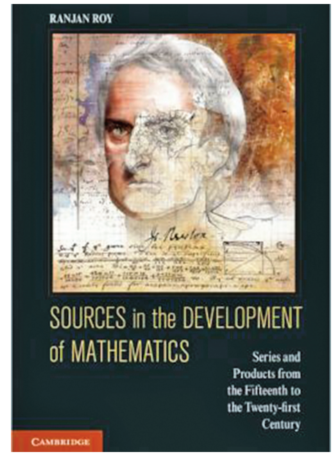
Sources in the Development of Mathematics

Reviewed by Tom Archibald



Sources in the Development of Mathematics
Ranjan Roy
Cambridge University Press, 2011
US$89.10, 994 pages
ISBN-13: 978-0521114707

For much of the twentieth century, writing about the history of mathematics emphasized above all the development of pure mathematics—and pure mathematics seen in a certain light. Abstract mathematics, of both the Bourbaki and non-Bourbaki varieties, dominated the cultural landscape of twentieth-century mathematics and formed a natural focus for historians studying that period. The creation of abstract structures, new worlds such as those of non-Euclidean geometry, and the apparently progressive evolution of rigor all posed special challenges for historians. These developments also had significant philosophical resonance, as the names Russell and Gödel suffice to underline, and the importance of the history of mathematics for philosophy was well illustrated (*remains* well illustrated) by the work of Lakatos.

The world changes, as does its mathematics, and historians looking at the mathematical world now tend to see a somewhat different picture. Many of the same issues remain fundamentally important, but they are no longer seen as providing a complete account of all important aspects of mathematical activity. The consensus about what mathematics is, as seen in twentieth-century historical writings about mathematics, no longer dominates. Historical writers today need to encompass a larger range of mathematical activity in their studies.

Tom Archibald is professor of mathematics at Simon Fraser University. His email address is `tarchi@sfu.ca`.

DOI: http://dx.doi.org/10.1090/noti1057

One way to achieve such a fresh and broad viewpoint is to seek unifying themes that cut across contemporary mathematical subfields. Ranjan Roy's erudite and broad study does just that, and in so doing provides an alternative thread through much historical mathematics of the last few centuries. Using series and products, both finite and infinite, as a leitmotif, Roy's forty-one chapters give us just short of one thousand pages in which mathematics is a concrete entity and mathematicians are working on a wide variety of problems. Calculations, formulas, manipulations, and brilliant insights into pattern are massed together by Roy to provide the "sources" of mathematical development described in the title. While most of the time the path of this thread remains close to analysis, the points of contact with many other major areas of mathematical research are plentiful, and the associations often both stimulating and surprising. The book is a personal view based on an evidently long immersion in both primary and secondary sources. It reads well and easily, and while there are many associations between the chapters, each one can be read as a self-contained vignette. It is broadly accessible, so that while some of the more recent or more technical developments described will perhaps be out of the reach of beginners, most people with mathematical training can read a great deal of the work and be both informed and entertained. It would be a great book to take on summer vacation (well, provided you are not kayaking or backpacking—it's a bit heavy).

The treatment is not strictly chronological over the course of the book, though chronology informs its organization: subjects that "started" later are typically treated later. Indeed, the care with which the subject has been divided in order to give both details and a big picture is one of the real services of the work. Given the titles of chapters, the contents often surprise in a good way. Let's take an example: the short (sixteen-page) chapter on inequalities. This begins with a survey section, discussing the arithmetic-geometric mean inequality and its treatment by Thomas Harriot (before 1631!), Descartes, Newton, Maclaurin, and Cauchy; then noting that Cauchy's method inspired Jensen, tying this to Rogers, Hölder, and Schwarz; and concluding with Hilbert and Riesz. The chapter continues with individual sections that provide additional detail on aspects of the mathematical work on this subject by Harriot, Maclaurin, Jensen, and Riesz. For example, the section on Harriot introduces his own notation, then gives his own proof of the arithmetic-geometric mean inequality, while the section on Riesz gives his proof of the Minkowski inequality. A concluding section (there is one of these for every chapter) points to primary and secondary literature of high quality. Along the way, many other mathematicians are both referred to and quoted; one can savor here, for example, Harald Bohr's remark that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove." The chapter also has five pages of exercises, usually with quite direct and explicit links to historical literature. This organization is typical of the book as a whole.

Since there are forty-one chapters, making a list of what is treated is not possible here. Roy is perhaps most widely known for his book with Richard Askey and George Andrews, so chapters on the hypergeometric series, q-series, and partitions are not surprising, nor are many other chapters that touch on closely related areas of classical analysis involving series and products. There is quite a bit of number theory (L-series, distribution of primes, transcendence), complex analysis (value distribution theory, for example), and there are even chapters on solvability by radicals and finite fields. We can remark on some things that are perhaps quite unexpected features. The first chapter is about power series in Kerala in the fifteenth century, for example—a nice addition. There are many places where items that are perhaps less well known than they should be are brought to light: I noted with pleasure Harkness and Morley's proof of the binomial theorem; Zolotarev's method for Lagrange inversion with remainder; Hermite on approximate integration; Lagrange's proof of Wilson's theorem; Bohr, Mollerup, and Artin on the Γ function. There are many, many more such instances. I just spent a semester teaching a seminar for seniors on classical analysis, and the book was tremendously useful as a resource.

Some of the longer survey chapters are in many cases attractive and motivating introductions to the subjects they treat. As an example, I'll mention the two chapters on elliptic functions in the eighteenth and nineteenth centuries. The two form a unified historical account at an introductory level to the basic ideas of the theory, culminating with a discussion of Eisenstein's application of the ideas to reciprocity laws. Much of the same material is covered (more succinctly, with more required input from the reader, and aiming squarely at modern results) by McKean and Moll in their lovely book *Elliptic Curves*. But in Roy's account the historical aspects are treated more fully, the pace is more leisurely, and the connections between the various developments are presented in a way that makes us more fully aware of the mathematical context of the developments at the time. Often Roy provides what feels like reliable insight into the innovations he describes, for example, with Euler's realization that $dt/\sqrt{1-t^4}$ could not be rationalized by substitution, since this would imply that the equation $z^2 = x^4 - y^4$ had integer solutions (as Euler knew from Fermat to be false).

Partly because this is material I find interesting—it is a *kind* of mathematics I particularly like and believe to be central to much mathematical endeavor that grew from these roots—I found the book charming. If I make a few minor points about what the work does not provide and indicate things that I could wish were otherwise, it does not detract from my overall view that this is a fine work. As an historian, I would love to have seen more thorough references, though the forest of citations that would result might have detracted from readability, and it can be argued that in some cases such references can be filled in with relative ease. For example, the statement (p. 83) that "some historians have suggested with some justification" that personal contact between Descartes and Faulhaber could have sent us to Kenneth Manders's paper of 2006, but it didn't, and the literature survey at the end of the chapter didn't either. Sometimes such statements contain an implicit historical analysis that may be correct but isn't provided: "one can see in the work of Newton, Leibniz, and others that they were implicitly aware of the method [of summation by parts]" (p. 195). A fair response to this kind of criticism is that "it is not that kind of book," but I decided to mention it given the many very welcome references to recent historical work that the author does bring to the reader's attention. There are a few typographical errors, even in proper names. Reisz and Frèchet should have been caught by the copy editors, who no longer seem to exist. The book's title might encapsulate the author's view of what he aims to do but unfortunately does not describe the work clearly; some variant of the subtitle using the words "series" and "products" would have been better in my view. To counter these negative remarks, allow me to compliment the figures and the index, both of which are well done and useful.

I recommend this book to a wide audience. Undergraduates can learn of the truly vast amount of material that lies alongside some of their more standard endeavors, many of which involve only elementary matters: sums, products, limits, calculus. Graduate students and nonspecialist faculty can wonder at the ingenuity of their predecessors and the connections between now disparate areas that are afforded by this very classical view. They'll also get lots of good ideas for teaching (and they may waste a good deal of time on the problems, as well). Historians, philosophers, and others should read this book, if only for the view of mathematics it propounds. And specialized researchers in the area of special functions and related fields should simply have a good time. All of these readers can benefit from the remarkable expository talents of the author and his careful choice of material. Among personal views of mathematics that use history as a key to understanding, Roy's book stands out as a model.