Problem Solving: Moving from Routine to Nonroutine and Beyond

Barry Garelick

An important part of the job of teaching math in K–12 is to stretch students—to teach them creative and personal engagement with the material. At some point this must involve expecting students to come up with previously unfamiliar steps on their own for new problems that do not lend themselves to known algorithms, prescribed methods, and predictable approaches. An effective way of doing this is to extend routine problems that students know how to solve into nonroutine problems.

Over the past two decades, however, disagreements between advocates of traditional or conventional math teaching and the math reform movement have resulted in a fragmented approach to teaching math. A key area of disagreement centers on the distinction between "exercises" and "problems". Math reformers generally believe that conventional math teaching consists mainly of routine problems that are nonthinking, repetitive, tedious and do not lead to students learning to solve nonroutine problems.

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DOI: http://dx.doi.org/10.1090/noti1050

One math reform approach has been to present students with a steady diet of "challenging problems" that neither connect with the students' lessons and instruction nor develop any identifiable or transferrable skills. The following problem from Hjalmarson and Diefes-Dux (2008) is one example: How many boxes would be needed to pack and ship one million books collected in a school-based book drive? In this problem the size of the books is unknown and varied, and the size of the boxes is not stated. While some teachers consider the open-ended nature of the problem to be deep, rich, and unique, students will generally lack the skills required to solve such a problem, skills such as knowledge of proper experimental approaches, systematic and random errors, organizational skills, and validation and verification.

Based on my experiences as both student and teacher, as well as the experiences of veteran math teachers, I submit that a substantial education in mathematics should steer a middle course between the proliferation of routine problems and reliance upon unique, complex projects. Students should learn to apply basic principles in a much wider variety of situations than typically presented in texts. Such problems, however, should not be as complex or as time consuming as the example above. A math problem is not necessarily useful just because it requires outside-of-the-box insight and/or inspiration and will generally not result in a problem-solving "habit of mind".

Extending Routine Algorithmic Problems to Nonroutine Problems

Problems that are extensions of routine problems, (as suggested by Henderson and Pingry (1953)) can effectively build upon prior knowledge and develop a larger problem-solving repertoire. Such extensions require students to synthesize ideas and procedures from two or more areas of learned procedures and are frequently multistep. For example, while most students have learned how to factor $a^2 - b^2$, they are likely to find the following difficult to factor: $a^2 - x^2 + 2bx - b^2$. This problem requires students to synthesize what they know about perfect square trinomials, how they are factored, and how these ultimately relate to factoring the difference between two squares. An intermediate problem may be introduced, such as $a^2 - (m - n)^2$, to serve as scaffolding for the more challenging version.

The factoring of the difference of two squares can be extended even further: Find two positive integers "a" and "b" such that $a^2 - b^2 = 2{,}011$. A student may first factor to obtain $(a + b)(a - b) = 2{,}011$. Some students may not be able to go any further. Others will see that expressing the right-hand side of the equation as the product of two factors xy puts them on the pathway to solving it. Students may be stumped and will overlook the fact that 2,011 and 1 can represent x and y.

Extending Routine Word Problems to Nonroutine Problems

A frequent criticism of word problems in textbooks is that they present a worked example/method for solving a particular type of problem, followed by a set of almost identical problems to solve. Students may experience the practice of applying a memorized technique and mechanically look for the data to plug in to the appropriate equations without having to read the problem. But what if you are given values with different units in a distance/rate problem? And what if you are given two legs and two different rates and need to find your average rate? Word problems can and should be varied for improving problem-solving ability.

This is, in fact, what is done in well-written algebra textbooks or in problem sets devised by teachers. Students are given instruction via worked examples and some initial practice problems. After that, the problems vary. For example, consider the following two distance/rate problems from Dolciani et al. (1962):

> A freight train left Beeville at 5 AM at 30 miles per hour. At 7 AM an express train traveling 50 miles per hour left the same station. When did the express overtake the freight?

> An airplane traveling 280 miles per hour leaves San Francisco 13½ hours after a steamship has sailed. If the plane overtakes the ship in 1½ hours, find the rate of the steamship.

While the variations between these two problems are slight (one asks for time traveled, the other speed), they present a challenge to beginning students. Varying the structure of the basic problem forces the student to read each problem carefully. They must identify what the question is, what data are provided, and how to transform it into an equation. The variation and difficulty increases so that the last problems of the set are more complex and nonroutine:

> A ship must average 22 miles per hour to make its ten-hour run on time. During the first four hours, bad weather caused it to reduce speed to 16 miles per hour. What should its speed be for the rest of the trip to keep the ship to its schedule?

Extending the nonroutine distance problem even further, students can be given the following problem in an Algebra 2 course:

> Two boys in a canoe are at a confluence of a lake and a river. A house upriver and a boulder on the other side of the lake are equidistant from the boys. Would it take the same amount of time to paddle across the lake to the boulder and back as it would to paddle to the house upriver and back (at the same speed relative to the water each way for both options)? If not, which option would take longer?

This problem does not allow a "plug-in" solution. Students must model the situation using symbols and apply their knowledge of expressing time as a function of rate and distance. Many students will assume that either route will take the same amount of time, reasoning that the amount that the canoe is slowed down by the current when traveling upstream is canceled by the additional speed the current imparts when traveling downstream. Proving or disproving the initial intuitive response, however, requires mathematics. Ultimately, it involves an inequality that does not lend itself to an algorithmic solution.

Building a Problem-Solving Repertoire

Students should routinely be presented with nonroutine problems. Even if a student fails to solve such a problem, seeing its solution opens up pathways in which they discover relationships between content and what they know about solving routine problems.

Sweller et al. (2010) state that problem solving cannot be taught independently of basic tools and basic thinking. Over time, students build up a repertoire of problem-solving techniques. Ultimately, the difference between someone who is good and someone who is bad at solving nonroutine problems is not that the good problem solver has learned to solve novel, previously unseen problems. It is more the case that, as students increase their expertise, more nonroutine problems appear to them as routine.

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In Memory of Professor Francisco Federico Raggi Cárdenas

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Professor Francisco Federico Raggi Cárdenas passed away June 12, 2012, in Mexico City. Professor Raggi was born in Mexico City in 1940. He obtained his undergraduate degree from Universidad Nacional Autónoma de México (UNAM), a master's degree from Harvard, and a doctorate from UNAM. In 1962 he joined the Institute of Mathematics of UNAM, where he worked until his passing. He held visiting positions in universities throughout the world, including institutions in Germany, the United States, Belgium, Canada, Spain, and Italy.

Professor Raggi taught modern algebra at all levels to many generations of students in Mexico. In particular, he had four Ph.D. students. Raggi was author or coauthor of five books that are prominently used in universities throughout Mexico. As he traveled constantly and taught at many places throughout Mexico, Raggi's commitment to the

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DOI: http://dx.doi.org/10.1090/noti1052

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