Book Review

Deep Questions on the Nature of Mathematics

Reviewed by Michael Heller

Meaning in Mathematics
Edited by John Polkinghorne
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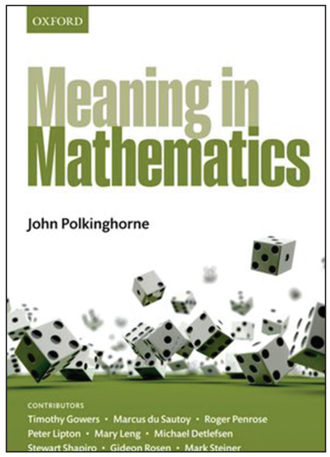
"Is mathematics a highly sophisticated intellectual game in which the adepts display their skill by tackling invented problems, or are mathematicians engaged in acts of discovery as they explore an independent realm of mathematical reality?" (p. 1). This is a question that has been asked many times and in manifold ways. This particular formulation is by John Polkinghorne, the editor of the volume under review. Two mathematicians (Timothy Gowers and Marcus du Sautoy), two mathematical physicists (Roger Penrose and John Polkinghorne), and six philosophers (Michael Detlefsen, Mary Leng, Peter Lipton, Gideon Rosen, Stewart Shapiro, and Mark Steiner) convened in Castel Gandolfo and in Cambridge (there is no mention when) to discuss the issue. The mathematicians spoke mainly about their experiences in doing mathematics, while both mathematical physicists argued strongly for discovering the "independent realm of mathematical reality," and the philosophers preferred to immerse themselves in sophisticated analyses and doctrinal subtleties. The question itself belongs to the category of questions that have never led to definite answers, but the very fact that they are asked so often has a certain metaphysical significance. It is well captured by John Polkinghorne (notably also in the form of a question): "The status of mathematics bears upon an answer to the



fundamental metaphysical question, 'What are the dimensions of reality?' Do they extend beyond the frontiers of a domain that is capable of being fully described simply in terms of exchanges of energy between material constituents, located within the arena of space-time?… Or, on the contrary, is it the case that true ontological adequacy requires that much more be said than physicalism can articulate?" (p. 27). The problem, when formulated in this way, opens up a vast array of subquestions and their ramifications. Indeed, the book is rich in terms of the breadth of its topics and multilayered nature. I will dwell upon just a few points in order to sample the content of the book.

First of all, what does it mean that mathematical objects exist? Gideon Rosen notices: "Anyone who accepts basic arithmetic must agree that there are two prime numbers between 15 and 20, and hence there are at least two numbers, and hence that there are numbers" (p. 113). But do they exist in the same way as guns and rabbits? The "full-strength realists" give a positive answer to this question (Hardy and Gödel are most often quoted as supporters of this view), whereas the "qualified realists" claim that mathematical objects do exist but "are somehow metaphysically 'second rate'" (p. 114). According to Rosen, a fact is fundamental if it is not grounded in further facts, and a thing is fundamental provided it is a constituent of a fundamental fact. "Then we may identify full-strength realism about mathematics

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with the thesis that some mathematical objects are fundamental things" (p. 124). On the other hand, Rosen defines qualified realism in the following way: "Qualified realism about Fs is the thesis that Fs exist, but no fundamental fact contains an F as a constituent" (p. 125).

But is there really a problem with the existence of mathematical objects? Mary Leng argues that "if accounting for the phenomenology of mathematical discovery requires us to posit any kind of 'reality' to ground our mathematical judgments, this reality is not a realm of mathematical objects, but rather, I claim, a realm of objective facts about logical consequences" (pp. 62–63). This view is consonant with mathematical structuralism, the standpoint according to which there are structures rather than objects that eventually exist in the Platonic realm, and any such structure can be regarded as a network of inferences, objects being only "places at which inferences intersect." Mary Leng substantiates her views by the reference to the "phenomenology of mathematical practice," which places her analysis in the context of discovery and distances her from ontological questions.

A more modest way of asking about the existence of mathematical objects or structures is to inquire as to the extent to which mathematics is objective. But what does "objective" mean? It is certainly not a univocal concept. Stewart Shapiro distinguishes (after Wright) several "notions or axes of objectivity," and "a given chunk of discourse can exhibit some of these and not others" (p. 100). Shapiro focuses on one such axis, which he terms "cognitive command". In his view, a given domain satisfies the criterion of cognitive command if any disagreement in the conclusion follows either from a divergence in the information input or in an error of the inference. This raises the question of the objectivity of logic. "If logic fails to be objective, can there be any objectivity anywhere? … Any attempt to characterize how the question of objectivity is to be adjudicated will presuppose logic" (p. 107). Shapiro hopes that the situation will become "more palatable" if we take into account the Kant-Quine thesis "that there is no way to sharply separate the parts of our best theories that are due to the way the world is and the parts that are due to the way that we, the human cognitive agents, are" (ibid.). It seems that we cannot aspire towards complete objectivity, but some of our compromises do not eliminate objectivity altogether. "In short, mathematics is objective, if anything is" (p. 100), or mathematics "is a paradigm of objectivity, one of the standards by which we measure other discourses" (p. 108).

If mathematical objects or structures exist objectively, the question arises about our "noetic access" to them. The problem was taken up in an extended study by Michael Detlefsen which was intended to shed light on the question of "whether there are features of our acquisition of mathematical knowledge that support a realist attitude towards mathematics" (p. 74). As is well known, the chief proponent of such an attitude was Kurt Gödel, and the study is centered around his views. The core of Gödel's argument was that "propositional contents concerning mathematical concepts are imposed or 'forced' on us as being true in a manner similar to that in which propositional contents are imposed on us by sensory experience" (p. 76). This is best explained "by seeing it as a consequence of a perception-like experience of a realm of beings whose existence and characteristics are independent of our mental acts and dispositions, both individual and generic" (ibid.). This view poses many questions, the most obvious of them being: How reliable is the analogy between the forcedness of mathematical propositions and that of ordinary sensory perceptions? How reliable is this forcedness as an indicator of an independent reality of mathematical objects? Detlefsen subjects these problems to a detailed analysis and discusses both ancient and modern alternatives. Let us jump to his final remarks: "If forcedness is symptomatic for experience, then the mathematical propositions forced on us as true ought to be seen as the experimental part of mathematics. Pursuing Gödel's analogy we would then be led to consider the possibility of an observation/theory divide of mathematics" (p. 94). This is not a conventional view, but it had its predecessors, among others, in the person of Bolzano.

The analogy between the physical world and the world of mathematics is considered by John Polkinghorne. Our encounter with the quantum world has generated discussions between realist, instrumentalist, and positivist interpretations of some problems in the foundations of physics, and, according to him, these discussions have their counterparts in the philosophy of mathematics. Similar arguments could be quoted on behalf of the independent reality of mathematical entities as supporting the realist standpoint in the philosophy of physics. However, in these matters we cannot expect definitive answers: "the character of the conclusions reached will be insightful and persuasive, rather than logically coercive" (p. 29).

In his short essay Roger Penrose summarizes his well-known views on "mathematical Platonism". A new element that he introduces to the discussion (as compared with the other authors in this book) is the problem of the mind as an intermediary between the realm of mathematics and the physical world. He also argues that "the very fact that our minds are capable of comprehending sophisticated mathematical arguments—at least in favourable circumstances —leads us to the conclusion that the operation of conscious minds cannot be entirely computational and, accordingly, that our minds cannot be the product of entirely computational physics” (p. 43). Another topic of Penrose’s, “getting more out of mathematics than what we put in,” is developed by Mark Steiner in a separate essay.

The analogy between mathematics and physics raises the question of explanation in mathematics. It is not enough to know that a given phenomenon occurs; one should also know why. Stewart Shapiro, quoting from the book *Inference to the Best Explanation* by the prematurely deceased Peter Lipton (a participant in the first colloquium), distinguishes three levels of explanation in physics: by scientific laws, by causality, and by referring to unification. It might seem that only “explanation by unification” is applicable to mathematics, but Lipton tries to extend causal explanations to mathematics as well: “The idea is that mathematical propositions stand in some sort of objective dependency relations to each other” (p. 57), and causality, in a broader sense, might be understood in this way.

Indeed, I think that the problem of causality (in a broader sense), by throwing some light on the relationship between mathematics and physics, could possibly contribute to the discussion on the foundations of mathematics. It seems that mathematics and physics are linked together not only by inspirations and applications but also on a much deeper foundational level. Let us suppose that a high-energy physicist detects disintegration of an elementary particle in a cascade of other particles and is happy to conclude that what is observed well agrees with the predictions of a theoretical model. The standard reaction to this effect would be to say that the theoretical model well agrees with a “physical reality”. This suggests that we have a “physical reality” and, independently, a mathematical structure, and it just happens that this “reality” is correctly described, up to a good approximation, by this mathematical structure. However, in the eyes of the method of physics, the entire process should be looked upon in a different way. The role of mathematics in physics is not only descriptive but also prescriptive. It is a mathematical structure that determines what we should understand by the elementary particle, its interactions with other elementary particles, and its disintegration channels. When a theoretical physicist computes the behavior of a cascade of elementary particles, he treats equations of quantum field theory as expressing a kind of software of the universe. In this sense, all of the causal powers latent in the physical world come from this software. Does this tell us something about the nature of mathematics itself? I leave this as an open question.

The last paragraph testifies to the fact that *Meaning in Mathematics* is a stimulating book; it provokes the reader to his or her own reflections on the subject. This is why the book is recommended to anyone unafraid of deep questions.

About the Cover

Parallel transport by Schild’s ladder

Reading through the article by Jason Osborne in this issue, we wondered whether there existed an intuitive explanation of parallel transport along an arbitrary path on a Riemannian manifold. Professor Google came up with two.

The first, and simplest, is the one illustrated on the cover. It constructs a series of approximations to parallel vectors by constructing approximate parallelograms along the path, essentially by laying off measuring rods along geodesics in the style of Einstein’s popular accounts of relativity. This is a straightforward idea attributed to the physicist A. Schild and explained succinctly in the classic book *Gravitation* by Misner, Thorne, and Wheeler. This scheme is first order in the size of the approximating parallelograms, and is used by these authors to prove, with more or less rigor and more or less clarity, basic properties of covariant differentiation.

The second and more interesting is by means of a modern version of the ancient Chinese machine called the “south-seeking chariot”. Unfortunately there are no extant models from old times, but in theory this machine maintains a pointer to a fixed direction no matter what path the chariot follows. The basic principle at work is an ingenious mechanism involving differential gears (among the oldest known) that keeps track of the difference in angle of rotation of the chariot’s two wheels. There is a large literature on this device, including much advice on how to build a working model. One good reference, containing a useful bibliography, is “The south-pointing chariot on a surface” by Bernard Linet. It can be found among the physics articles on the `arxiv`.

—*Bill Casselman, Graphics Editor*
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