting kinds of results are those that present new phenomena.

The first paper of Rudin's about function theory on the polydisc [5] was written jointly with me and comprises two rather disparate kinds of results. The first is a characterization of the rational inner functions on the polydisc, i.e., the rational functions of n complex variables that are holomorphic on the polydisc and that are unimodular on the distinguished boundary $\mathbb{T}^n$ (which is the Cartesian product of n copies of the unit circle in the plane). These are natural n-dimensional

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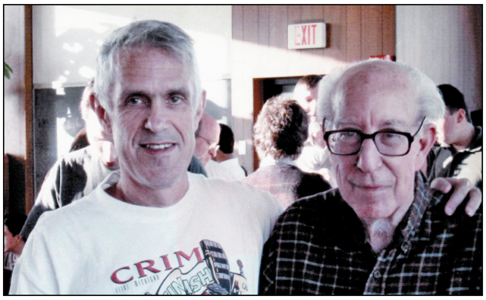

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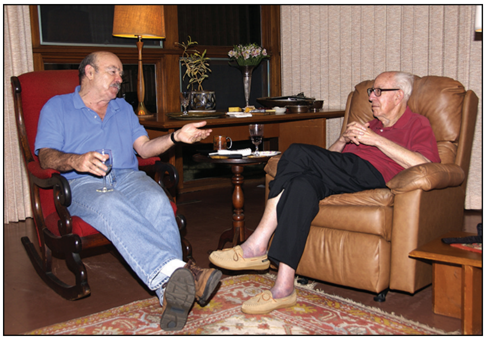

Photos by Yvonne Nagel.

Rudin and Pat Ahern, who wrote eight joint papers.

analogues of the finite Blaschke products. The second kind of result the paper contains concerns the extension to the polydisc of the *Rudin-Carleson Theorem*, which characterizes as the closed subsets of length zero of the unit circle the peak-interpolation sets for the disc algebra, that is, the sets on which every continuous function can be matched by a function continuous on the closed unit disc and holomorphic on its interior. This paper contains some examples of peak-interpolation sets in the torus $\mathbb{T}^n$ and was the first paper to address the question of this kind of interpolation in n dimensions. In spite of considerable subsequent work, there is still no characterization of the peak-interpolation sets for the polydisc or for any other domain in $\mathbb{C}^n$ with $n > 1$.

After this paper was written, Rudin continued to think about function theory on the polydisc and published in 1969 his book *Function Theory in Polydiscs*, which contained many results obtained in the preceding few years by him and others. The book contains the foundations of Hardy space theory on the polydisc; a discussion of the zero sets of bounded holomorphic functions and, more generally, of functions of the Hardy class; the theory of peak-interpolation sets; and a discussion of inner functions. All of these subjects are direct analogues of well-understood topics in classical function theory, but to this day not one of them exhibits the refined completion enjoyed by the classical theory.

About the time the polydisc book was published, Rudin's attention turned to function theory on the unit ball in $\mathbb{C}^n$. This is a considerably different subject from analysis on polydiscs, in good measure because of the greater degree of symmetry on the ball. Analysis on the ball is a rich subject in itself and also serves as a model for analysis on the general strictly pseudoconvex convex domain. Walter published in 1980 in Springer's Grundlehren series his *Function Theory on the Unit Ball of* $\mathbb{C}^n$, which is much longer than the polydisc book. There is some overlap in themes, e.g., inner functions, peak-interpolation sets, H^p-theory, zero-set problems. At the time this book was written, it was not at all clear that any nonconstant inner functions—bounded holomorphic functions with radial limits almost everywhere of modulus one—exist on the ball. The book contains several results of the general sort that a nonconstant inner function on the ball must behave in very complicated ways, but it neither proves nor disproves their existence. It was eventually shown by Alexandroff and by Hakim-Sibony-Løw that nonconstant inner functions do exist. It is curious that although inner functions on the disc play a fundamental role in many function-theoretic considerations, so far those on the ball have not found many applications. When it appeared, this book gave a panoramic view of almost the entire known theory of functions on the ball, and it has served as the standard reference since.

Walter's interests were much broader than suggested so far. For example, in [3] he gives a characterization of the algebraic varieties in $\mathbb{C}^n$, and in [4] he obtains the structure of a proper holomorphic map from the ball in $\mathbb{C}^n$ to a domain in $\mathbb{C}^n$. He also gave two series of CBMS lectures, the first on the edge-of-the-wedge theorem, the second on constructions of functions on the ball. Both were very well received.

Quite aside from the breadth and depth of his research efforts, one must notice the clarity and elegance of Walter's expository work. His style is concise yet clear and can well serve as a model of mathematical exposition.

I count myself very fortunate to have been a student of Walter Rudin, first in a wonderful year-long course in complex variables—his *Real and Complex Analysis* had not yet appeared—and subsequently as a doctoral student. While I studied with him for my doctorate, Walter was always very helpful, always ready with a technical suggestion about some point under consideration, and, more importantly always very encouraging, which is essential for beginning students. He and his wife, Mary Ellen, herself a distinguished mathematician, were very generous with their hospitality, both to their students and to their former students. What a pleasure it has been to enjoy their friendship over the last half-century.

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John Wermer

Walter Rudin has made important contributions to a very wide range of problems in analysis. Of special interest to me is some of Walter's work on problems linking the theory of commutative Banach algebras and complex function theory.

The following question was raised by Isadore Singer in the 1950s. The disk algebra $A = A(\mathbb{D})$ is the algebra of continuous complex-valued functions on the closure $\overline{\mathbb{D}}$ of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ which are holomorphic on the open disk. Singer asked, to what extent is $A(\mathbb{D})$ characterized by the following two properties:

(1) A is an algebra of functions continuous on the closed unit disk,
(2) the maximum principle holds with respect to the boundary of the disk.

Walter proved the following theorem: Suppose that B is an algebra of continuous functions on the disk such that

(i) the function $f(z) = z$ lies in B;
(ii) for every function f in B and every point $z_0 \in \mathbb{D}$,

$$|f(z_0)| \leq \max\{|f(w)| \mid |w| = 1 \text{ so } w \text{ belongs to the boundary of the disk}\}.$$

Then B is the disk algebra.

He gave extensions of this result, using the local maximum principle and replacing the disk by other domains, in [1].

Walter's work was influential in stimulating the development of the theory of function algebras. In the study of polynomial approximation in $\mathbb{C}^n$, one asks, let K be a compact set in $\mathbb{C}^n$ and let P(K) denote the uniform closure on K of polynomials in the complex coordinates. When is $P(K) = C(K)$? In 1926 J. L. Walsh proved (in *Math. Ann.* **96**) that for each continuous arc J in the complex plane, $P(J) = C(J)$. In [2] Walter gave a counterexample to the corresponding statement in $\mathbb{C}^2$.

For each commutative Banach algebra, one would like to identify all the closed ideals. This is usually difficult. Using results of Beurling on the invariant closed subspaces on H^2, Walter solved the corresponding problem for the disk algebra in his paper [3].

A smooth manifold in $\mathbb{C}^n$ is called *totally real* if no tangent space at a point of M contains a complex line (e.g., $\mathbb{R}^n$ is totally real in $\mathbb{C}^n$). Totally real submanifolds play an important role in the complex geometry of $\mathbb{C}^n$. Walter studied totally real embeddings of the 3-sphere in $\mathbb{C}^3$ jointly with Pat Ahern in [4], and totally real embeddings of the Klein bottle in $\mathbb{C}^2$ in [5].

Walter is well known in the mathematical world for his many basic textbooks, some of which are:

- *Functional Analysis. Second Edition*, McGraw Hill (1991)
- *Real and Complex Analysis. Third Edition*, McGraw Hill (1987)
- *Function Theory of the Unit Ball of* $\mathbb{C}^n$, Springer Verlag (1980)
- *Principles of Mathematical Analysis. Third Edition*, McGraw Hill (1976)
- *Function Theory in Polydiscs*, W. A. Benjamin (1969)

I own several of them and have found them excellent and of great value.

Walter Rudin has contributed to so many subjects that the above has touched on only a small part of his work.

One nonmathematical book of Walter's is his autobiography, *The Way I Remember It*, published by the American Mathematical Society. I found it of particular interest, since Walter and I are fellow refugees from Nazi-occupied Vienna in the late thirties, and so his fascinating story was of special concern to me. I remember one line from him. It was oral and may not be in the book. Walter said, "During the war, I served in the British Army. After that, nothing ever bothered me...."

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