Book Review

The Evolution of an Idea

Reviewed by Robyn Arianrhod

The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century
Yvette Kosmann-Schwarzbach, translated from the French by Bertram E. Schwarzbach
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Emmy Noether, the most famous female mathematician of the twentieth century, was the daughter of mathematician Max Noether. She received her doctorate at Erlangen in 1907 (when she was twenty-five) for a thesis on the topic of algebraic invariants. She soon became part of the brilliant circle led by Felix Klein and David Hilbert at Göttingen, and popular legend says she influenced even Einstein. Of course, legends tend to abound about pioneering female mathematicians, but in this case there is some substance to the claim, as *The Noether Theorems* shows. In the interest of clarifying the intended readership, let me hasten to add that this book is not a series of portraits of colorful personalities; rather, it is a deeply scholarly work, tersely written but generously footnoted, that celebrates Noether's most famous achievements. It begins with an English translation of Noether's paper "Invariant variational problems", published originally in German ("Invariante Variationsprobleme") in 1918, which will delight mathematicians and mathematical physicists who use these theorems without having read the original version.

But *The Noether Theorems* does more than celebrate Emmy Noether and her theorems: as the subtitle suggests, it traces what Yvette Kosmann-Schwarzbach calls the "evolution of ideas" about the relationship between mathematical invariance and conservation laws, and it will be of great interest to historians of mathematics. For those who simply wish to get a feel for the work and influence of this famous female mathematician but who are not expert in the mathematics behind these theorems (namely, Lagrangian dynamics, calculus of variations, and Lie groups), let me provide a short summary. A physical law has symmetries when it is unchanged or invariant under certain coordinate or gauge transformations. Noether showed that these symmetries indicate which quantities are conserved under the law. While Noether's theorems apply to laws of physics formulated in terms of a Lagrangian, the principle can be illustrated by a well-known example from undergraduate mathematics. A particle with momentum $\mathbf{p}$ moves in a conservative force field $\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$ according to Newton's law $d\mathbf{p}/dt = \mathbf{F}$. If V is invariant under translations in the x_1 direction, then the x_1-component p_1 of the momentum is constant, since $dp_1/dt = -\partial V/\partial x_1 = 0$, so p_1 is "conserved". Another relevant undergraduate result is that mechanical energy too is conserved under a "conservative" force, because the "work" integral of such a force is path independent. The mathematical role of the boundary in Noether's theorems is illustrated by line integrals that depend only on the end-points of the relevant path.

These elementary examples are more than simplifying analogies. They illustrate what Kosmann-Schwarzbach calls the "strange story" of the Noether theorems, which were generally overlooked in the early years precisely because their significance lies in their generalizing a host of already-known results like these. These known results had been built up slowly and separately over the centuries, and it took physicists some time to appreciate the importance of Noether's discovery that a single fundamental principle united mathematical symmetries and physical conservation laws.

Even the concept of energy itself had been long contested: when Leibniz first proposed the existence and conservation of the mysterious *vis viva*, which he represented by the formula mv^2, there was a heated and sometimes partisan debate about why we would need any consequence of force other than the Cartesian/Newtonian concept of momentum, mv! In 1740 another pioneering female mathematician, Émilie du Châtelet, played an

Robyn Arianrhod is an adjunct research fellow in the school of mathematical sciences at Monash University. Her email address is robyn.arianrhod@monash.edu.

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important role [1] in defending Leibniz's concept. Following Jean Bernouilli's lead, mathematicians eventually resolved the *vis viva* debate by showing that integrating $F = dp/dt$ with respect to t gives the expression for momentum, while integration with respect to x yields the formula for (what we now call) kinetic energy. However, identifying the latter as a "work" integral came much later (as did definitive experimental confirmation of the conservation of mechanical energy). Indeed, the quest to understand the concept of energy is another example of the fascinating evolution of scientific ideas. But I digress. Kosmann-Schwarzbach's historical survey begins with Lagrange, who first connected conservation laws with symmetries in the laws of dynamics.

He took the first step towards this connection in his *Mécanique Analytique* (1788) when he outlined what we now call Lagrangian dynamics, and showed that the equations of motion derived in this way lead naturally to the known conservation laws. Kosmann-Schwarzbach writes (p. 33),

> While, before Lagrange, the various conservation results had been taken to be first principles belonging to the foundations of dynamics, Lagrange viewed them as consequences of the equations of dynamics, an important shift of point of view.

I will return to this shift towards the mathematization of physics later in this review. Kosmann-Schwarzbach adds (p. 34),

> It is only in the second edition of *Mécanique Analytique* (1811) that Lagrange observed a correlation between symmetries and the principles of conservation of certain quantities, in particular, energy.

Others made incremental generalizations of these ideas, but it was Emmy Noether who took the final step, proving a general, purely mathematical connection between symmetry (or invariance) and formal "conservation laws". However, she realized immediately that her second theorem could be applied to the conservation of energy in the then-new theory of general relativity, her first theorem being applicable to special relativity and classical dynamics. The difference in the two theorems concerns the nature and dimension of the symmetry groups of transformations and the nature of the associated conservation laws. (The first theorem applies to theories whose conservation laws are "proper" because they can be derived from a flux integral over the boundary of a surface, which leads, via Gauss's law, to a "continuity equation"; the field equations give a divergence-free quantity that expresses the relevant local conservation law. Noether's first theorem showed, for example, that "proper" conservation of energy is a consequence of the invariance of the Lagrangian under time-translations. Her second theorem applies to gauge theories whose conservation laws are "improper" because the vanishing of ordinary divergences does not yield meaningful local conservation laws, but the second theorem identifies invariant identities that can be used to define a concept of conservation. See, for example, Bergmann [2, p. 194], Byers [3].) As Kosmann-Schwarzbach puts it (p. 26), "Noether thus emphasized an essential difference between special relativity and general relativity…."

The problem of energy in general relativity had perplexed both Einstein and Hilbert, and finding a global law remains problematic because of the difficulty of including gravitational energy as well as "local" conservation of the stress-energy tensor **T**. (See Kosmann-Schwarzbach, p. 127, and for more detail, see, e.g., Wald [4, p. 286].) Local conservation can be expressed in terms of the vanishing divergence of **T**, although use of the term "divergence" refers to the symbolic similarity with vector divergences, because in general relativity one is really talking about contracted covariant derivatives of the tensor **T**, as Kosmann-Schwarzbach points out (pp. 127–128). She also notes (p. 43) that, in 1924, Schouten and Struik showed that when Noether's second theorem is applied to the Lagrangian of general relativity, "The identities obtained were also consequences of the Bianchi identities, which were well known in Riemannian geometry."

The Bianchi identities also lead to the vanishing of the contracted covariant derivatives of **T** via Einstein's field equations, and herein lies an interesting tale, which Kosmann-Schwarzbach sketches very briefly: the story of Einstein's field equations versus Hilbert's. It is important in Emmy Noether's story, because she arrived in Göttingen, at Klein and Hilbert's invitation, in the spring of 1915, when Einstein and Hilbert were each trying to derive the gravitational field equations. They would both succeed in late November of that year, although Kosmann-Schwarzbach makes the point (pp. 40–41) that not only did Einstein lay the foundation for Hilbert's work, but he also (just!) has priority in the discovery of the field equations.

The fascinating thing about the Einstein-Hilbert field equations is the very different approaches of the two men, the physicist and the mathematician. I confess that I have always been drawn to Einstein's use of physical principles: the relativity principle, together with the constancy of the speed of light in special relativity, and the "equivalence principle" in general relativity. He illustrated the latter with the example of a freely falling lift: if you drop a ball, it will appear stationary in the falling lift, because it is falling under the influence of gravity at the same rate as the lift itself. So, by changing reference frames (from the ground to the falling lift), the "force" of gravity on the ball is transformed away. Similarly, according to Einstein [5, p. 114], "we are able to 'produce' a gravitational field merely by changing the system of coordinates." In this way he showed that gravitation could be treated in a relativistic theory, in which the laws of physics retain their form (that is, remain invariant) under coordinate transformations between reference frames. The mathematical language he used to make this idea precise was that of tensor calculus (which Ricci and Levi-Civita had pioneered in the late nineteenth century). By contrast, Hilbert's approach was to use a Lagrangian-style action principle.

Kosmann-Schwarzbach shows (p. 70) that, as early as 1910, Hilbert's colleague Klein, who had been working on the geometry of Lorentz groups which are fundamental in Einstein's special theory, claimed that one could, "*if one really wanted to*, replace the term 'theory of invariants with respect to a group of transformations' with the term 'relativity with respect to a group'." The italics are mine: what a perfect illustration of the priorities of Klein the mathematician, for whom the theory of relativity is a purely mathematical theory of invariants and groups, in contrast to Einstein's use of physical principles! In fact, in 1916 Einstein implicitly acknowledged the superior elegance of the Lagrangian/Hamiltonian formulation used by Hilbert but concluded [5, p. 118]:

> It is not my purpose in this discussion to represent the general theory of relativity as a system that is as simple and logical as possible, and with the minimum number of axioms; but my main object is to develop this theory in such a way that the reader will feel that the path we have entered upon is psychologically the natural one, and that the underlying assumptions will seem to have the highest possible degree of security.

In July 1915 Hilbert had invited Einstein to spend a week at Göttingen. Einstein likely met Emmy Noether there, and even if he did not, he soon became aware of her work. Although his and Hilbert's field equations were published at the end of 1915, in the following year they were still trying to clarify the relativistic concept of conservation of energy, and both appealed to Noether for clarification, which she provided. (Like Hilbert, Einstein used a Lagrangian (or rather, a Hamiltonian) mathematical approach in examining conservation of energy [5, pp. 145–149].) By July 1918 she had completed the "Noether theorems", which impressed Einstein so much that he wrote to Klein in December:

> I once again feel that refusing her the right to teach [because of her gender] is a great injustice. I would be very favorable to taking energetic steps [on her behalf] before the ministry. If you do not think that this is possible, then I will go to the trouble of doing it alone.

Kosmann-Schwarzbach gives this quote on page 72, although she doesn't say whether or not Einstein carried out his offer, but she notes (pp. 48–49) that the following year the Weimar Republic's new Ministry of Science, Arts and Education allowed Noether to take up an appointment as a (poorly paid) *Privatdozentin*, or assistant professor of mathematics, at Göttingen. She seems to have been popular there, but within just a few years she had moved on from the "Noether theorems" to other topics in algebra, in which she was "one of the most important mathematicians of her time" (Kosmann-Schwarzbach, p. 53). In 1933 the Nazis removed her from her teaching position—like Einstein, she was Jewish, and like him, she migrated to the U.S. in 1933. Sadly, she died in 1935.

Most of *The Noether Theorems*—five of its seven chapters—is an overview of the response to these two theorems from Noether's contemporaries (I have mentioned only Einstein's reaction here) and from later physicists and mathematicians. The book traces the influence of the theorems—which first waned, and then waxed after 1980—through a detailed, highly technical survey of various attempts to modernize and extend their applicability, as well as tracing recent progress in the mathematical understanding of symmetries and conservation laws in general. A notable example is the rise of gauge theories, in which "Noether's theorem is an essential tool" (p. 130). It is significant that Kosmann-Schwarzbach concludes that resistance to Noether's theorems had little to do with sexism or racism, because her later work in algebra was "immediately recognized and admired" (p. 146); rather, it was a product of the scientific styles and interests of those who were driving progress at that time. *The Noether Theorems* thus highlights the role of historical and personal contingencies in the evolution of scientific ideas in general.

For instance, a mathematical result such as Noether's may play a relatively small role in the process of creating physical theories but can become important in later refinements of the language of the theory. Kosmann-Schwarzbach quotes (p. 89) Vizgin's 1985 assessment that Hilbert's conservation law in general relativity is "a special case of Noether's second theorem, proved two and a half years later by Emmy Noether." Similarly, we learn (pp. 80–83) that in 1927, when Eugene Wigner used parity symmetry to derive the quantum mechanical law associated with parity conservation, apparently he had not heard of Noether's theorems. (In any case, these theorems would not in themselves have applied to the discrete symmetry he used, although discrete analogues were developed in the 1970s (Kosmann-Schwarzbach, pp. 80–81, 148). In 1972, however, Wigner claimed, "We physicists pay lip service to the great accomplishments of Emmy Noether, but we do not really use her work...." This quote is given on page 82, and, as I mentioned, *The Noether Theorems* then shows how this perception slowly changed, especially after 1980. The book concludes with a list of the many recent fields of application of these theorems, not only in quantum and classical mechanics and in relativity and quantum field theory but also in elasticity theory, fluid mechanics, geometric optics, the mechanics of nonholonomic systems, locomotion, and numerical analysis.

I think Kosmann-Schwarzbach's perspective is that physicists were unduly slow in recognizing the importance of mathematical concepts like Noether's. She mentions (p. 146) that, in a 1980 address, the physicist Chen Ning Yang quoted "an amusing letter from Faraday to Maxwell as 'a good example of [our] resistance to the mathematization of physics'," as he put it with good-humored self-deprecation.

Clearly Lagrange's mathematical approach—and therefore Hilbert's and Noether's—has been marvelously successful in physics. On the other hand, Einstein preferred to ground his theory in a "psychologically natural" physical framework, and it's interesting that his point of view is similar to Maxwell's. In his 1873 *Treatise on Electricity and Magnetism*, Maxwell gave a summary of Lagrangian dynamics (before using it to examine electromagnetic energy and its conservation), but he concluded his summary as follows [6, p. 210]:

> Lagrange and most of his followers, to whom we are indebted for these methods, have endeavoured to banish all ideas except those of pure quantity, so as not only to dispense with diagrams, but even to get rid of the ideas of velocity, momentum and energy....

He went on to say that mathematics has given science many ideas that would not have been possible otherwise, but that in exploring dynamics,

> we must have our minds imbued with dynamical truths as well as mathematical methods...[This is in order] to avoid inconsistency with what is already established, and also that when our views become clearer, the language we have adopted may be a help to us and not a hindrance.

Maxwell had a genius for choosing "helpful" mathematical language, notably his innovative use of differential vector calculus to represent the intuitive field concept created by the "mathematically illiterate" Faraday. Kosmann-Schwarzbach doesn't quote Yang's "amusing letter from Faraday to Maxwell," but it's worth noting that, despite his lack of formal education, Faraday instinctively knew that in trying to describe electromagnetic effects, mathematicians like Ampère placed too much faith in the action-at-a-distance mathematical methods of Newtonian gravity theory.

Of course, no matter which approach is more aesthetically or psychologically pleasing to us, a theory stands or falls on its ability to make suitably accurate predictions. But one of the fascinating things about scientific history is the often serendipitous nature of discovery, and to a mathematician surely the most thrilling of all such contingencies is the "unreasonable effectiveness" of mathematical language in physics, to use Wigner's well known but still evocative expression [7]. This effectiveness is evident in the "mathematized" approach of Lagrange, Hilbert, and Noether, as well as in Einstein's and Maxwell's (and many others') combination of experiment, physical intuition, and appropriate mathematical language.

But, to sum up *The Noether Theorems* in its own terms, it is an important study of the work of Emmy Noether, the evolution of ideas about conservation and symmetry, and the extraordinary fertility of mathematical language.

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