Stuck in the Middle: Cauchy's Intermediate Value Theorem and the History of Analytic Rigor

Michael J. Barany

Intermediate Values

With the restoration of King Louis XVIII of France in 1814, one revolution had come to an end, but another was just beginning. Historians often describe the French Revolution of 1789, along with its reactions and repercussions, as the start of the modern era. For many historians of mathematics, however, the modern era began with the Bourbon Restoration and the mathematics of Augustin-Louis Cauchy.

In a 1972 talk at a Mathematical Association of America sectional meeting, Judith Grabiner offered an important interpretation of this critical juncture in the history of mathematics [5]. Asking whether mathematical truth was time dependent, Grabiner argued that while truths themselves may not depend on time, our knowledge certainly does. She stressed the revolutionary character of Cauchy and his contemporaries' efforts to set mathematical analysis on a rigorous footing, which required applying a fundamentally new point of view to the problems and methods of their eighteenth-century forebears. One of Grabiner's leading illustrations of this point, both in her talk and in her 1981 book on Cauchy's calculus [6], is Cauchy's proof of the intermediate value theorem, that a continuous real-valued function f on an interval $[a, b]$ assumes every value between $f(a)$ and $f(b)$ on that interval.

Today the intermediate value theorem (IVT) is one of the first theorems about functions that advanced undergraduates learn in courses on mathematical analysis. These courses in turn are often the first places such students are comprehensively taught the methods of rigorous proof at the heart of contemporary mathematics. Here the theorem and its proof exemplify several important aspects of rigorous analysis. At first glance, the theorem seems obvious. Indeed, generations of mathematicians before Cauchy thought its idea so obvious as not to need explicit statement or justification. On the other hand, that the theorem can be proved with just some simple notions about continuity, convergence, and the system of real numbers is something quite remarkable. Students learn to take nothing for granted, to proceed systematically, and to use approximations and limiting principles to turn vague intuitions about the nature of functions into indubitable theorems.

Cauchy's undergraduate course in analysis at the École Royale Polytechnique, which he began teaching in 1816, was among the first to include a proof of the intermediate value theorem. His 1821 textbook [4] (recently released in full English translation [3]) was widely read and admired by a generation of mathematicians looking to build a new mathematics for a new era, and his proof of the intermediate value theorem in that textbook bears a striking resemblance to proofs of the

---

Michael J. Barany is a doctoral candidate in Princeton University's Program in History of Science. His writings on the history of science and mathematics can be found at `http://www.princeton.edu/~mbarany`. *His work is supported by a National Science Foundation Graduate Research Fellowship (Grant No. DGE-0646086). His email address is* `mbarany@math.princeton.edu`.

DOI: http://dx.doi.org/10.1090/noti1049

theorem that continue to be taught almost two centuries later.

With this in mind you might be surprised to learn that the theorem was proved *twice* in Cauchy's textbook, and that his more famous proof was relegated to an appendix, while Cauchy's main proof has been mostly forgotten by mathematicians and historians alike. Putting these two proofs side by side, we can add a new dimension to Grabiner's story by asking just what rigorous analysis meant for Cauchy at the dawn of modern mathematics. When Cauchy's language and methods are carefully dissected, he begins to look less like a far-sighted revolutionary who simply saw profound new meanings in old results. Instead, I argue (see also [1]) that Cauchy was "stuck in the middle": struggling to reclaim what he saw to be a neglected approach to mathematics while (perhaps inadvertently) pushing mathematicians toward a particular understanding of analytic rigor that would help define their future.

A Mathematical Revolution

Underneath the slogans of liberty, equality, and fraternity and behind the barricades and the bluster of the French Revolution, there was a massive transformation in the organization of the French state and society. For the world of mathematics, these transformations meant that, for the first time, a large cadre of elite military and civil engineers began to receive a common training in Paris in the most advanced mathematics of the day. These engineers took their mathematics and applied it to the pressing problems of the modern world: mass infrastructure, navigation, mining, energy, and war. The flagship institution where these students learned to draw, compute Taylor expansions, and see the world through mathematical eyes was the École Polytechnique (renamed the École Royale Polytechnique after Napoleon's defeat and the monarchy's return), and it was there that Cauchy made his mark as a student and then as an instructor.

Despite his acclaim beyond the walls of the École, Cauchy was not the most popular instructor among either his students or his fellow faculty. He regularly overran his allotted lecture time; his courses could be dense and difficult to follow; he revised the curriculum with abandon, disregarding the pleas of those teaching courses for which his was a prerequisite. His foes among the faculty grumbled that Cauchy, a devout Catholic and staunch supporter of the monarchy, was a bitter reactionary who owed his job more to the changing winds of national politics than to his brilliance as a teacher.[1]

[1] *On Cauchy's interconnected politics, religion, and pedagogy, see* [2].

In Cauchy's view, however, a restoration was just as much due for mathematics as it had been after the regrettable revolution in France, and it was no use arguing with the misguided mathematical Jacobins or Bonapartists who would have it otherwise. When Cauchy looked at the mathematics of the eighteenth century, he saw a discipline that had lost its discipline. Undoubtedly, the century had witnessed a host of marvelous mathematical innovations, but at what cost? Mathematicians such as Leonhard Euler freely toyed with nonconvergent series and ungrounded formal expressions and did not bat an eyelash when these produced absurd conclusions. Amidst the swirl of infinities, heuristics, imaginary numbers and more, it was hard to know what to believe.

At stake for Cauchy was the proper relationship between algebra and geometry. Geometry, as most saw it, was the ancient and noble science of magnitudes initiated by the Greeks and particularly associated with Euclid.[2] On the one hand, geometry referred to a specific body of problems and techniques associated with shapes and magnitudes. On the other hand, however, geometry was an emblem of philosophical exactitude and precision. Mathematicians and philosophers alike sought to proceed in the *more geometrico*, or geometric way, stating their assumptions carefully and reasoning systematically in order to produce results with absolute certainty.

Algebra, like geometry, could refer to a body of problems and techniques. From Viète and Descartes to Laplace and Lagrange, mathematicians (not all of them French) had developed the symbolic methods of algebra into a powerful tool for studying a wide range of mathematical phenomena, including those traditionally associated with geometry.[3] Also, like geometry, algebra had an implicit philosophical meaning that tied it to the unrestrained pursuit of mathematical ideas, regardless of whether each individual step had a clear geometric or physical interpretation. Eighteenth-century mathematicians saw in algebra a versatile tool for obtaining deep understandings of the world around them.

Algebra and geometry thus represented competing values. Algebraic mathematicians valued the profound mathematical truths their methods could reveal and ridiculed geometric mathematicians for their overzealous commitment to tedious proofs at the expense of vital creativity. Geometric

[2] *When Cauchy wrote his textbook, non-Euclidean geometries were still just over the horizon of mathematical theory.*

[3] *The specific body of theory and techniques we now call abstract algebra was, like non-Euclidean geometry, still just beginning to emerge as Cauchy wrote. Some of Cauchy's earliest work was on problems we might now consider in this area.*

mathematicians like Cauchy, by contrast, reviled the monstrosities that algebraic mathematics occasionally produced and sought protection in the rigorous certainties of their methods. Cauchy's textbook famously declared his desire to give his methods "all the rigor one requires in geometry, in such a way as never to resort to reasons drawn from the generality of algebra." Rarely, of course, were the values of algebra and geometry so sharply delineated. It was often in polemical writings such as Montucla's monumental history of mathematics [7, e.g., pp. 11, 270] or the introduction to Cauchy's textbook rather than in everyday mathematical work or in the École's courses on drawing and practical mathematics that these stakes loomed large. Nevertheless, the tension was real, and (at least as Cauchy saw it) algebra was winning.

Two Proofs

Cauchy's textbook introduces the intermediate value theorem by noting a "remarkable property of continuous functions of a single variable": that they can represent the geometric ordinates of continuous curves [4, p. 43]. Contrary to our present emphasis on the theorem in terms of the analytic properties of continuous functions, for Cauchy the theorem is foremost about the relationship between functions and geometry. This, we shall see, was not just his motivation but the central idea in his proof.

The theorem's statement is recognizable to readers today, even if the precise wording and notation appear unusual:

Theorem (Cauchy's IVT)**.** *If the function $f(x)$ is continuous with respect to the variable x between the limits $x = x_0$ and $x = X$ and if b designates a quantity between $f(x_0)$ and $f(X)$, one can always satisfy the equation*

$$f(x) = b$$

for one or several real values of x between x_0 and X.

But Cauchy's main proof of the theorem looks nothing like the proof we now associate with him. Here is a rather literal translation:

Proof. To establish the preceding proposition, it suffices to see that the curve whose equation is

$$y = f(x)$$

meets one or more times the straight line whose equation is

$$y = b$$

in the interval between the ordinates that correspond to the abscissas x_0 and X. Yet it is clear that this will take place under our hypotheses. Indeed, as the function $f(x)$ is continuous between the limits $x = x_0$ [and] $x = X$, the curve whose equation is $y = f(x)$ passing first through the point with coordinates $x_0, f(x_0)$ and second through the point with coordinates X and $f(X)$ will be continuous between these two points; and, as the constant ordinate b of the line whose equation is $y = b$ is found between the ordinates $f(x_0)$ and $f(X)$ of the two points under consideration, the line [corresponding to $y = b$] will necessarily pass between these two points so that it cannot avoid crossing the above-mentioned curve [corresponding to $y = f(x)$] in the interval. □

The first thing to notice is that, while Cauchy employs several variables and equations, he uses these symbolic expressions purely to describe curves and lines in a plane. There are no algebraic manipulations whatsoever, much less sequences, bounds, or limits. He presents the continuous function $f(x)$ as an unbroken curve connecting two points, and his proof hinges on a claim that a level line corresponding to the desired intermediate value must cross this curve. The argument is vague and unsystematic by our standards. Even though Cauchy has just given a definition of continuity,[4] his proof makes no use of it. Instead, the notion of continuity in this proof means simply that the function's corresponding curve remains unbroken.

Was Cauchy sloppy, lazy, or inconsistent with this proof? I have found nothing to suggest that he or his contemporaries had second thoughts about it.[5] Instead, we should see this as evidence of Cauchy's faith in geometric reasoning and his lingering distrust of algebra. Arguments based on unbroken planar curves were sensible and trustworthy to Cauchy in a way that arguments based on symbols and equations were not. Because

[4] *Cauchy's definition of continuity will also appear unusual to those expecting epsilons, deltas, quantifiers, and convergence. He defines continuous functions as those for which the difference $f(x+\alpha) - f(x)$ is infinitesimally small when α is. See* [4, pp. 34–35]. *There is considerable secondary literature on Cauchy's "infinitesimally small quantities" and their relationship to various ideas about variables, continuity, and convergence. In particular, while some have argued his notions were ultimately equivalent to ideas developed in either "epsilontic" or nonstandard analysis, most agree that Cauchy omitted or left implicit many important ideas and intuitions about his infinitesimally small quantities.*

[5] *One might be tempted to dismiss this proof as merely a pedagogically oriented plausibility argument, but Cauchy himself makes no such excuse for it, and such a move would be exceptional in a textbook meant to showcase his model of rigor. His allusion to the "direct and purely analytic" proof in the appendix is sometimes read as an admission that the above proof is inadequate, but this view substitutes later values of analytic rigor where Cauchy's own priorities are at best unclear. The best evidence of Cauchy's view remains the fact that he calls this argument a proof (something he does not do for every argument following a stated theorem) and places it prominently in the body of his textbook.*

he could visualize two curves crossing, he needed no further argument to establish his theorem. Here the "rigor of geometry" involved not just careful systematic reasoning but the use of a fundamentally geometric argument.

Cauchy's more famous proof of the intermediate value theorem comes in an appendix on solving equations numerically. Here the above theorem becomes a corollary to the first theorem of the appendix, which states that if a function is continuous between $x = x_0$ and $x = X$ and if $f(x_0)$ and $f(X)$ have opposite signs, then there is at least one root satisfying $f(x) = 0$ between x_0 and X. One applies this result to the function $f(x) - b$ to obtain the familiar theorem.

Grabiner [5, p. 362] is among many who note that the proof in Cauchy's appendix is based on a method of approximating roots that was well known in Cauchy's time. Cauchy supposes that the interval between x_0 and X has length h and divides the interval into m parts of length h/m for some m greater than 1. Inspecting values of f for consecutive terms of this sequence and picking one pair of such terms where the corresponding values of f have opposite signs, Cauchy then subdivides this new interval of length h/m into m parts of length h/m^2 and repeats the process to produce two sequences of x values. The first, denoted $x_0, x_1, x_2, \ldots$, is increasing, while the second, $(X, X', X'', \ldots)$, is decreasing, with corresponding terms in the two sequences coming closer and closer together.

From this, Cauchy concludes that the sequences have a common limit a. Without citing his earlier theorem that continuous functions map convergent sequences to convergent sequences, he then simply stipulates that the sequences

$$f(x_0), f(x_1), f(x_2), \ldots$$

and

$$f(X), f(X'), f(X''), \ldots$$

must both converge to $f(a)$. Finally, Cauchy claims that, because corresponding terms of these two sequences have opposite signs, $f(a)$ must have the value 0.

At first glance, this is a pure example of the rigorous algebraic analysis for which Cauchy is known today. Nevertheless, we can still see Cauchy's preference for geometric reasoning. On the one hand, Cauchy's proof does more than one would expect if the goal were simply to prove the existence of a root. Why, for instance, carry out the argument with an arbitrary value of m when simply halving the interval each time is sufficient? Cauchy's rhetoric makes it clear that he continues to see his procedure primarily as a tool of approximation rather than as an existence proof. Thus he makes repeated reference to the possibility of there being multiple roots and elaborates on this point in two of the three scholia that follow the proof. The first such scholium is even more directly about approximation: it notes that the average of the terms x_n and $X^{(n)}$ is at most a distance $\frac{1}{2}\frac{X-x_0}{m^n}$ from the desired root a—an observation that is extraneous (in the context of Cauchy's argument) to the question of whether such a root exists but is important when one cares about rates of convergence for approximations.

Why, for that matter, did Cauchy try to find a root instead of an arbitrary intermediate value b? The proof would need few modifications to fit this more general case. Making this theorem about roots rather than arbitrary values allowed Cauchy to preserve it as an argument about a curve meeting a line (in this case, the x-axis). At the same time, while finding arbitrary values was (simply put) a rather arbitrary thing to do, the engineers-in-training at the École would have had many occasions to find roots in the course of their work and studies.

On the other hand, Cauchy leaves several potentially important ideas between the lines. We know, for instance, what it means for terms to be pairwise of opposite signs, but what does this mean in the limit of a sequence? Proofs today typically specify one sequence of values as approaching the intermediate value from below and the other from above. Indeed, it is striking that, though the proof discusses sequences and values with opposite signs, the only symbolic inequalities in the entire argument are used for values of x and never for values of $f(x)$. For both x and $f(x)$, Cauchy refers frequently not just to values but to *quantities*, implying that they have geometric magnitudes.

This explains, in part, why Cauchy so freely makes claims about the convergence and limits of the sequences obtained in his proof. As a pure matter of algebraic abstractions, one needs a lemma to assert that the common limit of the sequences $f(x_0), f(x_1), \ldots$ and $f(X), f(X'), \ldots$ would equal $f(a)$. As a matter of geometry, however, the identity might not strike one as requiring a separate argument or citation.

It would not be until several decades after Cauchy's course was published that mathematicians would systematically attempt to define teratological functions that defied the intuitions associated with the usual mechanical problems of polytechnical mathematics. In Cauchy's time, mathematical analysis remained first and foremost the mathematical study of the world—a world filled with complex phenomena but also a world exhibiting profound regularities. Cauchy's course, for instance, assumes that all continuous functions are differentiable; indeed, all the continuous functions he might care to differentiate were more or less smooth. In a sense, then, Cauchy's preference for geometry reflected a desire to remain true to the world and to use only those mathematical techniques that genuinely reflected worldly magnitudes, even if this limited what he could say mathematically about that world.[6]

Stuck in the Middle

The peculiarities of Cauchy's proofs help us see that the rigor Cauchy prized was something quite different from the rigor we now associate with his name. When Cauchy objected to the mathematics of his predecessors, he did not find them lacking in their adherence to formal rules for symbolic manipulation. Quite the contrary, he felt mathematicians in the preceding century trusted such rules altogether too much. With this in mind, it is not surprising that Cauchy's project of reform was not, at its heart, based on carefully placed quantifiers, deftly manipulated sequences and inequalities, and meticulous logical exactitude.

To tame the dangerous fashion for algebra, Cauchy demanded a return to geometry in both its senses. He is remembered today for making his proofs systematic and logical, but his own proofs place a clearer emphasis on the geometry of magnitudes, not the geometry of methods. Cauchy sought to save mathematical analysis by ensuring that its powerful algebraic tools stayed true to the world of geometry, hence his insistence on convergence and his caution when defining imaginary and even negative numbers. Where he did not see any danger of symbols losing their geometric referents, as in his proofs of the intermediate value theorem, he could in fact be quite lax with their use.

In this sense, Cauchy's analysis appears surprisingly regressive. The rigor he advocated was a return to geometric reasoning that a century of mathematicians had rejected as stale, tedious, and counterproductive. His methods were difficult, and his students and colleagues frequently lamented their cumbersome impracticality. And yet, Cauchy seems now to have won the day.

How could such a reactionary mathematician so transform the mathematics of his generation in a way that now appears progressive and visionary? Cauchy realized he could not do away with algebra even in his own mathematics. Rather than throw algebra out entirely, he worked to endow algebra with the virtuous rigors of geometry by developing algebraic criteria to match geometric reasoning. In so doing, he advanced the idea that it was possible to have it both ways: to enjoy the power of algebraic thinking while still adhering to the discipline and certitude of geometry. All one needed was to put the demands of geometry in algebraic terms.

Thus Cauchy's rules for convergence and his attention to the limits of formal expressions' validity created new problems and new opportunities. He opened up ways of studying mathematical phenomena that remain vital nearly two centuries later. While his approach proved durable, its initial motivation could easily be forgotten. The ensuing generation of European mathematicians latched on to his disciplined way of studying the meanings of formal expressions while jettisoning his preoccupation with the geometry of magnitudes. From them, we have the beginnings of the set-theoretic foundations of analysis that undergraduates learn today.

Of course not even the set theorists had the final word on rigor. Mathematicians must constantly balance what methods are worthwhile, what arguments are convincing, and what values are worth conveying to students. Those dismayed by the lack of consensus on these points today or wishing that opponents could simply see that their positions are illogical, unrigorous, or counterproductive can take comfort in the fact that these debates are not just a normal part of the history of mathematics, but that they can help to spur new ideas and approaches, often in unexpected ways. Studying the history of mathematics, we can appreciate that in some ways we are all, like Cauchy, stuck in the middle between our discipline's past and its open-ended future.

[6] *I elaborate in* [1] *how this mathematical impulse relates to Cauchy's religious and political conservatism.*

References

[1] M. J. Barany, God, king, and geometry: Revisiting the Introduction to Cauchy's Cours d'analyse, *Historia Mathematica* **38**(3)(2011):368–388.

[2] B. Belhoste, *Augustin-Louis Cauchy: A Biography*, translated by F. Ragland. Springer, New York, 1991.

[3] R. E. Bradley and C. E. Sandifer, *Cauchy's Cours d'Analyse: An Annotated Translation*, Springer, New York, 2009.

[4] A.-L. Cauchy, *Cours d'Analyse de l'École Royale Polytechnique;* 1^{re} *Partie, Analyse Algébrique*, Debure, Paris, 1821.

[5] J. V. Grabiner, Is mathematical truth time-dependent? *The American Mathematical Monthly* **81**(4)(1974):354–365.

[6] ———, *The Origins of Cauchy's Rigorous Calculus*, The MIT Press, Cambridge, MA, 1981.

[7] J. F. Montucla, *Histoire des mathématiques, nouvelle édition, considérablement augmentée, et prolongée jusque vers l'époque actuelle*, Volume III, Henri Agasse, Paris, 1802.