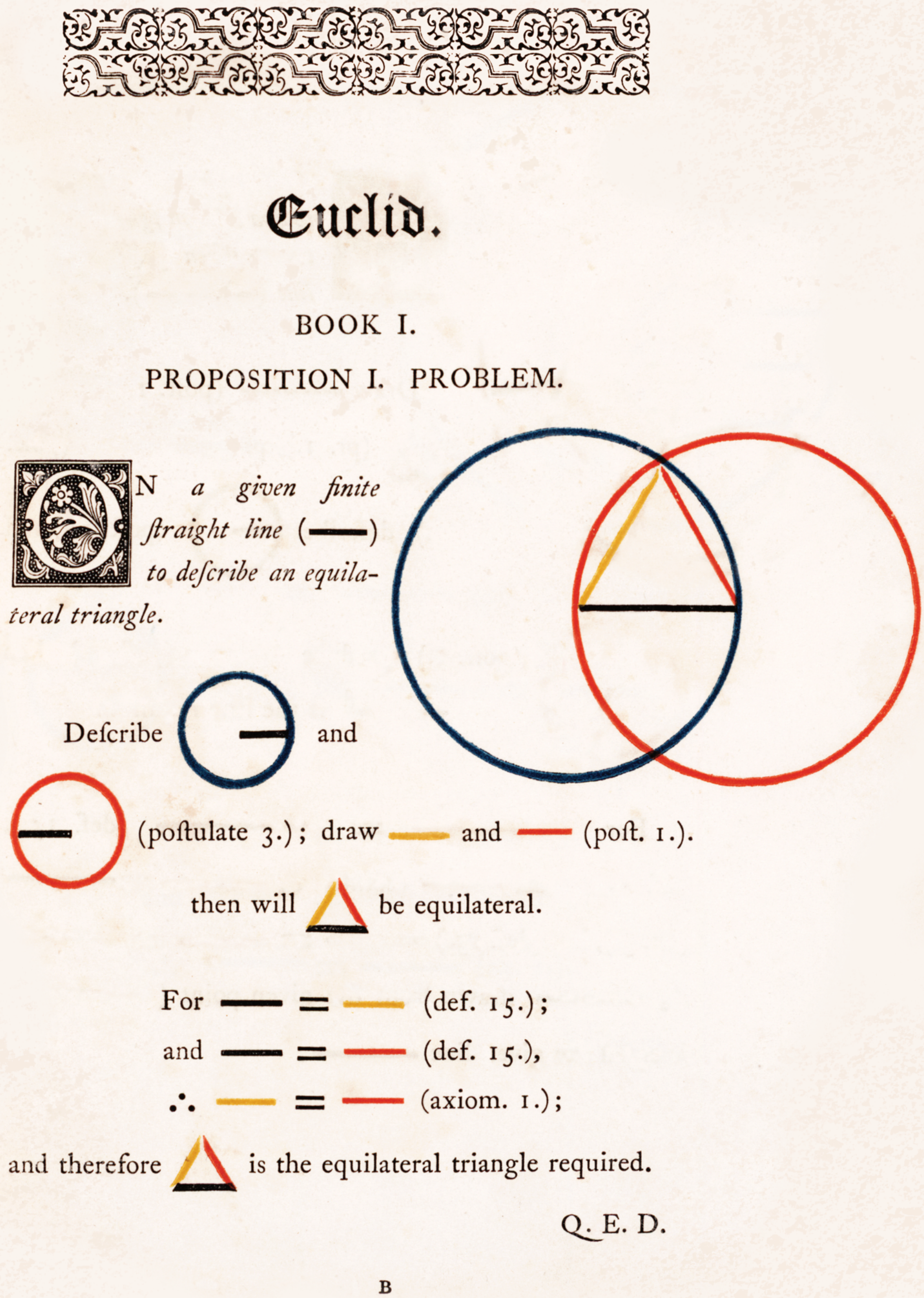
ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

January 2014 Volume 61, Number 1


Euclid.

BOOK I.

PROPOSITION I. PROBLEM.

On a given finite ſtraight line ( ) to deſcribe an equilateral triangle.

Deſcribe and (poſtulate 3.); draw and (poſt. 1.).

then will be equilateral.

For = (def. 15.);

and = (def. 15.),

∴ = (axiom. 1.);

and therefore is the equilateral triangle required.

Q. E. D.

B

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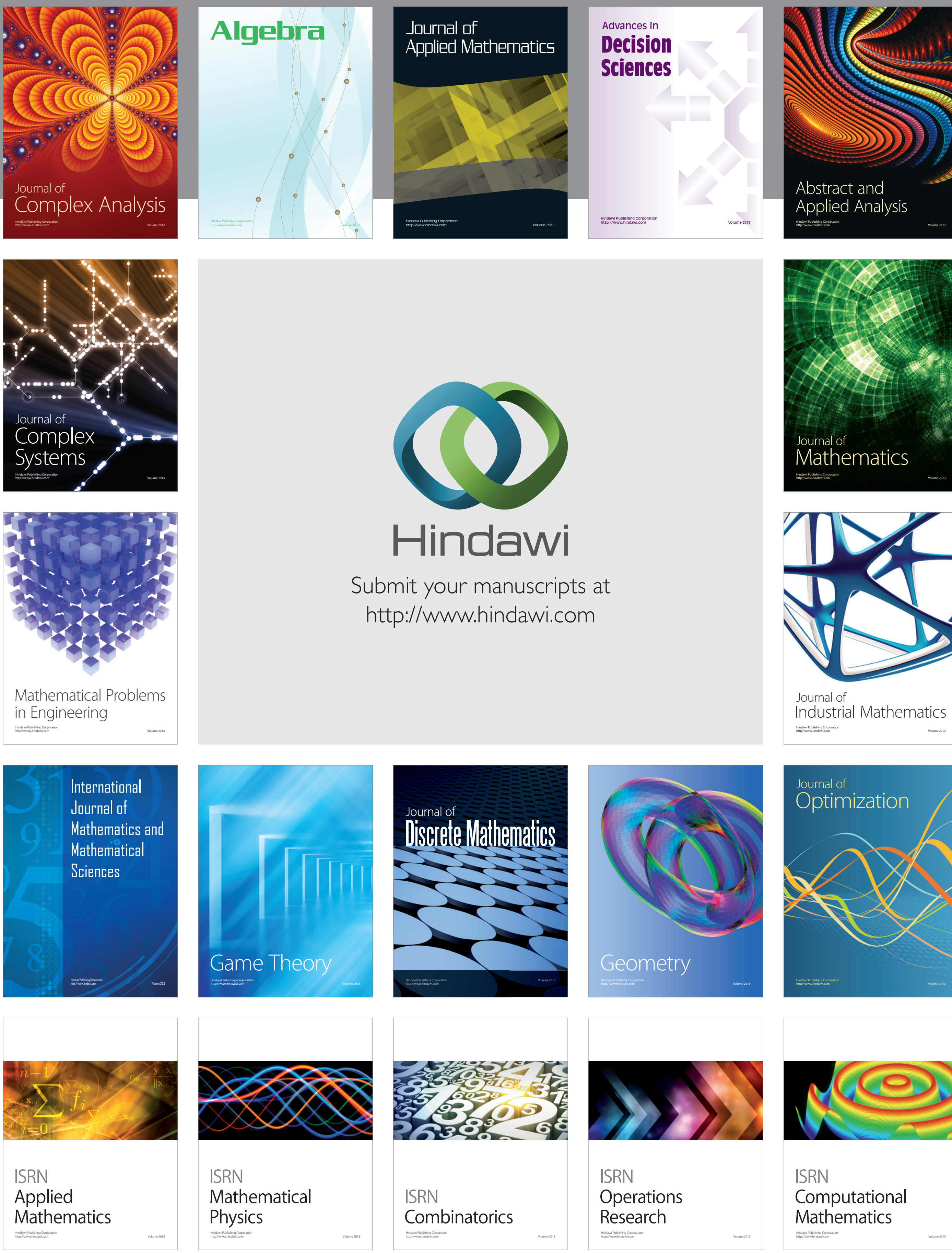
About the cover: Plain geometry (see page 88)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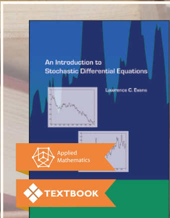
AMERICAN MATHEMATICAL SOCIETY

NEW AND NOTABLE at the 2014 JOINT MATHEMATICS MEETINGS

Explore some of the newest titles that will appear as part of the AMS book exhibit at this year's JMM.



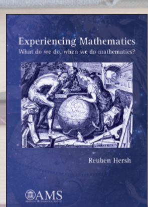
AN INTRODUCTION TO STOCHASTIC DIFFERENTIAL EQUATIONS

Lawrence C. Evans

This book covers the most important elementary facts regarding stochastic differential equations; it also describes some of the applications to partial differential equations, optimal stopping, and options pricing. This book's style is intuitive rather than formal, and emphasis is made on clarity. This book will be very helpful to starting graduate students and strong undergraduates as well as to others who want to gain knowledge of stochastic differential equations. I recommend this book enthusiastically.

—Alexander Lipton, Mathematical Finance Executive, Bank of America Merrill Lynch

2013; 151 pages; Softcover; ISBN: 978-1-4704-1054-4; List US$34; AMS members US$27.20; Order code MBK/82



EXPERIENCING MATHEMATICS

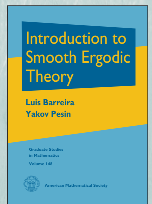
What do we do, when we do mathematics?

Reuben Hersh

The question "What am I doing?" haunts many creative people, researchers, and teachers. Mathematics, poetry, and philosophy can look from the outside sometimes as ballet en pointe, and at other times as the flight of the bumblebee. Reuben Hersh looks at mathematics from the inside; he collects his papers written over several decades, their edited versions, and new chapters in his book "Experiencing Mathematics", which is practical, philosophical, and in some places as intensely personal as Swann's madeleine.

—Yuri Manin, Max Planck Institute, Bonn, Germany

c2014; approximately 270 pages; Softcover; ISBN: 978-0-8218-9420-0; List US$39; AMS members US$31.20; Order code MBK/83

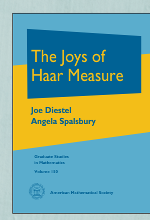


INTRODUCTION TO SMOOTH ERGODIC THEORY

Luis Barreira and **Yakov Pesin**

The first comprehensive introduction to smooth ergodic theory, consisting of two parts: the first introduces the core of the theory and the second discusses more advanced topics.

Graduate Studies in Mathematics, Volume 148; 2013; 277 pages; Hardcover; ISBN: 978-0-8218-9853-6; List US$65; AMS members US$52; Order code GSM/148

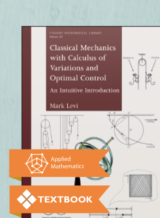


THE JOYS OF HAAR MEASURE

Joe Diestel and **Angela Spalsbury**

The aim of this book is to present invariant measures on topological groups, progressing from special cases to the more general.

Graduate Studies in Mathematics, Volume 150; 2013; approximately 327 pages; Hardcover; ISBN: 978-1-4704-0935-7; List US$65; AMS members US$52; Order code GSM/150



CLASSICAL MECHANICS WITH CALCULUS OF VARIATIONS AND OPTIMAL CONTROL

An Intuitive Introduction

Mark Levi

It is hard to imagine a more original and insightful approach to classical mechanics. Most physicists would regard this as a well-worn and settled subject. But Mark Levi's treatment sparkles with freshness in the many examples he treats and his unexpected analogies, as well as the new approach he brings to the principles. This is inspired pedagogy at the highest level.

—Michael Berry, Bristol University, UK

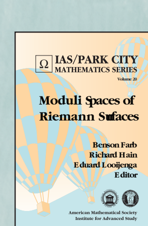
Student Mathematical Library, Volume 69; 2013; approximately 317 pages; Softcover; ISBN: 978-0-8218-9138-4; List US$42; AMS members US$33.60; Order code STML/69

PRIMALITY TESTING FOR BEGINNERS

Lasse Rempe-Gillen and **Rebecca Waldecker**

Provides a complete presentation of the proof of the AKS algorithm, without requiring any prior knowledge beyond general computational skills and the ability to think logically.

Student Mathematical Library, Volume 70; 2014; 244 pages; Softcover; ISBN: 978-0-8218-9883-3; List US$39; AMS members US$31.20; Order code STML/70



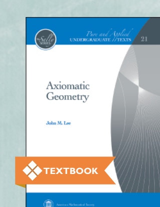
MODULI SPACES FOR RIEMANN SURFACES

Benson Farb, Richard Hain and **Eduard Looijenga**, Editors

This book presents the nine different lecture series comprising the Graduate Summer School of the 2011 IAS/Park City Mathematics Institute, which do not duplicate standard courses available elsewhere.

This series is co-published with the Institute of Advanced Study/Park City Mathematics Institute.

IAS/Park City Mathematics Series, Volume 20; 2013; 356 pages; Hardcover; ISBN: 978-0-8218-9887-1; List US$75; AMS members US$60; Order code PCMS/20



AXIOMATIC GEOMETRY

John M. Lee

Jack Lee's book will be extremely valuable for future high school math teachers. It is perfectly designed for students just learning to write proofs; complete beginners can use the appendices to get started, while more experienced students can jump right in. The axioms, definitions, and theorems are developed meticulously, and the book culminates in several chapters on hyperbolic geometry--a lot of fun, and a nice capstone to a two-quarter course on axiomatic geometry.

—John H. Palmieri, University of Washington

Pure and Applied Undergraduate Texts, Volume 21; 2013; 469 pages; Hardcover; ISBN: 978-0-8218-8478-2; List US$75; AMS members US$60; Order code AMSTEXT/21

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INSTITUTE FOR PURE AND APPLIED MATHEMATICS

Los Angeles, California

Hands-on Summer School:

ELECTRONIC STRUCTURE THEORY FOR MATERIALS AND (BIO)MOLECULES

July 21 - August 1, 2014

ORGANIZING COMMITTEE: Matthias Scheffler (Fritz-Haber-Institut der Max-Planck-Gesellschaft) and **Volker Blum** (Duke University)

Scientific Overview

Electronic structure theory (for total energies, forces, neutral and charged excitations, dynamics and transport, etc.) has reached a level where quantitative analyses and predictions of hitherto unknown properties and functions of materials are possible - including bulk materials, isolated molecules, surfaces, nanostructures, clusters, liquids, (bio)molecules in their environment, and more. Finding better or even novel functional materials is critical for nearly every aspect of our society. Key issues are, for example, the “energy challenge” and “managing the environment”.

This ten-day Hands-on Summer School introduces the basics of electronic-structure theory and teaches how actual calculations are performed. Morning lectures on the most important topics will be given by internationally renowned experts. In the afternoon (and evening) the participants will put this knowledge into practice. We will also discuss recent developments towards the "Materials Genome", and how to treat large amounts of data.

Participation

This summer school will provide a rare opportunity for researchers in mathematics, physics, chemistry, engineering, and related sciences to learn about recent research directions and future challenges in this area. Funding is available to support graduate students and postdoctoral researchers in the early stages of their career, as well as more senior researchers interested in undertaking new research in this area. Encouraging the careers of women and minority mathematicians and scientists is an important component of IPAM's mission and we welcome their applications. The application is available online, and is due March 31, 2014.

www.ipam.ucla.edu/programs/gss2014

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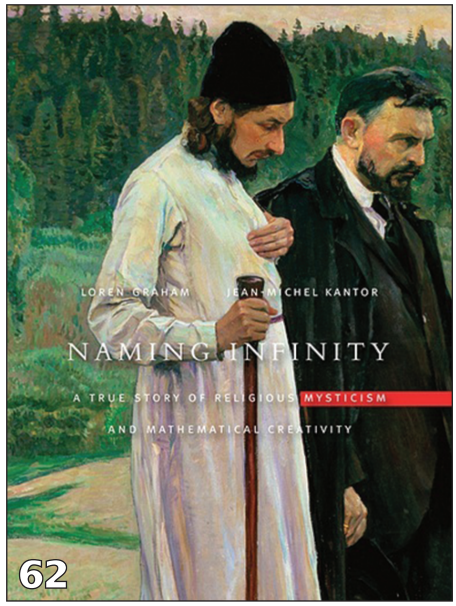
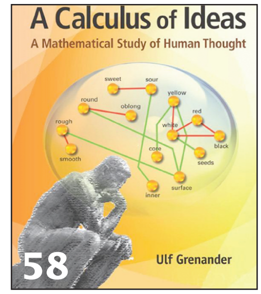
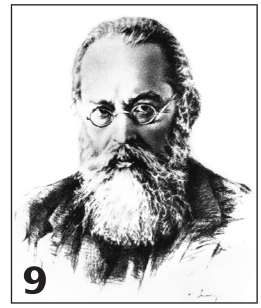
January 2014

Communications

Commentary







The January issue explores a number of important themes—mathematical, educational, and political. These include hypergeometric functions, the class group of a real quadratic field, the National Security Agency and security issues, and the common core standards in geometry. There is also a fascinating history of the Steklov Institute. Our *Doceamus* this month is about active learning in math courses. Happy reading.

—Steven G. Krantz, Editor

Features

Notices

of the American Mathematical Society

SUBSCRIPTION INFORMATION: Subscription prices for Volume 61 (2014) are US$569 list; US$455.20 institutional member; US$341.40 individual member; US$512.10 corporate member. (The subscription price for members is included in the annual dues.) A late charge of 10% of the subscription price will be imposed upon orders received from nonmembers after January 1 of the subscription year. Add for postage: Surface delivery outside the United States and India—US$27; in India—US$40; expedited delivery to destinations in North America—US$35; elsewhere—US$120. Subscriptions and orders for AMS publications should be addressed to the American Mathematical Society, P.O. Box 845904, Boston, MA 02284-5904 USA. All orders must be prepaid.

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[*Notices of the American Mathematical Society* (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2294 USA, GST No. 12189 2046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to *Notices of the American Mathematical Society,* P.O. Box 6248, Providence, RI 02940-6248 USA.] Publication here of the Society's street address and the other information in brackets above is a technical requirement of the U.S. Postal Service. Tel: 401-455-4000, email: `notices@ams.org`.

Departments

American Mathematical Society

CURRENT EVENTS BULLETIN

Friday, January 17, 2014, 1:00 PM to 5:00 PM

Room 310 Baltimore Convention Center
Joint Mathematics Meetings, Baltimore, MD

1:00 PM

Daniel Rothman
Massachusetts Institute of Technology

2:00 PM

Karen Vogtmann
Cornell University

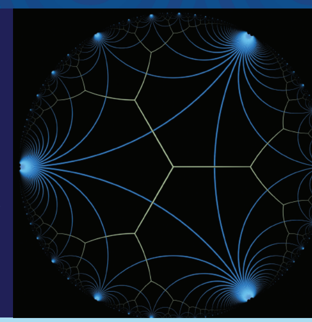


Image courtesy of: Jos Leys

Earth's carbon cycle: A mathematical perspective

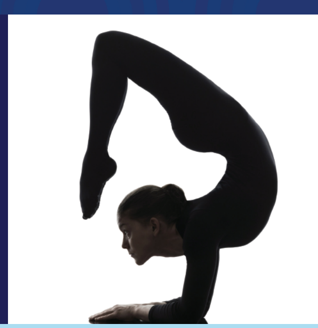
Mathematics to understand one of the great challenges to our society

The geometry of outer space

New geometric methods advance the theory of automorphism groups of free groups

3:00 PM

Yakov Eliashberg
Stanford University



4:00 PM

Andrew Granville
Université de Montréal

Primes get closer and closer together

Recent advances in symplectic flexibility

Flexible methods (known as Gromov's *h*-principle generalizing the work of Nash and Smale) played an important role in symplectic topology from its inception. Learn about the classic results and their new developments.

Infinitely many pairs of primes differ by no more than 70 million (and the bound's getting smaller every day)

-- Prime Twins?

-- Dunno, but at least they're close.

Organized by David Eisenbud, University of California, Berkeley

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Dear NSA: Long-Term Security Depends on Freedom

It has become impossible in the last few months to escape media reports about the National Security Agency. There are certainly many more pressing problems in the world, but this one is close to home for mathematicians. I think it is time to seriously consider the relationship that exists between academia and the NSA, both the potential for good and the need for caution.

Other people's physical safety often comes before personal freedoms. Traffic laws are one example. Religious freedom does not justify homicide, for another.

Thus it is arguably altruistic, when danger looms, to agree to give up another freedom: specifically, the freedom from unwarranted searches of private property and communications. In the wake of attacks such as occurred on September 11 more than a decade ago and in Boston this year, personal transparency looks like the price we pay in order to see our loved ones live in safety. I also wouldn't want to criticize without first expressing gratitude for the hard work and sacrifices that have prevented more attacks and kept more children from harm.

Constant surveillance, however, whether by algorithms or human agents, comes at a cost beyond merely the uncomfortable feeling of being watched. The existence of databases storing our private communication conveys a certain degree of power to the few with access. Without layers of equally powerful oversight, there is always a temptation for abuse. Even a hint of malignant use of power is enough to alienate many thoughtful citizens, including the very community of mathematicians that the NSA depends on for recruits.

It would be shortsighted for the NSA to push away our top scientists by appearing negligent. Leadership at the NSA evidently realizes the vital importance of public and scientific support. A portion of their effort is dedicated to improving all levels of math education and supporting open, unclassified math research in the United States. Many mathematicians earn NSA funding for their research, their students, and their universities through an annual grant competition administered by the American Mathematical Society. This program specifically avoids any secret research such as cryptology. (Full disclosure: my future students and I are currently expecting this support starting in the next fiscal year.)

This sort of spending by the intelligence agency has two main goals: it aims to secure goodwill, from academics at least, and it hopes to ensure that the U.S. will have enough ready-to-use brain power when future threats demand it. Meeting these goals demonstrates that an increase in security can come as a byproduct of (academic) freedom. Unfortunately, the converse is true as well. If power is abused (and freedoms unnecessarily curtailed), then security is undermined.

When I hear about a misuse of government power, my reaction is not to disband the agency entirely. Nor do I point to their constructive work and use it to excuse the bad. Instead, I want to see checks and balances go into action: an investigation by an impartial, separate, and equally powerful arm of government. If rules are broken, then there has to be a penalty. If the rules are unclear or inadequate or not enforced, then they need to be rewritten clearly and with provision for enforcement. If the rules themselves are unwise, unethical, shortsighted, or obsolete, then it is time for new rules. If the overreaching piece of government does not move in the direction of accountability, then no amount of constructive efforts and positive PR can save it. Here are some naive suggestions for new leadership at the NSA:

1) Invite more judicial/congressional oversight, perhaps by strengthening what currently exists. The oversight should have the power to suggest penalties for rules broken (intentionally or not) and to make recommendations regarding any rules that are unclear, inadequate for protecting privacy, or not easily enforced.

2) Reiterate publicly a willingness to abide by any changes that Congress approves.

3) Avoid the error of chasing away the best mathematical talent in the U.S. by allowing the appearance of unwarranted intrusions. We might need that talent when the next threat arises.

4) Continue to fund free and open research and education in mathematical sciences but be aware that it won't buy unconditional support from academia.

5) Be patient. Most of the above suggestions are what the great people at the NSA already want and are working for. The recent declassification of secret court opinions and internal audits, increased funding for oversight, and the efforts to get feedback from the public constitute a good beginning. There is required much patience on the part of both those with security clearance and those of us without it. We will have to wait for full information; they will have to wait for full trust to be restored.

—Stefan Forcey
University of Akron
`sf34@uakron.edu`

DOI: http://dx.doi.org/10.1090/noti1067

Second Joint International Meeting of the AMS and the Israel Mathematical Union

This meeting is co-hosted by the
Bar-Ilan University and the Tel-Aviv University.

June 16 - 19, 2014, Ramat Gan and Tel Aviv, Israel

Program - For more information and for the program of the meeting, please consult the webpage maintained by the local organizing committee at http://imu.org.il/Meetings/IMUAMS2014/index.html or at the AMS website www.ams.org.

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The Legacy of Vladimir Andreevich Steklov

Nikolay Kuznetsov, Tadeusz Kulczycki, Mateusz Kwaśnicki, Alexander Nazarov, Sergey Poborchi, Iosif Polterovich and Bartłomiej Siudeja

Vladimir Andreevich Steklov, an outstanding Russian mathematician whose 150th anniversary is celebrated this year, played an important role in the history of mathematics. Largely due to Steklov's efforts, the Russian mathematical school that gave the world such giants as N. Lobachevsky, P. Chebyshev, and A. Lyapunov, survived the revolution and continued to flourish despite political hardships. Steklov was the driving force behind the creation of the Physical–Mathematical Institute in starving Petrograd in 1921, while the civil war was still raging in the newly Soviet Russia. This institute was the predecessor of the now famous mathematical institutes in Moscow and St. Petersburg bearing Steklov's name.

Steklov's own mathematical achievements, albeit less widely known, are no less remarkable than his contributions to the development of science. The Steklov eigenvalue problem, the Poincaré–Steklov operator, the Steklov function—there exist probably a dozen mathematical notions associated with Steklov. The present article highlights some of the milestones of his career, both as a researcher and as a leader of the Russian scientific community.

V. A. Steklov in the 1920s.

Nikolay Kuznetsov heads the Laboratory for Mathematical Modelling of Wave Phenomena at the Institute for Problems in Mechanical Engineering, Russian Academy of Sciences. His email address is `nikolay.g.kuznetsov@gmail.com`.

Tadeusz Kulczycki and Mateusz Kwaśnicki are professors at the Institute of Mathematics and Computer Science, Wrocław University of Technology. Tadeusz Kulczycki's email address is `Tadeusz.Kulczycki@pwr.wroc.pl` *and Mateusz Kwaśnicki's email address is* `Mateusz.Kwasnicki@pwr.wroc.pl`.

Alexander Nazarov is a leading researcher at the Laboratory of Mathematical Physics, St. Petersburg Department of the Steklov Mathematical Institute, and a professor at the Department of Mathematical Physics, Faculty of Mathematics and Mechanics, St. Petersburg State University. His email address is `al.il.nazarov@gmail.com`. *His work was supported by RFBR grant 11-01-00825 and by the St. Petersburg University grant 6.38.670.2013.*

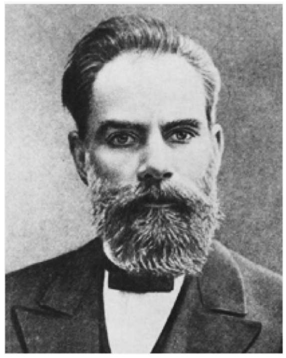
DOI: http://dx.doi.org/10.1090/noti1073

Sergey Poborchi is a professor in the Department of Parallel Algorithms, Faculty of Mathematics and Mechanics, St. Petersburg State University. His email address is `poborchi@mail.ru`. *His work was supported by RFBR grant HK-11-01-00667/13.*

Iosif Polterovich is a professor in the Département de mathématiques et de statistique, Université de Montréal. His email address is `iossif@dms.umontreal.ca`.

Bartłomiej Siudeja is a professor in the Department of Mathematics, University of Oregon. His email address is `siudeja@uoregon.edu`.

The authors are grateful to David Sher for proofreading the article.



Aleksandr Mikhaylovich Lyapunov in the 1900s.

The article is organized as follows. It starts with a brief biography of V. A. Steklov written by N. Kuznetsov. The next section, written by N. Kuznetsov, A. Nazarov, and S. Poborchi, focuses on Steklov's work related to several celebrated inequalities in mathematical physics. The remaining two sections are concerned with some recent developments in the study of the Steklov eigenvalue problem, which is an exciting and rapidly developing area on the interface of spectral theory, geometry and mathematical physics. The "high spots" problem for sloshing eigenfunctions is discussed in the section written by T. Kulczycki, M. Kwaśnicki, and B. Siudeja. In particular, the authors explain why it is easier to spill coffee from a mug than to spill wine from a snifter. An overview of some classical and recent results on isoperimetric inequalities for Steklov eigenvalues is presented in the last section, written by I. Polterovich.

Many topics to which Steklov contributed in a major way are beyond the scope of the present article. For further references, see [14], [24], [31] and [64].

A Biographical Sketch of V. A. Steklov

Vladimir Andreevich Steklov was born in Nizhni Novgorod on January 9, 1864 (= December 28, 1863, old style). His grandfather and great-grandfather on the father's side were country clergymen. His father, Andrei Ivanovich Steklov, graduated from the Kazan Theological Academy and taught history and Hebrew at the Theological Seminary in Nizhni Novgorod. Steklov's mother, Ekaterina Aleksandrovna (née Dobrolyubova), was a daughter of a country clergyman as well. Her brother, Nikolay Aleksandrovich, was a prominent literary critic and one of the leaders of the democratic movement that aimed to abolish serfdom in Russia.

At ten years of age, Steklov enrolled into the Alexander Institute (a gymnasium that had many notable alumni, including the famous Russian composer M. Balakirev) in Nizhni Novgorod. Steklov's critical thinking manifested itself at a very early age. In his diaries, Steklov describes how he was chastised by the school principal for a composition deemed "disrespectful" towards the Russian empress Catherine II.

> I said to myself: "Aha! It occurs to me that I have my own point of view on historical events which is different from that of my schoolmates and teachers. […] It was the principal himself who proved that I am, in some sense, a self-maintained thinker and critic." This was the initial impact that led to my mental awakening; I realized that I am a human being able to reason and, what is important, to reason freely. […] Soon, my free thinking encompassed the religion as well. […] Thus, the cornerstone was laid for my future complete lack of faith.

After graduating from school in 1882, Steklov entered the Faculty of Physics and Mathematics of Moscow University. Failing to pass an examination in 1883, he left Moscow and the same year entered a similar faculty in Kharkov. There he met A. M. Lyapunov, and this encounter became a turning point in his life. Steklov graduated in 1887, but remained at the university working under Lyapunov's supervision towards obtaining his Master's Degree. In the beginning of 1890, Steklov married Olga Nikolaevna Drakina, who was a music teacher; their marriage lasted for 31 years. In the fall of the same year, he was appointed Lecturer in Elasticity Theory. In 1891, the Steklovs' daughter Olga was born and, presumably, this event delayed the defence of his Master's thesis, *On the motion of a solid body in a fluid*, until 1893. The same year, Steklov began lecturing at Kharkov Institute of Technology, combining it with his work at the university; the goal was to improve his family's financial situation, given that his wife had to leave her job after giving birth to their child. The sudden death of their daughter in 1901 was a heavy blow to Steklov and his wife, and caused a six-month break in his research activities.

He was appointed to an extraordinary professorship in mechanics in 1896. The first in a series of full-length papers, which formed the core of his dissertation for the Doctor of Science degree, appeared in print the same year (not to mention numerous brief notes in *Comptes rendus*). The dissertation entitled *General methods of solving fundamental problems in mathematical physics* was published as a book in 1901 by the Kharkov Mathematical Society [53].

At the time of completing his DSc dissertation, Steklov began to publish his results in French. Since then, most of his papers were written in French — the language widely used by Russian mathematicians to make their results accessible in Europe. Unfortunately, this did not prevent some of his results from remaining unnoticed. In particular, this concerns the so-called Wirtinger's inequality which was published by Steklov in 1901 in *Annales fac. sci. Toulouse* (see details in the next section). Even before that, Steklov became very active in corresponding with colleagues abroad (J. Hadamard, A. Kneser, A. Korn, T. Levi-Civita, E. Picard, S. Zaremba, and many others were among his correspondents); these contacts were of great importance for him, residing in a provincial city. In 1902, Steklov was appointed to an ordinary professorship in applied mathematics and was elected a corresponding member of the Academy of Sciences in St. Petersburg the next year.

In 1903, the Steklovs went on a summer vacation to Europe. Some details of this trip are described in one of Steklov's letters to Lyapunov (see [60], letter 29). In particular, the meeting with J. Hadamard in Paris:

> Somehow, Hadamard found me himself; presumably, he had learned my address from A. Hermann [the well-known publisher]. Once he missed me, but the next day he came at half past eight in the morning when we had just awakened. He arrived to Paris to stay for two days examining for "baccalauréat" [at some lycée]; on the day of his returning to the countryside, where he spends summer, he called on me before examination. His visit lasted only half-an-hour, but he told as much as another person would tell in a whole day. He is a model Parisian, very agile and swift to react; he behaved so as we are old friends who had not seen each other for some time.

In 1908, the Lyapunovs and the Steklovs travelled to Italy together, where A. M. and V. A. participated in the Rome ICM. At the Cambridge ICM (1912), Steklov was elected a vice president of the congress (Hadamard and Volterra were the other two vice presidents). The Toronto ICM (1924) was the third and the last one for Steklov.

Let us turn to the Petersburg–Petrograd–Leningrad period of Steklov's life. In 1906, he succeeded (after several attempts) in moving to St. Petersburg. It is a remarkable coincidence that a group of very talented students entered the university the same year. In the file of M. F. Petelin, who was one of them, this fact was commented on by Steklov as follows:

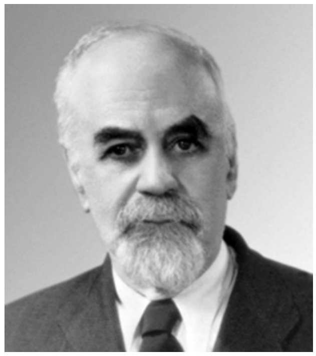
> I should note that the class of 1910 is exceptional. In the class of 1911 and among the fourth-year students who are about to graduate there is no one equal in knowledge and abilities to Messrs. Tamarkin, Friedmann, Bulygin, Petelin, Smirnov, Shohat, and others. There was no such case during the fifteen years of teaching at Kharkov University either. This favorable situation should be used for the benefit of the University.

Aleksandr Aleksandrovich Friedmann in the 1920s.

Steklov had done his best to nurture his students (see [66]). His dedication as an advisor was rewarded by the outstanding achievements of the members of his group, the most famous of which is Friedmann's solution of Einstein's equations in the general theory of relativity. The future fate of Steklov's students varied greatly; two of them (Bulygin and Petelin) died young. Tamarkin and Shohat emigrated to the USA and became prominent mathematicians there. It is worth noting that J.D. Tamarkin has more than 1500 mathematical descendants, and through him the St. Petersburg mathematical tradition had a profound impact on the American mathematics. Tamarkin's escape from the Soviet Union was quite an adventure. While secretly crossing the frozen Chudskoe Lake in order to reach Latvia, he was fired on by the Soviet border guards. As E. Hille wrote:

> One of J. D.'s best stories told how he tried to convince the American consul in Riga of his identity: the consul attempted to examine him in analytic geometry, but ran out of questions and gave up.

Friedmann and Smirnov became prominent scientists staying in Leningrad. Together with their colleagues and students (N. M. Günther, A. N. Krylov, V. A. Fock, N. E. Kochin, S. G. Mikhlin, and S. L. Sobolev to name a few) they organized the school of mathematical physics in Leningrad–St. Petersburg, the foundation of which was laid by Steklov.



Vladimir Ivanovich Smirnov in the 1950s.

Steklov's scientific career was advancing. In 1910, he was elected an adjunct member of the Academy of Sciences; two years later he was elected extraordinary and then ordinary academician within a few months. After his election to the executive committee of the Academy in 1916, Steklov reduced his work at the university and abandoned it completely in 1919, being elected vice president of the Academy. It would take too much space to describe everything he had accomplished at this post by the time of his unexpected and untimely death on May 30, 1926. (His wife died in 1920 from illness caused by undernourishment.) Therefore, only his role in establishing the Physical–Mathematical Institute—the predecessor of institutes named after him—will be outlined.

In January 1919, a memorandum was submitted to the Academy in which Steklov, A. A. Markov Sr., and A. N. Krylov proposed to establish a Mathematical cabinet as an initial stage in further development of the Academy's Department of Physics and Mathematics (the Physical cabinet existed in the Academy since its foundation in 1724, and it was reorganized into a laboratory in 1912). Later the same year, the initiative was supported, and Steklov became the head of the new institution named after P. L. Chebyshev and A. M. Lyapunov. In January 1921, Steklov submitted another memorandum, pointing out the necessity to merge the Physical laboratory and the recently established Mathematical cabinet. As he writes, "Mathematics and physics have now merged to such an extent that it is sometimes difficult to find the line that divides them." Nowadays, this viewpoint is shared by many mathematicians; however, at the time it was quite unusual.

The same January 1921, Steklov, S. F. Oldenburg (the Permanent Secretary of the Academy), and V. N. Tonkov (Head of the Military Medical Academy) visited V. I. Lenin in Moscow. In his recollections about Lenin, Maxim Gorky (the famous Russian writer close to the Bolsheviks), who had been present at this meeting, wrote (see also [67]):

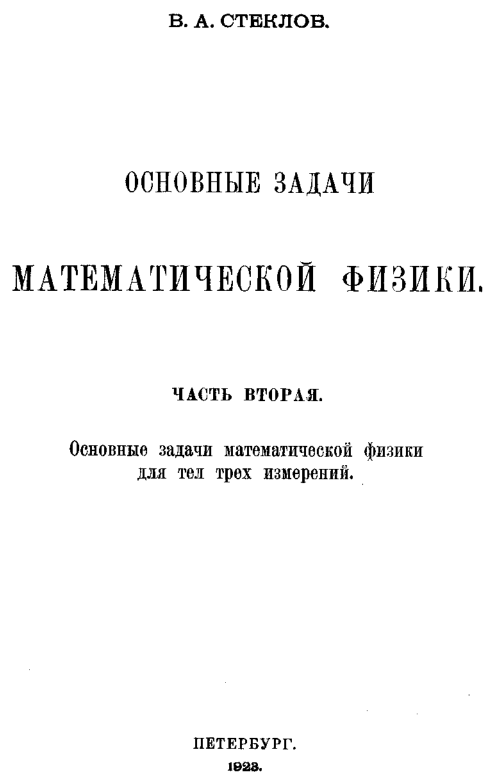
> They talked about the necessity to reorganize the leading scientific institution in Petersburg [the Academy]. After seeing off his visitors, Lenin said with satisfaction.
> 'What clever men! Everything is simple for them, everything is formulated rigorously; it is clear immediately that they know well what they want. It is a pleasure to work with such people. The best impression I've got from …'
> He named one of the most prominent Russian scientists; two days later, he told me by phone.
> 'Ask S[teklov] whether he is going to work with us.'
> When S[teklov] accepted the offer, this was a real joy for Lenin; rubbing his hands, he joked.
> 'Just wait! One Archimedes after the other, we'll gain support of all of them in Russia and in Europe, and then the World, willingly or unwillingly, will turn over!'

Indeed, after the Bolshevik government declared the so-called "New Economic Policy", the deadlock over Academy funding was broken thanks to the improving economic situation in the country. This resulted in the creation of the Physical–Mathematical Institute later in 1921, and Steklov was appointed its first director. In his recollections [59] completed in 1923, he writes:

> Another achievement of mine for the benefit of the Academy and the development of science in general is establishing the Physical–Mathematical Institute with the following divisions: mathematics, physics, magnitology and seismology. Its work is still in the process of being organized, the funding is scarce and difficult to obtain. The number of researchers is still negligible […], but a little is better than nothing.

After Steklov's death the institute was named after him. In 1934, simultaneously with relocation of the Academy from Leningrad to Moscow the Physical–Mathematical Institute was divided into the following two: the P. N. Lebedev Physical Institute and the V. A. Steklov Mathematical Institute (even before that the division of seismology became a separate institute). The Leningrad (now St. Petersburg) Department of the latter was founded in 1940, and it is an independent institute since 1995. To summarize, it must be said that the role of V. A. Steklov in Petrograd was similar to that of R. Courant who organized mathematical institutes, first in Göttingen and then in New York.

In 1922 and 1923, Steklov's monograph [58] was published; it summarizes many of his results in mathematical physics. Based on the lectures



В. А. СТЕКЛОВ.

ОСНОВНЫЕ ЗАДАЧИ

МАТЕМАТИЧЕСКОЙ ФИЗИКИ.

ЧАСТЬ ВТОРАЯ.

Основные задачи математической физики
для тел трех измерений.

ПЕТЕРБУРГ.
1923.

The title page of Steklov's monograph [58].

given in 1918–1920, this book is written in Russian, despite the fact that the corresponding papers originally appeared in French. More material was presented in the lecture course than was included in [58], and Steklov planned to publish the 3rd volume about his results concerning "fundamental" functions (that is, eigenfunctions of various spectral problems for the Laplacian) and some applications of these functions. Unfortunately, the administrative duties prevented him from realizing this project. However, one gets an idea about the probable contents of the unpublished 3rd volume from the lengthy article [55], in which Steklov developed his approach to "fundamental" functions (it is briefly outlined by A. Kneser [28], section 5). It is based on two different kinds of Green's function, and this allowed Steklov to apply the theory of integral equations worked out by E. I. Fredholm and D. Hilbert shortly before that. It is worth mentioning that [58] was listed among the most important mathematical books published during the period from 1900 to 1950 (see "Guidelines 1900–1950" in [45]; another item concerning mathematical physics published the same year is Hadamard's *Lectures on Cauchy's problem in linear partial differential equations*).

A great part of the material presented in the second volume of [58] is taken from the article [54], which is concerned with boundary value problems for the Laplace equation. It is Steklov's most cited work, but, confusingly, his initials are given incorrectly in many citations of this paper. Indeed, R. Weinstock found the following spectral problem

$$\text{(1)} \qquad \Delta u = 0 \ \text{ in } D, \quad \frac{\partial u}{\partial n} = \lambda \varphi u \text{ on } \partial D,$$

in [54], and for this reason he called it the *Stekloff problem*; here n is the exterior unit normal on ∂D and φ is a non-negative bounded weight function. In fact, Steklov introduced this problem in his talk at a session of the Kharkov Mathematical Society in December 1895; it was also studied in his DSc dissertation. Nowadays, it is mainly referred to as the Steklov problem, but, sometimes, is still called the Stekloff problem. In [69], Weinstock initiated the study of this problem, but, unfortunately, citing [54], he supplied Steklov's surname with wrong initials, which afterwards were reproduced elsewhere. Weinstock's result and its later developments are discussed in detail in the last section.

It must be emphasized that the legacy of Steklov is multifaceted (see [67]). He wrote biographies of Lomonosov and Galileo, an essay about the role of mathematics, the travelogue of his trip to Canada, where he participated in the 1924 ICM, his correspondence—published (see [60] and [61]) and unpublished; the recollections [59] and still unpublished diaries. Fortunately, many excerpts from Steklov's diaries are quoted in [66] and some of them appeared in [43]. The most expressive is dated September 2, 1914, one month after the Russian government declared war:

> St. Petersburg has been renamed Petrograd by Imperial Order. Such trifles are all our tyrants can do—religious processions and extermination of the Russian people by all possible means. Bastards! Well, just you wait. They will get it hot one day!

What happened in Russia during several years after that confirms clearly how right was Steklov in his assessment of the Tsarist regime. In his recollections [59] written in 1923, he describes vividly and, at the same time, critically "the complete bacchanalia of power" preceding the collapse of "autocracy and [Romanov's] dynasty" in February 1917 (old style), "the shameful Provisional government headed by Kerensky, the fast end of which can be predicted by every sane person", and how "the Bolshevik government [...] decided to accomplish the most Utopian socialistic ideas in the multi-million Russia." The list can be easily continued.

The pinpoint characterization of Steklov's personality was given by A. Kneser (see [28]):

> Everybody who maintained contact with Steklov was impressed by his personality. He was highly educated in the traditions of the European culture, but at the same time maintained distinctive features characteristic to his nation. He was not only a deep mathematician, but also a connoisseur of music and art. [...] Besides, he was a skillful mediator between scientists and the new government in Moscow. Thus, his role was crucial for the survival of the Russian science and its restoration (predominantly, in the Academy and its institutes) after the revolutions and the Civil War.

To conclude this section, we list some of Steklov's awards and distinctions. He was a member of the Russian Academy of Sciences, a corresponding member of the Göttingen Academy of Science, and a Doctor honoris causa of the University of Toronto. The mathematical Societies in Kharkov, Moscow, and St. Petersburg (in Petrograd, it was reorganized into the Physical–Mathematical Society) counted him among their members, as well as the Circolo Matematico di Palermo.

V. A. Steklov and the Sharp Constants in Inequalities of Mathematical Physics

In 1896, Lyapunov established that the trigonometric Fourier coefficients of a bounded function that is Riemann integrable on $(-\pi, \pi)$ satisfy the closedness equation. He presented this result at a session of the Kharkov Mathematical Society, but left it unpublished. The same year, Steklov had taken up studies of the closedness equation initiated by his teacher; Steklov's extensive work on this topic lasted for 30 years until his death. For this reason A. Kneser [28] referred to this equation as "Steklov's favorite formula". It should be mentioned that the term *closedness equation* was introduced by Steklov for general orthonormal systems, but only in 1910 (see brief announcements [56] and the full-length paper [57]).

The same year (1896), Steklov [50] proved that the following inequality (nowadays often referred to as Wirtinger's inequality)

$$\int_0^l u^2(x)\,\mathrm{d}x \leqslant \left(\frac{l}{\pi}\right)^2 \int_0^l [u'(x)]^2\,\mathrm{d}x \tag{2}$$

holds for all functions which are continuously differentiable on $[0, l]$ and have zero mean. For this purpose, he used the closedness equation for the Fourier coefficients of u (the corresponding system is $\{\cos(k\pi x/l)\}_{k=0}^{\infty}$ normalised on $[0, l]$). Inequality (2) was among the earliest inequalities with a sharp constant that appeared in mathematical physics. It was then applied to justify the Fourier method for initial-boundary value problems for the heat equation in two dimensions with variable coefficients independent of time. Later, Steklov justified the Fourier method for the wave equation as well. The fact that the constant in (2) is sharp was emphasized by Steklov in [52], where he gave another proof of this inequality (see pp. 294–296). There is another result proved in [52] (see pp. 292–294); it says that (2) is true for continuously differentiable functions vanishing at the interval's end-points, and again the constant is sharp. In the first volume of his monograph [58], Steklov presented inequality (2) along with its generalization.

The problem of finding and estimating sharp constants in inequalities attracted much attention from those who work in theory of functions and mathematical physics (see, for example, the classical monographs [20] and [49]). It is worth mentioning that in the famous book by Hardy, Littlewood, and Polya [20, section 7.7], inequality (2) is proved under either type of conditions proposed by Steklov; however, the authors call it Wirtinger's inequality and refer to the book of Blaschke (who was a student of Wirtinger) published in 1916 [5, p. 105], twenty years after the publication of Steklov's paper [50]. This terminology became standard. However, the controversy does not end here; we refer to [41] for other historical aspects of this inequality.

S. G. Mikhlin (he graduated from Leningrad University a few years after Steklov's death; see his recollections of student years [40]) emphasized the role of sharp constants in his book [39]. Let us quote the review [44] of the German version of [39]:

> [This book] is devoted to appraising the (best) constants—exact results or explicit (numerical) estimates—in various inequalities arising in "analysis" (=PDE). [...] This is the most original work, a bold attack in a direction where still very little is known.

In 1897, Steklov published the article [51], in which the following analogue of inequality (2) was proved:

$$\int_D u^2\,\mathrm{d}x \leqslant C \int_D |\nabla u|^2\,\mathrm{d}x. \tag{3}$$

Here ∇ stands for the gradient operator and the integral on the right-hand side is called the Dirichlet integral. The assumptions made by Steklov are as follows: D is a bounded three-dimensional domain whose boundary is piecewise smooth and u is a real C^1-function on $\bar{D}$ vanishing on ∂D. Again, inequality (3) was obtained by Steklov with the sharp constant equal to $1/\lambda_1^D$, where λ_1^D is the smallest eigenvalue of the Dirichlet Laplacian in D. In the early 1890s, H. Poincaré [46] and [47] obtained (3) using different assumptions, namely, u

Figure 1. High spot in a coffee cup.

Figure 2. High spot in a snifter.

has zero mean over D which is a union of a finite number of smooth convex two- and three-dimensional domains, respectively. In the latter case, the sharp constant in (3) is $1/\lambda_1^N$, where λ_1^N is the smallest positive eigenvalue of the Neumann Laplacian in D.

If u vanishes on ∂D (this is understood as follows: u can be approximated in the norm $\|\nabla u\|_{L^2(D)}$ by smooth functions having compact support in a domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, of finite volume), then (3) is often referred to as the *Friedrichs inequality*. In fact K. O. Friedrichs [13] obtained a slightly different inequality under the assumption that $D \subset \mathbb{R}^2$. Namely, he proved:

$$(4) \qquad \int_D u^2 \,\mathrm{d}x \leqslant C \left[\int_D |\nabla u|^2 \,\mathrm{d}x + \int_{\partial D} u^2 \,\mathrm{d}S \right],$$

where $\mathrm{d}S$ denotes the element of length of ∂D. Generally speaking, (4) holds for all bounded domains in $\mathbb{R}^n$ for which the divergence theorem is true (see [38], p. 24).

Inequality (3) for functions u with a zero mean value over D is equivalent to the following one (it is called the *Poincaré inequality*):

$$(5) \qquad \|u - \langle u \rangle\|_{L^2(D)} \leqslant C \|\nabla u\|_{L^2(D)}, \quad \text{where } \langle u \rangle = \frac{\int_D u(x)\,\mathrm{d}x}{\mathrm{meas}_n D},$$

here $\mathrm{meas}_n D$ is the n-dimensional measure of D. Note that the sharp constant here is $1/\sqrt{\lambda_1^N}$. Some requirements must be imposed on D for the validity of (5). Indeed, as early as 1933 O. Nikodým [42] (see also [38], p. 7) constructed a bounded two-dimensional domain D and a function with finite Dirichlet integral over D such that inequality (5) is not true. Another example of a domain with this property is given in the classical book [10] by Courant and Hilbert (see ch. 7, sect. 8.2).

In conclusion, we consider the following "boundary analogue" of inequality (5):

$$\|u - \langle u \rangle_G\|_{L^2(G)} \leqslant C \|\nabla u\|_{L^2(D)}, \quad \text{where } \langle u \rangle_G = \frac{\int_G u(x)\,\mathrm{d}S}{\mathrm{meas}_{n-1} G}.$$

Here D is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$, whereas G is a part of ∂D possibly coinciding with ∂D. The sharp constant in this inequality is equal to $1/\sqrt{\lambda_1^S}$, where λ_1^S is the smallest positive eigenvalue of the following mixed (unless $G = \partial D$) Steklov problem:

$$\Delta u = 0 \ \text{ in } D, \ \frac{\partial u}{\partial n} = \lambda u \ \text{ on } G, \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial D \setminus G.$$

If D is a special two- or three-dimensional domain with a particular choice of G, then the eigenvalues of this problem give rise to the *sloshing frequencies*, that is, the frequencies of free oscillations of a liquid in channels or containers; see, for example, [35, Chapter IX]. The sloshing problem is discussed in detail in the next section.

Spilling from a Wineglass and a Mixed Steklov Problem

The 2012 Ig Nobel Prize for Fluid Dynamics was awarded to R. Krechetnikov and H. Mayer for their work [37] on the dynamics of liquid sloshing. They investigated why coffee so often spills while people walk with a filled mug. In their study, oscillations of coffee are modeled by an appropriate mixed Steklov problem which is usually referred to as the *sloshing problem*. They realized that within this model one of the main reasons for spilling coffee can be described as follows. In a typical mug, the sloshing mode corresponding to the lowest eigenfrequency of the problem tends to get excited during walking.

However, there is another reason for spilling coffee from a mug of typical shape. Namely, a *high spot* is present on the boundary of the free surface, that is, the maximal elevation of the surface is always located on the mug's wall (see Figure 1), provided oscillations are free and their frequency is the lowest one. The latter effect (combined with that described in [37]) makes it even easier to spill coffee from a mug. On the other hand, in a

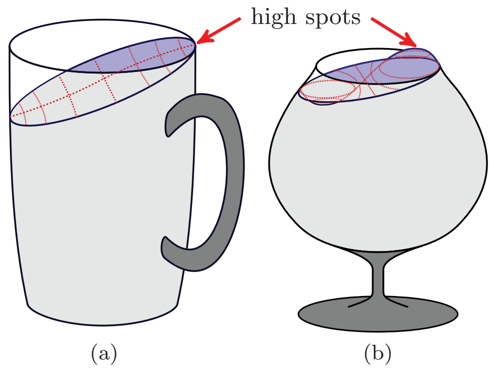


Figure 3. A schematic sketch of location of high spots in a coffee cup (a) and in a snifter (b).

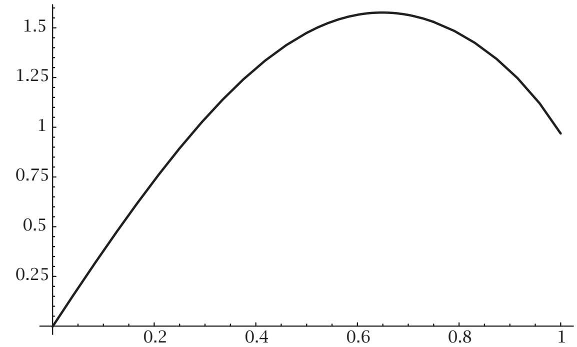


Figure 4. The highest radial free surface profile in an ice-fishing hole.

bulbous wineglass both antisymmetric sloshing modes corresponding to the lowest eigenfrequency are such that their maximal elevations (high spots) are attained inside the free surface, but not on the wall (see Figure 2). This reduces the risk of spilling from a snifter. Thus, the position of a high spot depends on the container's shape as is schematically shown in Figure 3 for a coffee cup (a) and a snifter (b); theorems guaranteeing these kinds of behavior are proved in [34].

The natural limiting case of bulbous containers is an infinite ocean covered by ice with a single circular hole (the corresponding sloshing problem is usually referred to as the *ice-fishing* problem). The question about the shape of the free surface when water oscillates at the lowest eigenfrequency in an ice-fishing hole was answered in [30]. This shape is similar to that in a snifter (see Figures 2 and 3 (b)). The highest free surface profile existing in radial directions of an ice-fishing hole was computed numerically and is plotted in Figure 4. One finds that the maximal amplitude is attained at some point located approximately $\frac{2}{3}r$ away from the hole's center (r is the hole's radius). This amplitude is over 50% larger than at the boundary.

Let us turn to the exact statement of the sloshing problem which is the mathematical model describing small oscillations of an inviscid, incompressible and heavy liquid in a bounded container. The liquid domain W is bounded by a free surface (its mean position F is horizontal) and by the wetted rigid part of ∂W, say B (bottom). Choosing Cartesian coordinates (x, y, z) so that the z-axis is vertical and points upwards, we place the two-dimensional domain F into the plane $z = 0$.

The water motion is assumed to be irrotational and the surface tension is neglected on F. In the framework of linear water wave theory, one seeks sloshing modes and frequencies as eigenfunctions and eigenvalues, respectively, of the following mixed Steklov problem:

$$\Delta\varphi = 0 \ \text{ in } W, \quad \frac{\partial\varphi}{\partial z} = \nu\varphi \ \text{ on } F, \tag{6}$$

$$\frac{\partial\varphi}{\partial \boldsymbol{n}} = 0 \ \text{ on } B, \quad \int_F \varphi \, \mathrm{d}x\mathrm{d}y = 0. \tag{7}$$

The last condition is imposed to exclude the eigenfunction identically equal to a non-zero constant and corresponding to the zero eigenvalue that exists for the problem including only the Laplace equation and the above boundary conditions.

In terms of (ν, φ) found from problem (6), (7), the velocity field of oscillations is given by

$$\cos(\omega t + \alpha)\nabla\varphi(x, y, z).$$

Here α is a certain constant, t stands for the time variable and $\omega = \sqrt{\nu g}$ is the radian frequency of oscillations (as usual, g denotes the acceleration due to gravity). Furthermore, the elevation of the free surface is proportional to $\sin(\omega t + \alpha)\,\varphi(x, y, 0)$, and so high spots are located at the points, where the restriction of $|\varphi|$ to F attains its maximum values.

It is known that if W and F are Lipschitz domains, then problem (6), (7) has a sequence of eigenvalues:

$$0 < \nu_1 \leqslant \nu_2 \leqslant \dots \nu_n \leqslant \dots, \qquad \nu_n \to \infty.$$

For all n, $\varphi_n \in H^1(W)$, whereas their restrictions to F form (together with a non-zero constant) a complete orthogonal system in $L^2(F)$. In hydrodynamics, eigenfunctions corresponding to ν_1 play an important role because the rate of their decay (which is caused by non-ideal effects for real-life liquids) is least.

Modelling a mug by the following vertical-walled container $W = \{(x, y, z) : x^2 + y^2 < 1,\ z \in (-h, 0)\}$, in which case $F = \{(x, y, 0) : x^2 + y^2 < 1\}$ (cf. [37]), one finds all solutions of problem (6), (7) explicitly. In particular, there are two linearly independent

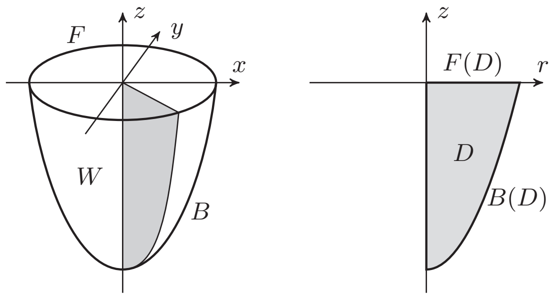


Figure 5. A body of revolution (left) and its radial cross-section (right).

eigenfunctions

$$\varphi_1 = J_1(j'_{1,1}r)\sin\theta\cosh j'_{1,1}(z+h),$$
$$\varphi_2 = J_1(j'_{1,1}r)\cos\theta\cosh j'_{1,1}(z+h),$$

corresponding to $\nu_1 = \nu_2 = j'_{1,1}\tanh j'_{1,1}h$. Here (r, θ, z) are the cylindrical coordinates such that θ is counted from the x-axis (see the left-hand side of Figure 5), J_1 is the Bessel function of the first kind and $j'_{1,1} \approx 1.8412$ is the first positive zero of J'_1. It is clear that $\varphi_1(x, y, 0)$ is an odd, increasing function of y, and so it attains extreme values (high spots) at the boundary points $(0, 1)$ and $(0, -1)$; similarly, φ_2 has its high spots at $(1, 0)$ and $(-1, 0)$. Moreover, all linear combinations of φ_1 and φ_2 have high spots on the boundary of F.

Using the finite element method, one can obtain approximate positions of high spots for more complicated domains. We used FEniCS [36] to implement a trough (see [33] for the corresponding rigorous result), which is short and has a hexagonal cross-section. Such a trough is shown in Figure 6, where several level surfaces of the lowest-frequency mode φ_1 are also plotted. It is clear

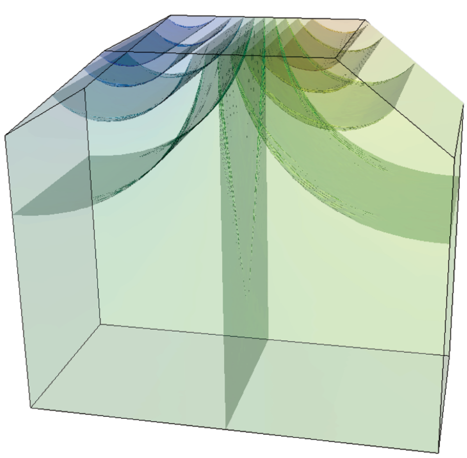


Figure 6. A short trough and level surfaces of its fundamental eigenfunction.

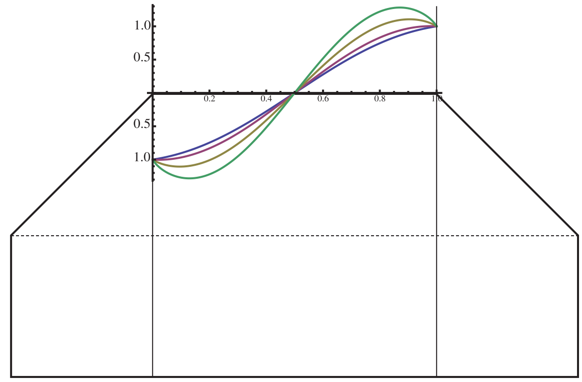


Figure 7. The cross-section of a channel is given by solid segments. The length of the free surface (respectively, bottom) is equal to 2 $(2\,(x+1)$**, respectively); the depth of the channel (respectively, its rectangular part) is equal to** $x+1$ **(**x**, respectively). The free surface profile plotted in blue corresponds to an isosceles trapezoid which is 50% wider at the top than at the bottom; other profiles correspond to the shown hexagonal cross-section with** $x = 0.1, 1, 10$**.**

that the maximum of $\varphi_1(x, y, 0)$ is not on the boundary.

Since the previous example is essentially two-dimensional (see [33]), we exploited this reduction to obtain numerically more accurate free surface profiles plotted in Figure 7. The blue curve corresponds to an isosceles trapezoid which is 50% wider at the top than at the bottom; its maximum is on the boundary (as for a coffee mug). Other profiles correspond to hexagons with different slopes of side walls. Notice that the point of maximum moves towards the center for more horizontal slopes, whereas the high spot becomes more pronounced.

Photos of water oscillations in bulbous and other containers were made (examples are given in Figures 1 and 2). It proved difficult to illustrate the high spot effect by photographing in a conventional way because of the nonlinearity caused by relatively large amplitude of oscillations and non-ideal nature of liquid. Therefore, along with photos shown in Figures 1 and 2, we also photographed a reflection of a dotted piece of paper on a slightly disturbed surface of the liquid (see Figure 8, bottom). Images produced using sufficiently long exposure time mostly consist of blurred segments with just a few clearly visible dots (see Figure 8, top). The reason for this is the fact that planes tangent to the water surface oscillate almost everywhere creating a segment path for each dot. The exceptional points are those where the sloshing surface has its local extrema, and so the corresponding tangent planes

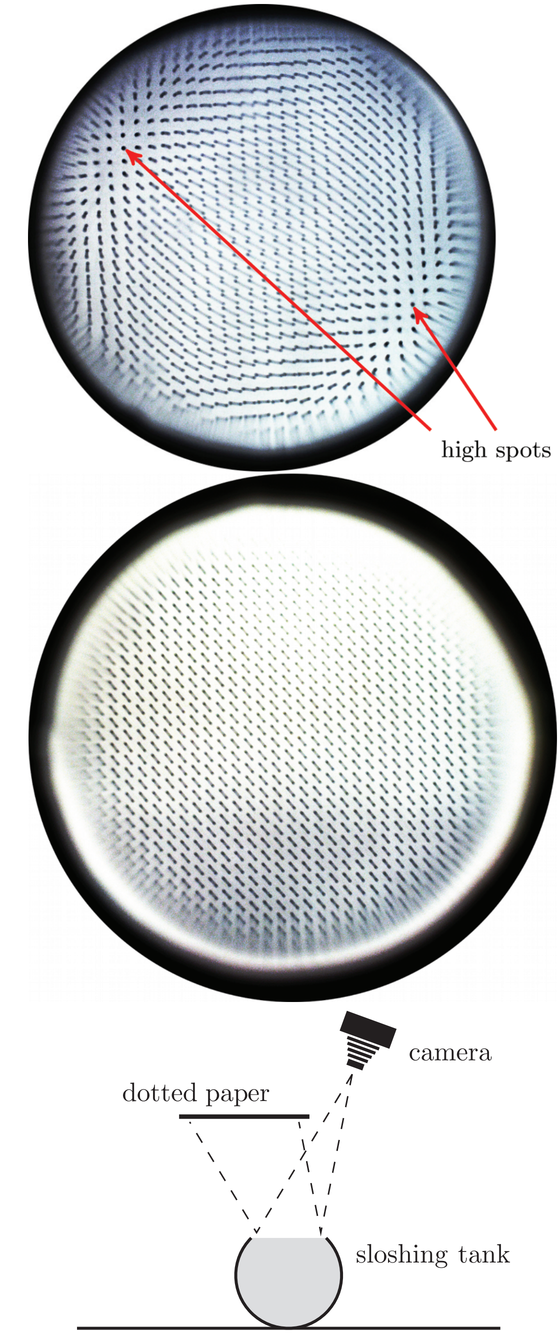


Figure 8. (top) Image of reflected light from the sloshing surface of water in a fish bowl. (middle) Similar image for a cocktail glass, showing no points with vanishing gradient. (bottom) Setup for photos.

are always horizontal, which makes these dots sharp. The image obtained for a bulbous container (a fish bowl) has two clearly visible points of extrema located away from the boundary (see Figure 8, top), which is in agreement with Figures 2 and 3 (b). The similar image for a conical tank (a cocktail glass) consists exclusively of almost the same blurred segment paths for all dots (see Figure 8, middle), and this agrees with Figures 1 and 3 (a).

Let us turn to discussing results proved rigorously for bodies of revolution. If W is obtained by rotating a two-dimensional domain D that has a horizontal segment on the top and is attached to the z-axis around this axis (see Figure 5), then the free surface F is a disk in the (x, y)-plane. We assume that the fundamental modes are antisymmetric; many domains have this property (see below). Nevertheless, examples of rotationally symmetric fundamental eigenfunctions also exist. For example, this takes place for the following domain: halves of a ball and a spherical shell joined by a small vertical pipe so that all of them are coaxial.

Under antisymmetry assumption about the fundamental modes, there are two of them

$$\varphi_1 = \psi(r, z) \cos\theta, \qquad \varphi_2 = \psi(r, z) \sin\theta,$$

that correspond to $\nu_1 = \nu_2$ and are linearly independent; here $\psi(r, z)$ is defined on D.

In [34] (see Theorems 1.1 and 1.2), the following is proved. If W is a convex body of revolution confined to the cylinder $\{(x, y, z) : (x, y, 0) \in F,\ \ z \in \mathbb{R}\}$ (this condition was introduced by F. John in 1950), then three assertions hold: (i) $\nu_1 = \nu_2$; (ii) the corresponding eigenfunctions φ_1 and φ_2 are antisymmetric; (iii) the high spots of these modes are attained on ∂F.

On the other hand (see [34], Proposition 1.3), if the angle between B and F is bigger than $\frac{\pi}{2}$ and smaller than π then $\varphi_1(x, y, 0)$, $\varphi_2(x, y, 0)$ attain their extrema inside F as is shown in Figure 8, top.

The proof of (ii) and (iii) is based on the technique of domain deformation used by D. Jerison and N. Nadirashvili [26], who studied the *hot spots* conjecture. The latter was posed by J. Rauch in 1974 (see a description by I. Stewart in his *Nature* article [62], and Terence Tao's Polymath project [65] for current developments). Roughly speaking, the hot spots conjecture states that in a thermally insulated domain, for "typical" initial conditions, the hottest point will move towards the boundary of the domain as time passes. The mathematical formulation of the hot spots conjecture is as follows: *every fundamental eigenfunction of the Neumann Laplacian in an n-dimensional domain D attains its extrema on ∂D*. It was proved for sufficiently regular planar domains (see, for example, [3] and [26]), disproved for some domains with holes (see, for example, [7]) and is still open for arbitrary convex planar domains.

There is a remarkable relationship between high and hot spots (see, for example, [32], Proposition 3.1). If $\varphi_1, \ldots, \varphi_k$ are the sloshing eigenfunctions corresponding to an eigenvalue ν in a vertical cylinder $W = \{(x, y, z) : (x, y) \in F, z \in (-h, 0)\}$, then μ is an eigenvalue of the Neumann Laplacian in $D = F$, if and only if $\nu = \sqrt{\mu} \tanh \sqrt{\mu} h$. Moreover, every eigenfunction $\psi(x, y)$ corresponding to μ is defined by some φ_j, $j = 1, \dots, k$, in the following way: $\varphi_j(x, y, z) = \psi(x, y) \cosh \sqrt{\mu}(z + h)$. This relationship between the two problems implies, in particular, that the results obtained in [3] and [26] for planar convex domains with two orthogonal axes of symmetry (e.g., ellipses) can be reformulated as follows. In a vertical-walled, cylindrical tank with such free surface, high spots of fundamental sloshing modes are located on the free surface's boundary.

Isoperimetric Inequalities for Steklov Eigenvalues

It was mentioned above that Steklov's major contribution to mathematical physics appeared in 1902, namely, the article [54], in which he introduced an eigenvalue problem (1). This problem with the spectral parameter in the boundary condition turned out to have numerous applications. Moreover, the Steklov problem—as it is referred to nowadays—provides a new playground for exciting interactions between geometry and spectral theory, exhibiting phenomena that could not be observed in other eigenvalue problems. For the sake of simplicity, it is assumed throughout this section that $\varphi \equiv 1$ in formula (1).

In two dimensions, problem (1) can be viewed as a "cousin" of the Neumann problem in a bounded domain D. Indeed, the latter problem describes the vibration of a homogeneous free membrane, whereas the Steklov problem models the vibration of a free membrane with all its mass concentrated along the boundary (see [2], p. 95). Steklov eigenvalues

$$0 = \lambda_0 < \lambda_1(D) \leqslant \lambda_2(D) \leqslant \lambda_3(D) \leqslant \cdots \nearrow \infty$$

correspond to the frequencies of oscillations. As in the Neumann case, the Steklov spectrum starts with zero, and in order to ensure discreteness of the spectrum it is sufficient to assume that the boundary is Lipschitz.

Isoperimetric inequalities for eigenvalues is a classical topic in geometric spectral theory that goes back to the ground-breaking results of Rayleigh–Faber–Krahn and Szegő–Weinberger on the first Dirichlet and the first non-zero Neumann eigenvalues. The problem is to find a shape that extremizes (minimizes for Dirichlet and maximizes for Neumann) the first eigenvalue among all shapes of fixed volume. In both cases, the unique extremal domain is a ball, similarly to the classical isoperimetric inequality in Euclidean geometry.

Szegő's proof of the isoperimetric inequality for the first Neumann eigenvalue on a simply connected planar domain D is based on the Riemann mapping theorem and a delicate construction of trial functions using eigenfunctions on a disk [63]. In 1954 (the same year Szegő's paper was published), R. Weinstock [69] realized that this approach could be adapted to prove a sharp isoperimetric inequality for the first Steklov eigenvalue. Weinstock showed that the first nonzero Steklov eigenvalue is maximized by a disk among all simply connected planar domains of fixed *perimeter*. Note that for the Steklov problem, the perimeter is proportional to the mass of the membrane, like the area in the Neumann problem. In fact, Weinstock's proof is easier than Szegő's, because the first Steklov eigenfunctions on a disk are just coordinate functions, not Bessel functions as in the Neumann case. In a way, Weinstock's argument is a first application of the "barycentric method" that is being widely used in geometric eigenvalue estimates.

The analogy between isoperimetric inequalities for Neumann and Steklov eigenvalues is far from being complete, which makes the study of Steklov eigenvalues particularly interesting. For instance, as was shown by Weinberger [68], Szegő's inequality for the first Neumann eigenvalue can be generalized to arbitrary Euclidean domains of any dimension. At the same time, Weinstock's result fails for non-simply connected planar domains: if one digs a small hole in the center of a disk, the first Steklov eigenvalue of the corresponding annulus, normalized by the perimeter, is bigger than the normalized first Steklov eigenvalue of a disk [19].

Another major distinction from the Neumann case is that for simply connected planar domains, sharp isoperimetric inequalities are known for *all* Steklov eigenvalues. It was shown in [16] that the inequality $\lambda_n(D)\, L(\partial D) \leqslant 2\pi n$, $n = 1, 2, 3 \dots$, proved in [23] is sharp, with the equality attained in the limit by a sequence of domains degenerating to a disjoint union of n identical disks; here $L(\partial D)$ denotes the perimeter of D. For Neumann eigenvalues, a similar result holds for $n = 2$ [15], but the situation is quite different for $n \geqslant 3$ (see [1] and [48]).

In 1970, J. Hersch [22] developed the approach of Szegő in a more geometric direction. He proved that among all Riemannian metrics on a sphere of given area, the first eigenvalue of the corresponding Laplace–Beltrami operator is maximal for the standard round metric. Note that the first eigenspace on a round sphere is generated by coordinate functions, which allows one to prove Hersch's theorem in a similar way as Weinstock's inequality [17]. The result of Hersch stimulated a whole direction of research on extremal metrics for Laplace–Beltrami eigenvalues on surfaces.

Recently, Fraser and Schoen (see [11] and [12]) extended the theory of extremal metrics to Steklov eigenvalues on surfaces with boundary. They have studied extremal metrics for the first Steklov eigenvalue on a surface of genus zero with l boundary components, and proved the existence of maximizers for all $\ell \geqslant 1$. Weinstock's inequality covers the case $\ell = 1$, but already for $\ell = 2$ the result is quite unexpected. The maximizer is given by a "critical catenoid"; it is a certain metric of revolution on an annulus such that the first Steklov eigenvalue has multiplicity three. Interestingly enough, this is the maximal possible multiplicity for the first eigenvalue on an annulus (see [12], [27] and [25]). The critical catenoid admits the following characterization: it is a unique free boundary minimal annulus embedded into a Euclidean ball by the first Steklov eigenfunctions [12].

Maximizers for higher Steklov eigenvalues on surfaces, as well as sharp isoperimetric inequalities for eigenvalues on surfaces of higher genus, are still to be found. Some bounds were obtained in [11], [29], [18], [21] in terms of the genus and the number of boundary components. At the same time, it was shown in [9] that there exists a sequence of surfaces of fixed perimeter, such that the corresponding first eigenvalues of the Steklov problem tend to infinity.

Apart from the two-dimensional vibrating membrane model discussed above, there is another physical interpretation of the Steklov problem, which is valid in arbitrary dimension. It describes the stationary heat distribution in a body D, under the condition that the heat flux through the boundary is proportional to the temperature. In this context, it is meaningful to use the volume of D as a normalizing factor. Weinstock's inequality combined with the classical isoperimetric inequality implies that the disk maximizes the first Steklov eigenvalue among all simply connected planar domains of given area. Generalizations of this result were obtained in [6] and [4]. In particular, it was shown that in any dimension, the ball maximizes the first Steklov eigenvalue among all Euclidean domains of given volume.

Yet another interpretation of the Steklov spectrum involves the concept of the Dirichlet-to-Neumann map (sometimes called the Poincaré–Steklov operator), which is important in many applications, such as electric impedance tomography, cloaking, etc. The Dirichlet-to-Neumann map acts on functions on the boundary of a domain D (or, more generally, of a Riemannian manifold), and assigns to each function the normal derivative of its harmonic extension into D. The spectrum of this operator is given precisely by the Steklov eigenvalues. Since the Dirichlet-to-Neumann map acts on ∂D, it is natural to normalize the eigenvalues by the volume of the boundary. If the volume of ∂D is fixed, the corresponding Steklov eigenvalues λ_n of a Euclidean domain D can be bounded in terms of n and the dimension [8]. However, no sharp isoperimetric inequalities of this type are known at the moment in dimensions higher than two.

Mathematical notions often lead a life of their own, independent of the will of their creators. When Steklov introduced the eigenvalue problem that now bears his name, he was motivated mainly by applications. It is hard to tell whether he could foresee the interest in the problem coming from geometric spectral theory. There is no doubt, however, that the past and future work of many mathematicians on isoperimetric inequalities for Steklov eigenvalues owes a lot to Steklov's insight.

References

[1] P. Antunes and P. Freitas, Numerical optimization of low eigenvalues of the Dirichlet and Neumann Laplacians, *J. Optim. Theor. Appl.*, **154** (2012), 235–257.

[2] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, 1980, 228 pp.

[3] R. Bañuelos and K. Burdzy, On the "hot spots" conjecture of J. Rauch, *J. Funct. Anal.*, **164** (1999), 1–33.

[4] Binoy and G. Santharam, Sharp upper bound and a comparison theorem for the first nonzero Steklov eigenvalue, Preprint arXiv:1208.1690v1 [math.DG] 8 Aug 2012.

[5] W. Blaschke, *Kreis und Kugel*, Verlag von Veit & Comp., 1916, x+169 pp.

[6] F. Brock, An isoperimetric inequality for eigenvalues of the Stekloff problem, *Z. Angew. Math. Mech.* **81** (2001), 69–71.

[7] K. Burdzy, The hot spots problem in planar domains with one hole, *Duke Math. J.*, **129** (2005), 481–502.

[8] B. Colbois, A. El Soufi, and A. Girouard, Isoperimetric control of the Steklov spectrum, *J. Func. Anal.*, **261** (2011), 1384–1399.

[9] B. Colbois and A. Girouard, The spectral gap of graphs and Steklov eigenvalues on surfaces. Preprint arXiv:1310.2869 [math.SP] 10 Oct 2013.

[10] R. Courant, D. Hilbert, *Methoden der mathematischen Physik. II*, Springer, 1937, 549 pp.

[11] A. Fraser and R. Schoen, The first Steklov eigenvalue, conformal geometry, and minimal surfaces, *Adv. Math.*, **226** (2011), 4011–4030.

[12] ———, Sharp eigenvalue bounds and minimal surfaces in the ball, Preprint arXiv: 1209.3789 [math.DG] 3 Apr 2013.

[13] K. O. Friedrichs, Eine invariante Formulierung des Newtonschen Gravitationsgesetzes und des Grenzüberganges vom Einsteinschen zum Newtonschen Gesetz, *Math. Ann.*, **98** (1927), 566–575.

[14] B. G. Galerkin, N. M. Günther, R. O. Kuz'min, I. V. Meshcherski, V. I. Smirnov, *Pamyati V. A. Steklova (In Memoriam of V. A. Steklov)*, Leningrad, Acad. Sci. USSR, 1928, 92 pp. (In Russian.)

[15] A. Girouard, N. Nadirashvili and I. Polterovich, Maximization of the second positive Neumann eigenvalue for planar domains, *J. Diff. Geom.*, **83** (2009), 637–662.
[16] A. Girouard and I. Polterovich, On the Hersch–Payne–Schiffer inequalities for Steklov eigenvalues, *Funk. Anal. Prilozh.*, **44**, no. 2 (2010), 33–47. (In Russian, English transl. *Func. Anal. Appl.*, **44** (2010), 106–117.)
[17] ______, Shape optimization for low Neumann and Steklov eigenvalues, *Math. Meth. Appl. Sci.*, **33** (2010), 501–516.
[18] ______, Upper bounds for Steklov eigenvalues on surfaces, *Electron. Res. Announc. Amer. Math. Soc.*, **19** (2012), 77–85.
[19] ______, Spectral geometry of the Steklov problem (in preparation).
[20] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge, University Press, 1934, 324 pp.
[21] A. Hassannezhad, Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem, *J. Func. Anal.*, **261** (2011), 3419–3436.
[22] J. Hersch, Quatre propriétés isopérimétriques de membranes sphériques homogènes, *C. R. Acad. Sci. Paris*, Sér. A-B, **270** (1970), A1645–A1648.
[23] J. Hersch, L. Payne and M. Schiffer, Some inequalities for Stekloff eigenvalues, *Arch. Rat. Mech. Anal.*, **57** (1974), 99–114.
[24] V. A. Il'in, On solvability of mixed problems for hyperbolic and parabolic equations, *Uspekhi Mat. Nauk* **15**, no. 2 (1960), 97–154. (In Russian, English transl. *Russian Math. Surveys* **15** (2) (1960), 85–142.)
[25] P. Jammes, Prescription du spectre de Steklov dans une classe conforme, Preprint arXiv:1209.4571v1 [math. DG] 20 Sep 2012.
[26] D. Jerison and N. Nadirashvili, The "hot spots" conjecture for domains with two axes of symmetry, *J. Amer. Math. Soc.*, **13** (2000), 741–772.
[27] M. Karpukhin, G. Kokarev, and I. Polterovich, Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces, Preprint arXiv:1209.4869v2 [math.DG] 20 Mar 2013.
[28] A. Kneser, Wladimir Stekloff zum Gedächtnis, *Jahresber. Deutsch. Math. Verein.*, **38** (1929), 206–231.
[29] G. Kokarev, Variational aspects of Laplace eigenvalues on Riemannian surfaces, Preprint arXiv:1103.2448v2 [math.SP] 21 May 2011.
[30] V. Kozlov and N. Kuznetsov, The ice-fishing problem: the fundamental sloshing frequency versus geometry of holes, *Math. Meth. Appl. Sci.*, **27** (2004), 289–312.
[31] J. Král, *Integral Operators in Potential Theory. Lect. Notes Math.* **823**, 1980, i+171 pp.
[32] T. Kulczycki and N. Kuznetsov, "High spots" theorems for sloshing problems, *Bull. Lond. Math. Soc.*, **41** (2009), 495–505.
[33] ______, On the "high spots" of fundamental sloshing modes in a trough, *Proc. R. Soc.*, A**467** (2011), 1491–1502.
[34] T. Kulczycki and M. Kwaśnicki, On high spots of the fundamental sloshing eigenfunctions in axially symmetric domains, *Proc. Lond. Math. Soc.*, **105** (2012), 921–952.
[35] H. Lamb, *Hydrodynamics*, Cambridge, University Press, 1932, xv+738 pp.
[36] A. Logg, K.-A. Mardal and G.Wells, *Automated Solution of Differential Equations by the Finite Element Method. The FEniCS book. Lecture Notes in Comput. Sci. and Engin.*, **84**, Berlin, Springer-Verlag, 2012, xiii+723 pp.
[37] H. Mayer and R. Krechetnikov, Walking with coffee: Why does it spill? *Phys. Rev. E*, **85** (2012), 046117 (7 pp.)
[38] V. G. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Springer-Verlag, 2011, 866 pp.
[39] S. G. Mikhlin, *Constants in Some Inequalities of Analysis*, Wiley-Intersci., 1986, 108 pp.
[40] ______, On the history of mathematics at Leningrad State University at the end of the 1920s, *Proc. St. Petersburg Math. Soc.*, **2** (1993), 300–308. (In Russian, English transl. *AMS Translations*, Ser. 2, **159** (1994), 207–212.)
[41] D.S. Mitrinović and P.M. Vasić, An integral inequality ascribed to Wirtinger, and its variations and generalizations, *Public. fac. électrotech. Univ. Belgrade*, no 272, (1969), 157–170.
[42] O. Nikodým, Sur une classe de fonctions considérées dans l'étude du problème de Dirichlet, *Fund. Math.*, **21** (1933), 129–150.
[43] J. J. O'Connor and E.F. Robertson, Vladimir Andreevich Steklov, *MacTutor History of Mathematics*. Available online at http://www-history.mcs.st-and.ac.uk/history/Mathematicians/Steklov.html
[44] J. Peetre, MR655789 (84a:46076).
[45] J.-P. Pier, (ed.) *Developments of Mathematics 1900 – 1950*, Birkhäuser, Basel, 1994, xviii+729 pp.
[46] H. Poincaré, Sur les équations aux deŕiveés partielles de la physique mathematique, *Am. J. Math.*, **12** (1890), 211–294.
[47] ______, Sur les équations de la physique mathematique *Rend. Circ. Mat. Palermo*, **8** (1894), 57–156.
[48] G. Poliquin and G. Roy-Fortin, Wolf–Keller theorem for Neumann eigenvalues, *Ann. Sci. Math. Québec*, **36** (2012), no. 1, 169–178.
[49] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951, xvi+279 pp.
[50] V. A. Steklov, The problem of cooling of an heterogeneous rigid rod, *Communs Kharkov Math. Soc.*, Ser. 2, **5** (1896), 136–181. (In Russian.)
[51] ______, On the expansion of a given function into a series of harmonic functions, *Communs Kharkov Math. Soc.*, Ser. 2, **6** (1897), 57–124. (In Russian.)
[52] W. Stekloff, Problème de refroidissement d'une barre hétérogène, *Ann. fac. sci. Toulouse*, Sér. 2, **3** (1901), 281–313.
[53] V.A. Steklov, General methods of solving fundamental problems in mathematical physics, Kharkov Mathematical Society (1901) (in Russian). Available online at http://nasledie.enip.ras.ru/unicollections/list.html?id=42034006
[54] W. Stekloff, Sur les problèmes fondamentaux de la physique mathematique, *Annales sci. ENS*, Sér. 3, **19** (1902), 191–259 and 455–490.
[55] ______, Théorie générale des fonctions fondamentales, *Annales fac. sci. Toulouse*, Sér. 2, **6** (1904), 351–475.
[56] W. Stekloff [V. A. Steklov], Sur la condition de fermeture des systèmes de fonctions orthogonales, *Comp. rend. Acad. sci. Paris*, **151** (1910), 1116–1119. (Russian version *Bull. Imp. Acad. Sci.*, Sér. 6, **5** (1911), 754–757.)

[57] W. Stekloff, Sur la théorie de fermeture des systèmes de fonctions orthogonales dépendant d'un nombre quelconque des variables, *Mém. Acad. Sci. St. Pétersbourg, Cl. Phys. Math.*, Sér. 8, **30**, no. 4 (1911), 1–87.

[58] V. A. Steklov, *Osnovnye Zadachi Matematicheskoy Fiziki (Fundamental Problems of Mathematical Physics)*, Vol. 1, Petrograd, 1922, iv+285 pp., Vol. 2, Petrograd, 1923, ii+ 285 pp. (In Russian.)

[59] ______, Vospominaniya (The Recollections), *Nauchnoe Nasledstvo (Scientific Legacy)*, **17**, "Nauka", Leningrad, 1991, pp. 235–299. (In Russian.) Available online at http://mathedu.ru/memory/steklov.djvu

[60] ______, Perepiska s A. M. Lyapunovym, A. N. Krylovym, N. M. Krylovym, A. A. Markovym (Correspondence with A. M. Lyapunov, A. N. Krylov, N. M. Krylov, A. A. Markov Sr.), *Nauchnoe Nasledstvo (Scientific Legacy)*, **17**, "Nauka", Leningrad, 1991, pp. 19–234. (In Russian.)

[61] V. A. Steklov and A. Kneser, *Nauchnaya Perepiska (Scientific Correspondence), 1901–1925*, "Nauka", Moscow, 1980, 81 pp. (in Russian).

[62] I. Stewart, Mathematics: Holes and hot spots, *Nature*, **401** (1999), 863–865.

[63] G. Szegő, Inequalities for certain eigenvalues of a membrane of given area, *J. Rat. Mech. Anal.*, **3** (1954), 343–356.

[64] P. K. Suetin, V. A. Steklov's problem in the theory of orthogonal polynomials, *J. Soviet Math.*, **12** (1979), 631–682.

[65] T. Tao et al., Polymath7 research threads 4: the Hot Spots Conjecture. Available online at `http://polymathprojects.org/2012/09/10/polymath7-research-threads-4-the-hot-spots-conjecture/`

[66] E. A. Tropp, V. Ya. Frenkel', and A. D. Chernin, *Aleksandr Aleksandrovich Fridman: Zhizn' i Deyatel'nost'*, "Nauka", Moscow, 1988, 304 pp. (In Russian, English transl. *Alexander A. Friedmann: the Man Who Made the Universe Expand*, Cambridge University Press, 1993, 277 pp.)

[67] V. S. Vladimirov and I. I. Markush, *Academician Steklov: Mathematician Extraordinary*, Translated from Russian by C. A. Hainsworth and R. N. Hainsworth, "Mir", Moscow, 1983, 126 pp.

[68] H. F. Weinberger, An isoperimetric inequality for the N-dimensional free membrane problem, *J. Rat. Mech. Anal.*, **5** (1956), 633–636.

[69] R. Weinstock, Inequalities for a classical eigenvalue problem, *J. Rat. Mech. Anal.*, **3** (1954), 745–753.

Common Core State Standards for Geometry: An Alternative Approach

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Introduction

Forty-five states, including California, have adopted common core standards in mathematics for kindergarten through grade twelve. The CCSS (Common Core State Standards) [3] are designed to provide strong, shared expectations and, furthermore, allow the adopting states to collectively create and share tested tools such as assessments, curricula, and professional development programs. *Achieve* [1] has made this observation and further analyzed the CCSS in relation to the NAEP (National Assessment of Educational Progress) framework. Their findings suggest that the CCSS as a whole are mathematically more demanding than the current mathematics curricula. This is particularly true for the geometry standards due to their emphasis on mathematical proof, which, as has been widely established, is one of the most difficult concepts for students [8], [10].

Geometry often represents a high school student's first formal introduction to abstract mathematical reasoning. That is, the student is asked to (a) reason about such abstract concepts as points, lines, and triangles; (b) understand that certain geometric objects can be defined only in terms of their relation with each other; and (c) prove theorems about the Euclidean structure based on a small set of basic concepts and axioms. Not only is this type of reasoning fundamental for more advanced courses in mathematics, it is also prevalent in many areas of science and engineering where one reasons about simplified or ideal models. For example, in the applications of the theory of gravity, one thinks of mass being concentrated in a single point. Similarly, in the kinetics of motion, solid objects are almost always considered as points at their "center of (inertial) mass", and the most fundamental insight in all of kinetics is that objects proceed in straight line motion unless acted on by a force. The idea that *point*, *line*, *plane*, and *space* cannot be characterized independently of each other is analogous to situations where a physical quantity can be measured only by observing its interaction with other quantities (e.g., spin can be measured only by observing its interaction with other magnetic fields). Finally, the hypothetico-deductive method is undoubtedly the basis of all science. The practice of proving theorems on the basis of a small set of axioms serves as a cognitive precursor to the vital scientific practice of deriving one law from more fundamental laws; the derivation of Kepler's laws from Newton's laws—one of the greatest triumphs of calculus—is an example. Because high school geometry plays a key role in the development of our students' ability to reason about abstract models in mathematics, science, and engineering, we believe that it is absolutely essential that teachers and curriculum developers attend to the question of how to motivate abstract geometric concepts and reasoning to their students.

A close analysis of the narrative of the CCSS in high school geometry (hereafter, CCSS-Geometry) has revealed potentially serious problems with their future implementations. Our concerns were further validated by some of the initial curricular material developed, presentations given by teachers and curriculum developers in teacher conferences, and conversations with teachers who have participated in regional professional developments targeting the CCSS-Geometry. Collectively, these materials and activities represent a particular interpretation of the CCSS-Geometry that is pedagogically unsound. Given its current

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DOI: http://dx.doi.org/10.1090/noti1070

widespread use, we call this interpretation the *standard approach*.

Upon this realization, we developed an alternative interpretation of the CCSS-Geometry grounded in the *intellectual need* of the students. The notion of intellectual need was defined technically and discussed at length in [9] and will become clearer as this article unfolds. Generally speaking, intellectual need is an expression of a natural human behavior: When people encounter a situation that is incompatible with or presents a problem that is unsolvable by their existing knowledge, they are likely to search for a resolution or a solution and construct, as a result, new knowledge. Such knowledge is meaningful to them because it is a product of their personal need and connects to their prior experience. This human nature is the basis for what we call the *necessity principle*: *For students to learn what we intend to teach them, they must have a need for it, where "need" refers to intellectual need, not social or economic needs* [7], [9]. Pedagogically, this principle translates into three concrete instructional steps: (1) Recognize what constitutes an intellectual need for a particular population of students (high school students in our case), relative to the concept to be learned. (2) Present the students with a problem or sequence of problems that corresponds to their intellectual need and from whose solutions the concepts can be elicited. (3) Help students elicit the concepts from solutions to these problems.

The Standard Approach

The CCSS-Geometry consists of goals together with narrative. We agreed with the goals (e.g., *reason abstractly, construct viable arguments, use appropriate tools strategically, attend to precision, look for and make use of structure*, and *look for and express regularity in repeated reasoning)*, and we are particularly pleased with the new emphasis on geometry proof and construction (*prove geometric theorems, make geometric constructions*). We are concerned, however, with the narrative and especially with how it is currently being interpreted. Our characterization of the standard approach is, thus, based on this narrative and other resources, such as test items, presentations given in teacher conferences about the CCSS-Geometry, and some of the Common Core-based texts currently under examination for adoption by school districts. The ultimate goal of this article is to point out potential pitfalls with this approach and offer an alternative one.

Our main concerns can be summarized as (1) lack of attention to students' intellectual needs, (2) premature introduction to and overemphasis on plane transformations, and (3) lack of clarity about the importance of separating the analytic study and the synthetic study of Euclidean geometry.

Lack of Attention to Students' Intellectual Needs

It is hard to imagine any educator disagreeing with the CCSS goal: "During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs." The question of critical importance is, How do we bring students to see an intellectual need to formalize their informal geometry experiences into formal definitions and to reason deductively when justifying or refuting claims? That is to say, how do we problematize students' current experiences so that they come to understand and appreciate precise definitions and careful proofs? The conceptual basis for this question is Piaget's theory of learning (see, for example, [13]). In essence, according to this theory, the means—the only means—of learning is problem solving. In Piaget's terms, a failure to assimilate results in a disequilibrium, which in turn leads the mental system to seek equilibrium to reach a balance between the structure of the mind and the environment. Learning transpires when such balance occurs. While Piaget himself concentrated on the development of mathematical knowledge at the early stages, his theory was extended and applied to advanced topics going into undergraduate mathematics.

The question of intellectual necessity does not seem to be part of the repertoire of considerations by current Common Core-based curricula. The transition from informal to formal is done largely descriptively by merely restating concepts and assertions assumed to be understood intuitively into formal definitions, theorems, and proofs. Basically, this is precisely the approach that has been in use for decades, which, as we now know from status studies on students' conceptions of geometry and proof, has failed miserably. In aggregate, these studies show that "students accept examples as verification, do not accept deductive proofs as verification, do not accept counterexamples as refutation, accept flawed deductive proofs as verification, accept arguments on bases of other than logical coherence, offer empirical arguments to verify, cannot write correct proofs" ([10], p. 59).

A case in point is the ability to characterize objects and prove assertions in terms of mathematical definitions, what might be called *definitional reasoning.* Evidence exists to indicate that this way of thinking is difficult to acquire. For example, in Van Hiele's model [14], only those high school students who reach the highest stage of geometric reasoning can reason in terms of definitions (see [2]). College students too experience difficulty reasoning in terms of definitions. For example, asked to define "invertible matrix", many linear algebra students stated a series of equivalent properties (e.g., "a square matrix with a nonzero determinant", "a square matrix with full rank", etc.) rather than a definition. The fact that they provided more than one such property is an indication they are not definitional reasoners [6]. The standard approach seems to take for granted definitional reasoning. For example, taking the CCSS-Geometry's statement, "Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc" [3] literally in geometry curricula, as some current Common Core-based textbooks do, is likely to be unproductive for most students. Just telling students that a particular term is undefined does not guarantee that they will consider it as such and use it as a basis for defining other concepts.

Over a century ago, a great mathematician, with deep pedagogical sensitivity, pointed to the challenge of definitional reasoning: "What is a good definition? For the philosopher or the scientist, it is a definition which applies to all objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils.…" (Poincaré, 1952, p. 117).

Premature Introduction to and Overemphasis on Plane Transformations

Central to the CCSS-Geometry is the concept of *plane transformation*. In the standard approach the transition from middle school geometry to high school geometry is to be carried out through the rigid motions of *translation*, *reflection*, and *rotation* and the motion of *dilation*. In middle school geometry, these motions are delivered informally, and in high school they are defined as functions on the plane. In both levels, the motions are merely described, not intellectually necessitated through problems the students understand and appreciate, for example, by helping high school students see the power of reasoning in terms of plane transformations when solving geometry construction problems, as we will see in the next section.

This intellectual-need-free pedagogy has been dominant in current mathematics curricula. One of its characteristics is that it takes certain difficult conceptualizations for granted. Relevant to our discussion here is the overwhelming evidence that students have enormous difficulties with the concept of function and that the process of acquiring this concept is inevitably challenging (see, for example, [5], [12]). Yet, plane transformations are central in the CCSS-Geometry. The formal definitions of plane transformations require the application of advanced conceptualizations of functions, right at the start of the introduction of deductive geometry. And not just any functions! Students must understand transformations as functions from the plane to the plane. Thompson [12] makes a strong argument that this is a tremendous intellectual achievement for students. They think that transformations act only on the points you happen to have represented and that any transformation moves at most a few points, but of course this is not the case. For example, when one envisions rotating a triangle in a plane, one must be able to think of the rotation as a function acting on the plane, that is, rotating the entire plane, not just the triangle.

Why is there a focus on plane transformations? one might ask. An answer to this question can be found in [3]: namely, that rigid motions create continuity from middle school geometry to high school geometry. Once the motions of translation, reflection, and rotation are defined formally, they provide the foundation for the concept of superposition and, in turn, for *congruence* and congruence criteria; and once the motion of *dilation* is defined formally, it provides the foundation for *similarity* and the similarity criteria.

> In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. …During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures. …Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent. [3]

Thus, the main purpose of the focus on plane transformations throughout middle school and high school is merely to establish the concept of congruence and similarity and the criteria for triangle congruence and triangle similarity. Support for this conclusion can be found in [15]:

> One cannot overstate the fact that the CCSS do not pursue "transformational geometry" per se. Transformations are merely a means to an end: they are used in a strictly utilitarian way to streamline the existing school geometry curriculum. One can see from the high school geometry standards of the CCSS that, once translations, rotations,

> reflections, and dilations have contributed to the proofs of the standard triangle congruence and similarity criteria (SAS, SSS, etc.), the development of plane geometry can proceed along traditional lines if one so desires.

It didn't seem to bother Euclid to define congruence through the image of picking a figure up and laying it on top of another, and it is unlikely that the same action would bother high school students right at the start of their geometry course. The amount of "curricular space" needed to establish congruence and similarity through plane transformations is substantial, as can be seen from the following data: The CCSS-Geometry for high school Euclidean geometry (not including trigonometry) appears under eight headings, three of which (close to 40 percent) are devoted to transformations in the plane. (This does not include the preparation expected in the middle school geometry!) Those eight headings are: (1) Experiment with transformations in the plane (154 words), (2) Understand congruence in terms of rigid motions (98 words), (3) Prove geometric theorems (129 words), (4) Make geometric constructions (77 words), (5) Understand similarity in terms of similarity transformations (126 words), (6) Prove theorems involving similarity (47 words), (7) Understand and apply theorems about circles (88 words), and (8) Find arc lengths and areas of sectors of circles (31 words). This word count shows that the narrative devoted to transformational geometry occupies over 50 percent of the entire narrative allocated to Euclidean geometry.

The standard approach, thus, would require enormous effort and time to be spent on plane transformations—their definitions, compositions, and properties—which will inevitably shift the attention from deductive reasoning, the main objective of the CCSS-Geometry.

Lack of Clear Distinction between Analytic Geometry and Synthetic Geometry

The CCSS-Geometry indicates the importance of studying Euclidean geometry both analytically and synthetically and, further, points to the role of analytic geometry as a tool to connect algebra and geometry. The CCSS-Geometry, however, is silent about the sequencing of and relative emphasis on the two studies. The result is that, in the name of *integrated math*, some current Common Core-based textbooks blend analytic geometry with synthetic geometry, putting emphasis on the former and giving limited attention to the latter. While both geometries are important, synthetic geometry has special roles in school mathematics. Beyond the problem-solving skills one develops from studying Euclidean geometry synthetically, one also develops a crucial way of thinking: *the desire to know what makes a theorem true, not just that it is true.* Compare, for example, the insight one gets from a synthetic proof of a concurrency theorem (e.g., "The three medians in a triangle are concurrent") to the insight one gets from an analytic proof of the same theorem. In this respect, the distinction between the two geometries is analogous to, for example, the distinction between bijective proofs and generating function proofs of combinatorial identities and formulas for counting various classes of combinatorial objects. In many cases, bijective proofs provide more insight and understanding of the theorem at hand than proofs by manipulating formal series.

A Nonstandard Approach

This leaves us with a challenging question: What might be an alternative approach to the standard approach that would address all these concerns and achieve the ultimate goals of the CCSS-Geometry for all students? Our answer to this question was to develop, implement, and test a yearlong geometry unit in deductive geometry that positions the mathematical soundness of the content taught and intellectual need of the student at the center of the instructional effort. For reasons which will become clear shortly, we call this approach a *nonstandard approach* and the unit that is based on it *Conversations with Euclid*.

Many features differentiate the *Conversations with Euclid* unit from the curricula that follow the standard approach. We begin with four features as advance organizers to the presentation of the unit: (a) *The Conversations with Euclid* unit intellectually necessitates the abstract nature of geometric objects, including the so-called undefined terms such as *point*, *line*, and *plane* and attends to precise definitions and proofs without dwelling on axioms and postulates; (b) only when these critical skills are advanced significantly does it elicit plane transformations, and it does so by bringing the students to see their power to simplify solutions to geometry problems; (c) only when this power is sufficiently realized by the students does the unit use plane transformations to establish the congruence and similarity criteria; and (d) the unit studies Euclidean geometry synthetically. Following this presentation, we list other fundamental features of the *Conversations with Euclid* unit.

The Conversations with Euclid Unit: A Synopsis

We piloted the *Conversations with Euclid* unit during the summers of 2012 and 2013 with forty-two in-service secondary school teachers. At the center of the unit is a thought experiment, the goal of which is the process of gradually necessitating the abstract nature of geometric objects and with it the notion of geometric proof. The thought experiment, if extended, can be thought of as an allegory of the historical development of geometry—from Euclid (323–283 BC) to Hilbert (1862–1943). However, the stage of the thought experiment that is relevant to high school geometry is limited in scope, but it serves as a pedagogical tool and mathematical foundation for the entire Euclidean geometry.

In what follows we present a few sporadic segments from the unit's lessons to illustrate how the *necessity principle* was implemented. These segments alone do not capture the logical flow of the lessons nor do they give a sufficient portrait of the rich classroom debate that was generated as the unit was taught. They do, we hope, provide an image for the nonstandard approach we are advocating.

Due to space limitations, it was necessary to omit most of the narrative within the segments. The parts that are direct quotes from the units appear in italics to separate them from the rest of the discussion.

Necessitating the Abstract Nature of Geometric Concepts

The first lesson of the unit begins with a story about a group of mathematicians engaging in a certain project, and the students are asked to participate in the project's dilemmas and resolutions.

Imagine an intelligent alien who possesses none of our visual, kinesthetic, or tactile senses and, therefore, does not share any of the images we have for our physical world. A team of four mathematicians, Natalie, Jose, Eli, and Evelyn, began a thought experiment of communicating with such an alien, who they named Euclid.[1]

The mathematicians began with two questions: (1) *What aspects of humans' life should they begin describing to Euclid? And* (2) *What symbols and language to use to communicate to Euclid these aspects of life? After some discussion they decided that, at this stage of the thought experiment, it is best to focus only on the first question. And as to the second question, they decided that for now they should proceed as if Euclid were a person who spoke English and could think logically.*

But humans' life is very complex—where should they start? Again, after some discussion, they narrowed their focus to humans' physical surroundings. But even this turned out to be very complex; for example, imagine telling someone who is blind and has no movement or touching sensations what a tree is. So the mathematicians decided to narrow their thought experiments to four questions:

1. *What are the most basic physical objects of our surroundings?*
2. *What are humans' images for these objects?*
3. *What is a useful way to imagine these objects when doing math?*
4. *How can we share with Euclid the way we think of these objects when we do math?*

You might ask, what is meant by the term "image"? An image of an object is [a discussion of this term appears here.]*…we share no physical experiences with Euclid, so the communication with him is expected to be very challenging. This is the reason the mathematicians decided to focus on very simple objects as a start. …*

ACTIVITY: WHILE THE MATHEMATICIANS ARE THINKING...

Question: *What are some of the basic objects of our surroundings that you recommend the team should start with?*

To the teacher:

As students make suggestions, keep reminding them that we are only interested in the spatial description of the objects, not their functions. It is not necessary for the students to come up with and agree upon a particular set of objects. What is important is that the students realize how difficult the assignment is. This will prepare them for the next mathematicians' decision to focus on the simplest objects of humans' physical surroundings: point, line, and plane.

The mathematicians decided to start with what seemed to them the most elementary objects of our physical world: planes, lines, and points. Once this decision was made, they turned to the next two questions in their list:

2. *What are humans' images of point, line, and plane?*
3. *What is a useful way to imagine point, line, and plane when we do math?*

It was necessary for the mathematicians to answer these two questions before engaging in the fourth question.

4. *How can we share with Euclid the way we think of point, line, and plane when we do math?*

ACTIVITY: WHILE THE MATHEMATICIANS ARE THINKING …

Discuss in your small group Questions 2 *and* 3.

To the teacher:

It should not be difficult for the students to agree that humans' image of a plane is a flat surface extending indefinitely in its four directions: north, south, east, and west; that humans' image of line is a thin thread stretched tight and extending indefinitely in its two directions; and that humans' image of point is a dot.

The goal of the discussion of Question 3 is merely to prepare students for the dialogue that follows. In this dialogue, students will learn an explanation as to why it is necessary and useful to go one step further by imagining point, line, and plane as objects with no thickness.

Although students should not see this dialogue until after this discussion, the teacher can use the ideas in the dialogue to navigate the discussion.

At this point the students are presented with a debate among the four mathematicians

[1] *The names are used in dialogues among the four mathematicians debating various questions.*

justifying the need and usefulness of thinking of *point*, *line*, and *plane* as objects with no thickness. The students are also asked to add their own justifications. Following this, the students are asked to suggest ways to communicate to Euclid these images. The goal is to bring them to experience the difficulty of this task so they appreciate the mathematicians' idea to describe the spatial relations among these objects rather than each object individually. After this discussion (in our pilot, this was an intense discussion among the teachers), the students are presented with the mathematicians' ideas. For example, to convey to Euclid the images (1) "Points and lines do not have any thickness", (2) "There are as many lines as pairs of points", (3) "Lines do not bend" and (4) "Planes are flat", the mathematicians chose the following statements, respectively: (1) "Through any point, infinitely many lines can be drawn", (2) "A line can be drawn between any two points", (3) "Only one line can be drawn between two points", and (4) "If a line shares two points with a plane, the line must be on the plane".

In sum, the central theme behind the above activities was that these statements convey an idealized (perfect) version of the points, lines, and planes we experience in our surroundings. Part of the summary of Lesson 1 concerning points and lines is:

> *First, the points and lines we draw on a sheet of paper, for example, have width, no matter how small we draw them. The points and lines we conveyed through these statements have no thickness. Second, while we can imagine drawing infinitely many lines through a point, no one can do so with a pencil on a sheet of paper, no matter how sharp the pencil is. Third, we can imagine extending a segment indefinitely in both directions, but in reality any line is of a limited length. Fourth, we can imagine segments of any length. When communicating with Euclid, we should always remember that, so far, these are the only images he has for our physical world; if we don't, he wouldn't understand what we say to him about objects made of points, lines, and segments.*

We eventually call statements such as statements 1–4 postulates. However, the goal is not to provide complete rigorous axiomatic foundations to geometry. Rather, the goal is to necessitate the idea that in geometry we deal with idealized physical objects, not the actual objects in our surroundings, and so geometric figures we draw on a sheet of paper are merely signs representing our memory images of spatial perceptions. As can be seen in the dialogues below, this theme is central in the development of the unit. The theme is absolutely essential to the development of geometric proofs for the crucial reason that the incorporation of geometric figures in proofs is possible only if one understands the axioms that underlie these figures. (More on this point on page 36.)

Necessitating Geometric Proofs and Constructions

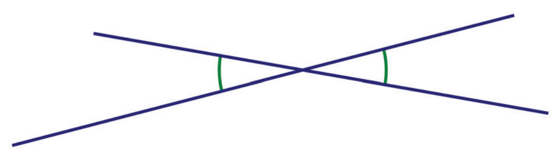
To necessitate the idea behind geometric proofs, the following question is raised: Can Euclid think logically as we do? The following activity, which appears in the unit as the 2nd Conversation with Euclid (in Lesson 2), illustrates the general approach taken to advance the concept of proof and abstract definitions of geometric objects.

2nd Conversation with Euclid

We: *Euclid, we are going to define for you a new concept, called vertical angles. We wonder if you observe any property of these angles.*

Euclid: *I am ready.*

We: *When two lines cross, four angles are formed. The pairs of angles that do not share a side are called vertical. For us, vertical angles look like this.*

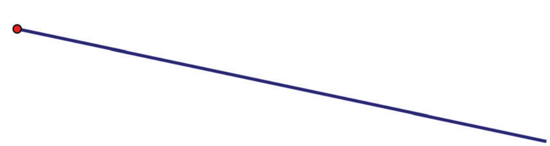


Euclid: *First, you know I am blind, so I have no idea how it looks to you. Second, I don't understand your definition. You have never told me what an angle is!*

We: *Oh, yes—you are right—we forgot. An angle is the figure formed by two rays which begin at the same point; this point is called the vertex of the angle.*

Euclid: *Again, you are using a geometry term I am not familiar with: What do you mean by ray?*

We: *Sorry, Euclid. We keep forgetting you don't know many of the terms we know. After this lesson we will define for you a collection of terms which will be in use for some time in our conversations. These are terms we learned in middle school. As to ray, take a segment and extend it in one direction. What you get is called a ray. And just for our record, to us a ray looks this:*



Euclid: *Now I know three new terms: ray, angle, and vertical angles. You are asking me what I observe about vertical angles—right?*

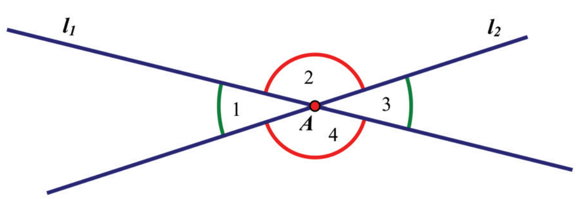
We: *Right.*

ACTIVITY: WHILE EUCLID IS THINKING…
Predict what Euclid might come up with. (You may experiment with the geometry software available to you.)

Euclid: *I say that any two vertical angles are equal.*
We: *What makes this true?*

ACTIVITY: PREDICT EUCLID'S PROOF

Euclid: *Imagine two lines intersecting; they create two pairs of vertical angles. Take any one of these angles and one of the angles that is adjacent to it. Now…*
We: W*ait a second, Euclid—we need to draw these lines, so we can follow you. We are humans, remember? Unlike you, we sometimes need to draw figures to assist us in our thinking.*
Euclid: *I can't see these figures.*
We: *Yes, we know; these figures are just for us. So for example, when you say "line" or "point," we imagine a line and a point, just like the way you do, and at the same time we draw them on a piece of paper for ourselves.*
Euclid: *Okay.*
We: *We drew two lines, l_1 and l_2 through a point A.*
Euclid: *Good—you can do that because of Postulate 2. Since one can draw unlimited number of lines through a point, one can definitely draw two lines through a point.*
[The following figure was described to Euclid and then he continued.]

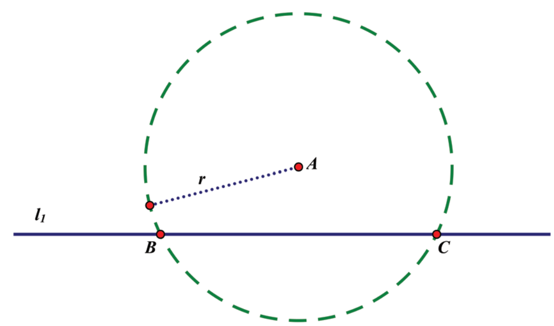


Euclid: $\sphericalangle 1+\sphericalangle 2=180^\circ$ *because* $\sphericalangle 1$ *and* $\sphericalangle 2$ *are adjacent.* $\sphericalangle 2+\sphericalangle 3=180^\circ$ *because* $\sphericalangle 2$ *and* $\sphericalangle 3$ *are adjacent. From here,* $\sphericalangle 1=180^\circ-\sphericalangle 2$ *and* $\sphericalangle 3=180^\circ-\sphericalangle 2$*. Hence* $\sphericalangle 1=\sphericalangle 3$.
We: *Bravo, Euclid, you proved what you claimed to be true.*

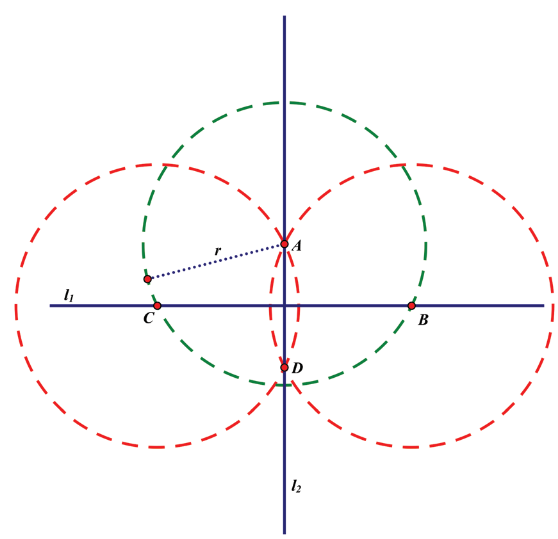
ACTIVITY: COMPARE EUCLID'S PROOF TO YOUR PROOF.

7th Conversation with Euclid
We: *Let's now go back to the questions you asked in Conversation 3* (not appearing in this presentation).
Euclid: *Oh, yes: You taught me how to imagine constructing a line: I take two points and imagine a line between them, as stated in Postulate 3. You also taught me how to imagine constructing a circle: I take a point in the plane and imagine all the points in the plane that are equidistant from that point. But how do I imagine constructing two perpendicular lines? You never taught me that. Nor did you ever teach me how to bisect an angle.*
We: *Okay. Let's start with how to construct two perpendicular lines:*
1. *Take any line. Call it l_1.*
2. *Take a point A not on the line.*
3. *Draw a circle whose center is at A and which intersects the line at two points, B and C. Call the radius of this circle r.*

Euclid, pardon us; we have to pause for a few minutes. In order to keep track of our construction, we draw for ourselves each of the steps in the construction. We realize that the only two constructions you can perform are (1) *creating a line through two given points and* (2) *creating a circle with given center and radius. We invented two mechanical instruments to draw these objects. One is called a straightedge, which we use to draw lines, and the other is called a compass, which is used to draw circles.*



Euclid: *Go ahead—I'll wait….*
We:
4. *Draw a circle with center B and radius r, and draw another circle with center C and radius r. These two circles intersect at a point; call it D.*



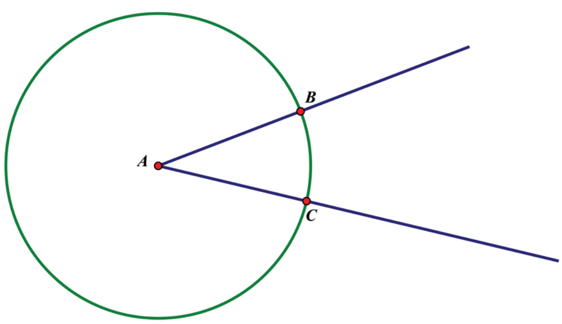
5. *Draw a line l_2 between A and D.*
We claim that l_1 is perpendicular to l_2 .

Euclid: *Why is that?*
We: *Take the triangles ABD* and *ACD. They are equal in three sides: AB=AC, DB=DC, and AD is common. Hence, by the SSS Theorem they are congruent.*
Euclid: *Wow—this is exciting!! Can I do the next one* (bisecting an angle)*?*
We: *Go ahead.*

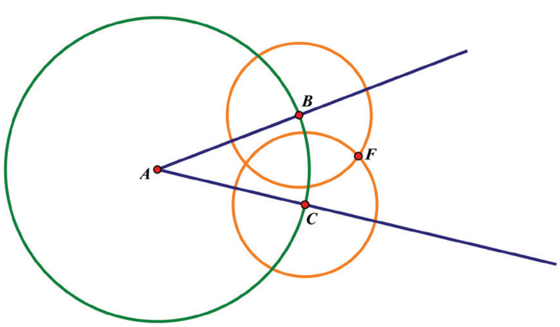
ACTIVITY: WHILE EUCLID IS THINKING ….
Predict what Euclid might come up with. (You may use geometry software available to you should you wish to do so.)

Euclid:
1. *Take any angle and mark its vertex by A.*
2. *Construct a circle with center at A.*
3. *The circle intersects the sides of the angle at two points, say, B and C.*

We: *Hold on… We need to draw this:*



Euclid: *Now, construct a circle with center at B, and another circle with the same radius at C, so that the two circles meet at a point. Call this point F. I say FA is on the angle bisector of angle A.*

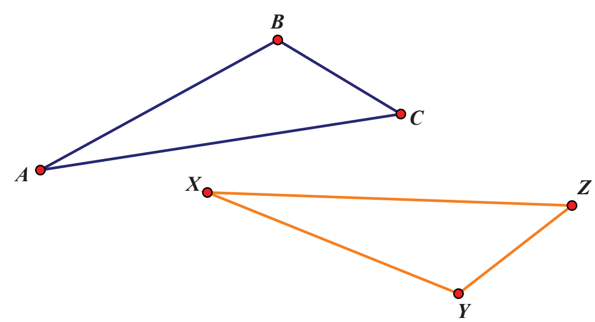


ACTIVITY: PREDICT EUCLID'S NEXT STEP

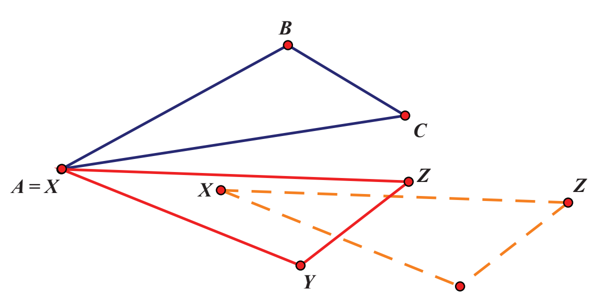
Now, construct the segments BF and CF. The triangles ABF and ACF are congruent by the SSS Theorem, since by construction, AB=AC, BF=CF, and AF is common to the two triangles. Hence $\sphericalangle BAF = \sphericalangle FAC$.
We: *Done!*

Necessitating Transformational Geometry
Our treatment of transformational geometry is completely different from that of the *standard approach*. While in the standard approach, plane transformations appear early in the curriculum, in the nonstandard approach they are defined at the completion of four chapters: *congruence*, *triangle inequality*, *parallelograms*, and *circles*, focusing extensively on proofs and Euclidean constructions. The following two conversations are five chapters apart: the first aims to raise the need for rigid motions, and the second is a dialogue demonstrating our approach to eliciting them.

5th Conversation with Euclid
We: *The congruence criterion we stated earlier* [in Conversation 3] *about congruent triangles is true. Let's restate it: If two sides and the angle enclosed by them in one triangle are equal respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.*
Euclid: *Why would this be true?*
We: *Here is our first attempt to convince you of this congruence criterion. Tell us if it makes sense to you. Please bear with us, as we have to draw pictures to help us keep track of our description.*
Euclid: *Go ahead, I am getting used to your habits of mind….*
We:
1. *Say we are given two triangles ABC and XYZ with AB = XY, AC = XZ, and $\sphericalangle A = \sphericalangle X$*

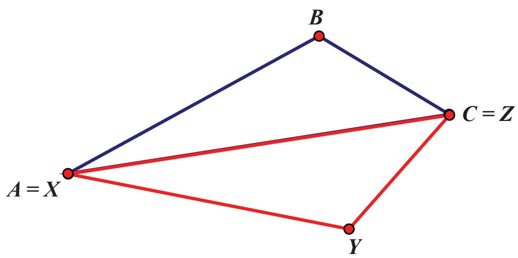


2. *Slide the vertex X onto the vertex A.*

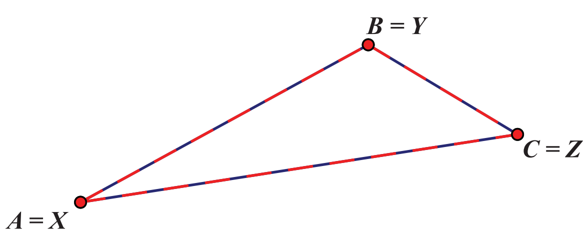


3. *Rotate triangle ABC as needed so that ray $\overrightarrow{AC}$ coincides with ray $\overrightarrow{XZ}$.*

4. *Since AC = XZ, the point Z coincides with the point C.*



5. *Reflect △XYZ with respect to its side XZ.*
6. *Since ∢A = ∢X, ray AB coincides with ray XY.*
7. *Since AB = XY, the point Y coincides with the point B.*

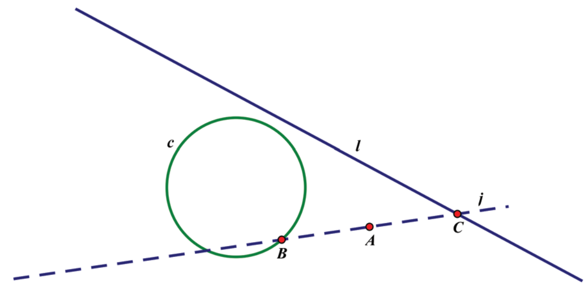


8. *Hence, △ABC ≅△XYZ.*

Does this make sense to you, Euclid?
Euclid: *Not at all. I understood that somehow you made the two triangles coincide, but I didn't understand how you did that. The terms slide, rotate, and reflect are foreign to me.*
We: *You are right. We are planning to define these terms to you sometime in the future and show you more precisely how they can be used to prove this congruence criterion. For your information, Euclid, we have two additional criteria for congruent triangles. They too will be proven in the future.*

Five chapters later, we begin to elicit plane transformations. This is done by bringing the students to experience the power and efficacy of reasoning in terms of plane transformations in solving geometry problems. We illustrate this approach with a segment from the lesson dealing with the *half-turn* motion, which follows the lesson on *translation*. The lesson opens with the following plane-geometry construction problem:

Problem: *A line l, circle c, and point A are given. Construct a line j through A so that the following two conditions hold:* (a) *j intersects both l and c (call these intersections C and B, respectively) and* (b) *AB = AC.*



Students work in small groups on this problem, and arrive (some with the teacher's help) at the following rather nontrivial solution.

Solution:

1. *Through O, the center of the circle c, construct a line n perpendicular to line l, and let F be the intersection point of l and n.*

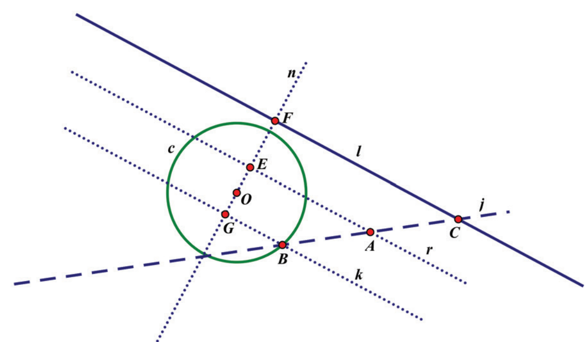
2. *Through the point A, construct a line r parallel to l, and let E be the intersection point of lines r and n.*

3. *Let G be a point on n so that EF = EG.*

4. *Through G, construct a line k parallel to l, and let B be the intersection of line k and circle c.*

5. *Let j be the line through B and A, and let C be the intersection of lines j and l.*

6. *Since BCFG is a trapezoid and AE is its midsegment (why?), line j is the desired line.*

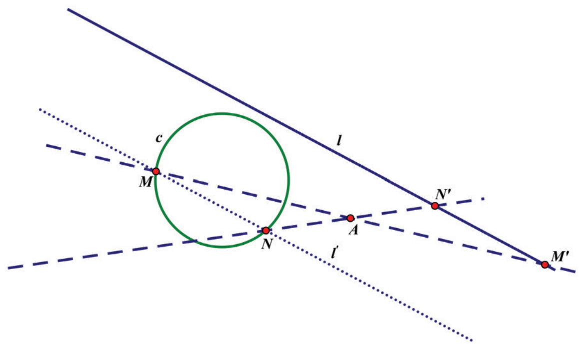


QED[2]

At this point, the teacher presents a different solution in the voice of an imaginary student. She tells the students that this same problem was solved in a similar way in Ms. Carlson's geometry class and when the discussion of the solution was ended, Alec, one of students in this class, said:

Alec: *Nice solution, Ms. Carlson. But I have a different solution. Earlier we learned translation and proved that the translation of a line is a line and of a circle is a circle. Can we also rotate figures?*

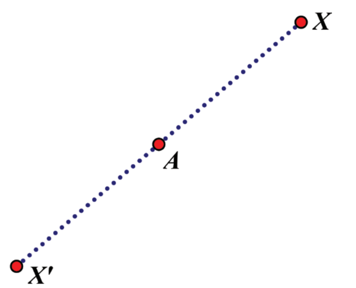
[2] *The solution is followed by a group project to determine the number of solutions to the problem.*



If yes, rotate line l about the point A 180°*; we get a line l´. If l´ intersects circle c in two points, say, M and N, then the lines MA and NA are the desired lines, simply because MA = AM´and NA = AN´.*

Teacher: *Wow! Let's clarify what it means to rotate a line 180° about a point.*

When we rotate a point X 180° about a point A, we get a point X´, where X, A, X´ are collinear and XA=X´A.



Rotating a line l 180° about a point A means rotating each of the points of the line 180° about a point A. We get a line l´.

Bravo, Alec.

ACTIVITY: PREDICT EUCLID'S QUESTIONS ABOUT ALEC'S SOLUTION AND ANSWER THEM

The kinds of questions students are expected to raise in the name of Euclid are similar to those Euclid raised in the previous lesson on *translation* (see below). As with the half-turn motion, the translation motion too emerged from a solution to a geometry construction problem involving a *translation* of a circle. After the definition of translation was stated to Euclid, the following conversation ensued.

nth Conversation with Euclid[3]

Euclid: *From what you have said so far, I conclude that you are claiming that the translation of a circle along a given vector is a circle congruent to the original circle—why?*

We: *Mmm… good question, Euclid. It is so obvious to us, but we understand why you need a proof for this fact. Give us some time to think how to answer your question.*

ACTIVITY: HOW SHOULD WE RESPOND TO EUCLID?

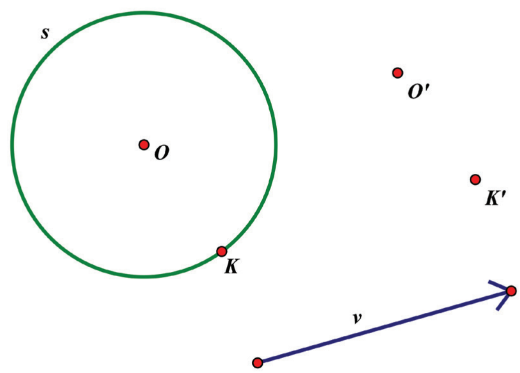
Euclid: *While you were thinking how to answer my question, I came up with an answer of my own.*

1. *Let s be a circle with center O and radius r, and let v be any vector.*
2. *Translate s and O a distance |v| along the vector v, to get s´and O´, respectively. I will show that the distance of any point on s´ from O´ is r.*
3. *Let K´ be any point on s´. K´ is the translation of a point K on s.*

By now you perhaps lost me. Go ahead and do your drawings, so you can follow what I say.

We: *Thank you, Euclid, for being considerate.*

We: *We are ready.*



Euclid:

4. *Consider the quadrilateral OO´K´K. By the definition of translation, we have OO´ ‖KK´‖v and OO´=KK´=|v|, and therefore, OO´K´K is a parallelogram.*
5. *Hence O´K´=OK=r.*

QED

ACTIVITY: IS EUCLID'S PROOF COMPLETE?

Finally, at the end of the chapter on plane transformations, we return to the 5th Conversation with Euclid to close what was left open by proving the congruence criteria using rigid motions.

Characteristics of the Nonstandard Approach

The nonstandard approach is intended to help students realize the need to state basic assertions about our idealized physical reality (axioms) as well as basic theorems, which they then use to solve geometry problems. These problems are designed to lead the students to derive more facts and theorems in a way that is logically coherent. The students are naturally led to make their arguments and definitions increasingly precise as time goes on so that they can effectively communicate with Euclid. The sample of lesson segments we have just presented raises several questions concerning the *Conversations with Euclid* unit and its implementation.

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[3] *The exact number of this conversation is yet to be determined.*

What Can Be Assumed about Euclid's Knowledge? Speaking to Euclid, an "alien" who does not share our taken-as-shared meanings may seem confusing for students, since it is not always clear what can be assumed about his knowledge and what might be considered a fair assumption. For many years I have been using the "game" of conversing with an "alien" as a pedagogical technique to advance students' ability to formulate and formalize mathematical ideas in geometry. In each case, I find the experience pleasantly surprising. Not only do students enthusiastically engage in the "game" but also very quickly—in a lesson or two—develop a clear sense about "the rules of the game", that this is a "mathematical game", not a "language arts game". For example, students seldom ask whether Euclid knows what is meant by terms such as "true", "false", "understand", "sorry", or any of the kinds of natural language phrases appearing in the above conversations with Euclid. Students do, however, often raise questions about Euclid's knowledge of certain geometric facts known to them from previous classes (e.g., "The sum of the angles in a triangle being 180°"). What is particularly pleasing is that students often raise questions of a subtle nature, for example: Does Euclid know what is meant by "different sides of a line"? "a point between two points"? "direction"? "algebra"? etc.

What Is the Underlying Structure of the Conversations with Euclid Unit? The issue of structure is particularly critical in the case of geometry. It is perhaps the only place in high school mathematics where a (relatively) complete and rigorous mathematical structure can be taught. Deductive geometry can be treated in numerous ways and at different levels of rigor. The *Conversations with Euclid* unit uses Euclid's *Elements* as a framework. In a program consistent with this framework, subtle concepts and axioms, such as those related to "betweenness" and "separation", are dealt with intuitively, but the progression from definitions and axioms to theorems and from one theorem to the next is coherent, logical, and exhibits a clear mathematical structure. Furthermore, the unit sequences its lessons so that neutral geometry—a geometry without the Parallel Postulate—precedes Euclidean geometry—a geometry with the Parallel Postulate.

Another central theme concerning structure is that the *Conversations with Euclid* unit puts more emphasis on the form of reasoning than on the content to which the reasoning is applied. This is done by continually drawing students' attention to the cause of facts—what makes a fact to be the way it is—rather than to the fact itself. By this, we try to create and establish the norm that a claim such as "In an isosceles triangle the angles opposite the equal sides are congruent" is less interesting than understanding how the fact that a triangle is isosceles causes the angles opposite its equal sides to be congruent. In other words, the focus is more on *form* than on *content*.

How Does the Conversations with Euclid Unit Deal with the Role of Figures in Geometry? The conversations with a fictitious "alien" who is devoid of visual, kinesthetic, and haptic perceptions has proven to be enormously effective in necessitating the role of figures in geometry proofs, namely, that the geometric figures we draw on a sheet of paper and incorporate in our proofs are mere signs of ideal spatial perceptions which can only be communicated with Euclid by means of agreements (axioms).

Beyond the realization of the need to define objects accurately and objectively and justify assertions logically, we want students to use figures to generate conjectures. This is critical because we do not want Euclid's constraints to limit the students' intuition and observations of their physical surroundings. The separation between Euclid's images and students' own images helps to combat one of the most prevalent difficulties students have with geometry proofs: namely, student's reliance on actual figures in justifying mathematical assertion [8], [10], e.g., a geometric relation is true because it looks so.

What Is the Nature of the Intellectual Need for the Conversations with Euclid? The intellectual need underlying the *Conversations with Euclid* unit is primarily, but not limited to, the *need for communication*,[4] which comprises two reflexive acts: *formulating* and *formalizing*. *Formulating* is the act of transforming strings of spoken language into mathematical language. *Formalizing* is the act of externalizing the exact intended meaning of an idea or a concept or the logical basis underlying an argument. The two acts are reflexive in that as one formalizes a mathematical idea, it is often necessary to formulate it, and, conversely, as one formulates an idea, one often encounters a need to formalize it. The preconceptualization that orients us humans to these acts when we learn mathematics is the act of conveying, exchanging, and defending ideas by means of a spoken language and gestures, which are defining features of humans.

What Is the Context of the Conversations with Euclid? Typically, the conversations in the unit are presented on an overhead projector by the teacher or handed out on paper to the students. However, their distribution is carried out strategically to avoid introduction of material prematurely, before the students have fully realized the need for an idea. In the first few lessons when solving

[4]*The* need for communication *is one of five intellectual needs (*need for certainty, need for causality, need for communication, need for computation, and need for structure*),* which characterize mathematical practice (*see* [9]).

geometry problems in class, the teacher plays the role of Euclid, but gradually the students take on his persona.

Final Observation

The idea of *Conversations with Euclid* was created years ago when I realized that my students in college geometry courses had serious difficulties dealing with geometric properties outside their own imaginative space. For example, they invoked imageries of "betweenness", "infinite length", "order", etc., in finite geometries devoid of these properties. The *Conversations with Euclid* technique was instrumental in helping students separate their own imageries from those implied by the geometry at hand.

Teachers who participated in the pilot experiment with the *Conversations with Euclid* unit found it effective. Here are several of their responses.

Teacher A: "The point of geometry (and all other math) is to improve our understanding and deductive reasoning. From the lessons we received, this goal is obtainable. Because we started with nothing and were able to prove so much, it allowed me to use logic and reasoning to deductively prove many geometrical theorems I never previously understood."

Teacher B: "I always knew that the way geometry is traditionally taught didn't give students the opportunity to reason deductively. We typically move through endless theorems and expect students to apply them algebraically. However, the "why" is often left out. Students are terrible at proofs because we never teach ways of thinking and reasoning. I now have a better idea of how to teach deductive reasoning."

Other teachers have been using the *Conversations with Euclid* unit in their own classes, and have reported success.

Teacher C reported that her students were so intrigued by the *Conversations with Euclid* game that they composed a rap song about his perceived personality.

Teacher D reported the results of a semiformal study she conducted to compare students' achievement on three types of curricula in her school: the *Conversations with Euclid* unit (137 students), a common district geometry unit (180 students), and an accelerated geometry unit (91 students). All the students took the same external district geometry end-of-course (EOC) exam at the end of fall of 2012. The EOC tested thirteen standards. Students with the *Conversations with Euclid* unit performed significantly better than those with the common geometry unit and about the same or better than those in the accelerated geometry curriculum.

When Teacher D had to resort to the common geometry unit in the spring of 2013 due to the institution's curricular constraints, her students protested and expressed their wish to return to their conversations with Euclid.

References

[1] Achieve, *Comparing the Common Core State Standards in Mathematics and the NAEP Framework*, 2012. Retrieved from http://www.achieve.org/files/CCSSMathandNAEP.pdf.

[2] W. F. Burger, and J. M. Shaughnessy, Characterizing the van Hiele levels of development in geometry, *Journal for Research in Mathematics Education* **17** (1986), no. 1, 31–48.

[3] CCSS, *Common Core State Standards for Mathematics*, 2012. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

[4] L. E. Deci and M. R. Ryan, *Handbook of Self-Determination Research*, The University of Rochester Press, Rochester, NY, 2004.

[5] P. Gomez and C. Carulla, Students' conceptions of cubic functions. In M. van den Heuvel-Panhuizen (ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 57–64). Utrecht, The Netherlands: Freudenthal Institute, 2001.

[6] G. Harel, Students' understanding of proofs: A historical analysis and implications for the teaching of geometry and linear algebra, *Linear Algebra and Its Applications* 302–303 (1999), 601–613.

[7] ________, A DNR perspective on mathematics curriculum and instruction, Part II. *ZDM—The International Journal on Mathematics Education* **40** (2008), 893–907.

[8] G. Harel and L. Sowder, Toward a comprehensive perspective on proof, in F. Lester (ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, National Council of Teachers of Mathematics, 2007.

[9] G. Harel, Intellectual need, in *Vital Direction for Mathematics Education Research*, K. Leatham (ed.), Springer, 2013.

[10] D. Reid and C. Knipping, *Proof in Mathematics Education: Research, Learning and Teaching*, Sense Publishing, Boston, 2010.

[11] Hans Niels Jahnke, *The Conjoint Origin of Proof and Theoretical Physics* - 246, pp. 161, 382, 383 (2011).

[12] A. Sierpinska, On understanding the notion of function, in G. Harel and E. Dubinsky (eds.), *The Concept of Function, Aspects of Epistemology and Pedagogy* (Vol. 25, pp. 25–58). USA: Mathematical Association of America, 1992.

[13] P. W. Thompson, Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula, in E. Silver (ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Erlbaum, Hillsdale, NJ, 1985, pp. 189–243.

[14] P. M. Van Hiele, Levels of thinking, how to meet them, how to avoid them, *Pre-session Meeting of the Special Interest Group for Research in Mathematics Education,* National Council of Teachers of Mathematics, Seattle, WA, 1980.

[15] H. Wu, *Teaching Geometry According to the Common Core Standards,* 2012. Retrieved from http://math.berkeley.edu/~wu/Progressions_Geometry.pdf.

Infrastructure: Structure Inside the Class Group of a Real Quadratic Field

M. J. Jacobson Jr. and R. Scheidler

Cows and Fields

Suppose that you are a wise ancient Greek, and that you have been given the task of counting the Sun God's cattle on the island of Sicily. They are too numerous to count manually, and your only clue is a puzzle that relates the number of cows and bulls of a particular color to those of another. How can you determine the size of your herd from such cryptic information?

A precise version of this numerical puzzle is the well-known cattle problem of Archimedes [2]. A number of accounts of its solution appear in the literature [31, 21, 19]. Unfortunately for the hypothetical Greek sage, this problem does not have a small and completely elementary solution—it would take more than 2,000 years before the first solution was discovered by Amthor [1]. Had a solution been available, the Greek would likely have been shocked to find that the Sun God had somehow managed to pack a herd of more than 10^{206544} animals onto Sicily!

One of the main ingredients in Amthor's solution was solving the Diophantine equation

$$x^2 - 4729414y^2 = 1$$

for integers x and y. This equation is a specific instance of the (incorrectly named) Pell equation

$$x^2 - Dy^2 = 1 \ , \tag{0.1}$$

where D is assumed to be a positive nonsquare integer. Lenstra [21] and Williams [31] have written excellent expository articles on the history and modern methods for solving this equation; see also Jacobson and Williams [19] for a more exhaustive treatment. The infrastructure plays an important role in this story, so we begin our discussion here.

One of the simplest methods for solving an instance of the Pell equation is based on continued fraction expansions of quadratic irrationals. Following [19, Ch. 3, p. 57], define $\delta = (q + \sqrt{D})/r$, where

$$r = \begin{cases} 2 & \text{if } D \equiv 0 \text{ or } 1 \pmod 4\,, \\ 1 & \text{otherwise} \end{cases}$$

and

$$q = \begin{cases} 1 & \text{if } D \equiv 1 \pmod 4\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Put $P_0 = q$, $Q_0 = r$, $q_0 = \lfloor \delta \rfloor$, $G_{-1} = r$, $G_0 = rq_0 - q$, $B_{-1} = 0$, $B_0 = 1$, and apply the recurrences

$$\begin{aligned} P_{i+1} &= q_i Q_i - P_i\,, \\ Q_{i+1} &= \frac{D - P_{i+1}^2}{Q_i}\,, \\ q_{i+1} &= \left\lfloor \frac{P_{i+1} + \sqrt{D}}{Q_{i+1}} \right\rfloor, \\ G_{i+1} &= q_{i+1} G_i + G_{i-1}\,, \\ B_{i+1} &= q_{i+1} B_i + B_{i-1}\,, \end{aligned}$$

for $i = 0, 1, 2, \ldots$ until we find the least positive index p for which $Q_p = r$.

For example, consider $D = 193$. Applying the formulas from above, the continued fraction expansion of $(q + \sqrt{D})/r = (1 + \sqrt{193})/2$ is obtained as follows:

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DOI: http://dx.doi.org/10.1090/noti1064

n	P_n	Q_n	q_n	G_n	B_n
-1				2	0
0	1	2	7	13	1
1	13	12	2	28	2
2	11	6	4	125	9
3	13	4	6	778	56
4	11	18	1	903	65
5	7	8	2	2584	186
6	9	14	1	3487	251
7	5	12	1	6071	437
8	7	12	1	9558	688
9	5	14	1	15629	1125
10	9	8	2	40816	2938
11	7	18	1	56445	4063
12	11	4	6	379486	27316
13	13	6	4	1574389	113327
14	11	12	2	3528264	253970
15	13	2			

We see that $Q_{15} = 2$. Set $p = 15$, and notice that r divides both G_{p-1} and B_{p-1}. We find that

$$(G_{p-1}/r)^2 - D(B_{p-1}/r)^2 = 1764132^2 - 193 \cdot 126985^2 = -1.$$

Hence, $(1764132, 126985)$ is a solution of the negative Pell equation $x^2 - Dy^2 = -1$ for $D = 193$. Continuing the continued fraction expansion of $(1 + \sqrt{193})/2$ yields a repetition of the cycle of (P_n, Q_n) pairs. Stopping at the second occurrence of $Q_n = 2$ yields

n	P_n	Q_n	q_n	G_n	B_n
29	11	12	2	12448646853698	896073208080
30	13	2			

Setting $p = 30$, we find again that r divides both G_{p-1} and B_{p-1}. This time, we obtain the identity

$$(G_{p-1}/r)^2 - D(B_{p-1}/r)^2 = 6224323426849^2 - 193 \cdot 448036604040^2 = 1.$$

Thus, $(t, u) = (6224323426849, 448036604040)$ is a solution of $x^2 - 193y^2 = 1$. It can be proved that the first solution obtained in this manner is the fundamental (i.e., minimal) solution of (0.1) [19, Ch. 3], and all solutions (T, U) can be obtained by computing $T + U\sqrt{D} = (t + u\sqrt{D})^n$ for integers $n \geq 1$ [19, Corollary 1.10].

In addition to the agricultural connection to fields through Archimedes' cattle problem, further examination of the continued fraction method for solving Pell equations reveals another connection to fields, specifically real quadratic number fields. Consider the quantity

$$\epsilon_D = \frac{G_{p-1} + B_{p-1}\sqrt{D}}{r} = \frac{3528264 + 253970\sqrt{193}}{2}$$

obtained after the first period in the continued fraction expansion from the example. It belongs to the real quadratic field $\mathbb{Q}(\sqrt{D}) = \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$. The *norm* of an element $\alpha = a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ is the rational number $\mathcal{N}(\alpha) = \alpha\overline{\alpha} = a^2 - b^2D$, where $\overline{\alpha} = a - b\sqrt{D}$ denotes the *conjugate* of α. For the quantity ϵ_D from our example, we obtain

$$\mathcal{N}(\epsilon_D) = \frac{G_{p-1} + B_{p-1}\sqrt{D}}{2}\,\frac{G_{p-1} - B_{p-1}\sqrt{D}}{2} = -1 \ .$$

Because $\mathcal{N}(\epsilon_D)$ is an integer, ε_D is an algebraic integer in $\mathbb{Q}(\sqrt{D})$, and because it divides 1, ε_D is a nontrivial unit in the subring $\mathcal{O}_D = \{a + b\delta \mid a, b \in \mathbb{Z}\}$ of $\mathbb{Q}(\sqrt{D})$. In fact, ε_D is the smallest unit exceeding 1 in $\mathcal{O}_D$ and is called the *fundamental unit* of $\mathcal{O}_D$, as all units in this ring can be expressed[1] as $\pm\varepsilon_D^k$ for some $k \in \mathbb{Z}$.

The fundamental solution to the Pell equation is the smallest power of ε_D with integer coefficients and norm equal to 1. In the example, one can verify that

$$\varepsilon_{193}^2 = 6224323426849 + 448036604040\sqrt{193}$$

and, as shown above,

$$\mathcal{N}(\varepsilon_{193}^2) = 6224323426849^2 - 193 \cdot 448036604040^2 = 1,$$

so $(6224323426849, 448036604040)$ is the fundamental solution of the Pell equation for $D = 193$. In general, ε_D^k for some $k \in \{1, 2, 3, 6\}$ yields the fundamental solution of the Pell equation. Thus computing the fundamental unit of $\mathcal{O}_D$ also yields all solutions to (0.1).

There are further connections to $\mathbb{Q}(\sqrt{D})$. Consider the quantities $\theta_j = (G_{j-2} + B_{j-2}\sqrt{D})/2$ for $j = 1, 2, \ldots, p-1$. Like ε_D, they all belong to $\mathcal{O}_D$. The set of principal $\mathcal{O}_D$-ideals $(\theta_j) = \theta_j\mathcal{O}_D$ forms the *infrastructure* of $\mathbb{Q}(\sqrt{D})$. Its remarkable group-like properties led to faster algorithms for solving the Pell equation, and other applications. These properties, and additional applications of the infrastructure, are the focus of this paper.

We begin our discussion of the infrastructure with a brief study of ideals in the ring of integers of a real quadratic field. The infrastructure is formally introduced, and its group-like properties explained, in the section "What Is the Infrastructure?". Further uses (and nonuses) of the infrastructure are discussed in the section "What Else Can (and Can't!) You Do with Infrastructure?", and practical implementation issues arising in these applications are the subject of the section "Practical Issues". We offer a glimpse beyond infrastructure in real quadratic fields in the section "Beyond Infrastructure in Real Quadratic Fields" and into relevant open problems in the section "Open Problems", concluding with suggested sources for further reading in the section "Further Reading".

[1] *This is a special case of Dirichlet's unit theorem for an arbitrary algebraic number field.*

What Is the Infrastructure?

The infrastructure is a collection of certain $\mathcal{O}_D$-ideals of a real quadratic field $\mathbb{Q}(\sqrt{D})$. For simplicity, assume henceforth that D is squarefree, and $q = r - 1$ (with q and r as defined in the section "Cows and Fields"). Then $\mathcal{O}_D$ is the *maximal order* or *ring of integers* of $\mathbb{Q}(\sqrt{D})$, i.e., the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{D})$. As seen earlier, $\mathcal{O}_D$ is precisely the two-dimensional $\mathbb{Z}$-module in $\mathbb{Q}(\sqrt{D})$ with basis $\{1, (r-1+\sqrt{D})/r\}$. The nonzero ideals of $\mathcal{O}_D$ are exactly the two-dimensional $\mathbb{Z}$-submodules of $\mathcal{O}_D$ with bases of the form $\{SQ/r, S(P+\sqrt{D})/r\}$, where $S, Q, P \in \mathbb{Z}$, r divides Q, and rQ divides $D - P^2$. An $\mathcal{O}_D$-ideal $\mathfrak{a}$ is *primitive* if it cannot be written as an integer multiple of another $\mathcal{O}_D$-ideal; we can take $S = 1$ in this case, and write $\mathfrak{a} = [Q, P]$. The conjugate of an ideal $\mathfrak{a}$ is the ideal $\overline{\mathfrak{a}}$ containing the conjugates of the elements in $\mathfrak{a}$. If $\mathfrak{a} = [Q, P]$, then $\overline{\mathfrak{a}} = [Q, -P]$.

The product of two $\mathcal{O}_D$-ideals $\mathfrak{a}, \mathfrak{b}$ is defined to be the collection of all finite sums of products $\alpha\beta$ with $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$. This is easily seen to be an $\mathcal{O}_D$-ideal, so the set of all $\mathcal{O}_D$-ideals forms an infinite monoid under multiplication with identity $\mathcal{O}_D$. Efficient formulas for computing a $\mathbb{Z}$-basis of the product of two ideals given in $\mathbb{Z}$-basis representation exist, based on the composition formulas of Gauß for binary quadratic forms [19, Sec. 5.4]. A *principal* ideal consists of all the $\mathcal{O}_D$-multiples of some fixed element $\theta \in \mathcal{O}_D$, called a *generator* of the ideal, and is denoted by (θ). The principal ideals form an infinite submonoid of the ideals.

The *(ideal) class group* Cl_D, discovered by Gauß in the context of binary quadratic forms [14], is an algebraic object of great interest in its own right. It is defined as the group of nonzero ideal equivalence classes under multiplication, where $\mathfrak{a}$ and $\mathfrak{b}$ are *equivalent* if there exist nonzero $\alpha, \beta \in \mathcal{O}_D$ such that $(\alpha)\mathfrak{a} = (\beta)\mathfrak{b}$. The class group Cl_D is a finite abelian group whose order is called the *(ideal) class number* and is denoted by h_D. One way to prove that Cl_D is finite is through the definition of reduced ideals. A primitive ideal $\mathfrak{a} = [Q, P]$ is said to be *reduced* if Q is a minimum in $\mathfrak{a}$, i.e., if there exists no nonzero $\alpha \in \mathfrak{a}$ such that $|\alpha| < Q$ and $|\overline{\alpha}| < Q$. Given any primitive ideal $\mathfrak{a} = [Q, P]$ of $\mathcal{O}_D$, an equivalent reduced ideal can be found by expanding the continued fraction of $(P + \sqrt{D})/Q$ [19, Ch. 5]. Furthermore, it can be shown that if $\mathfrak{a}$ is reduced, then the coefficients Q and P are bounded (roughly) by $\sqrt{D}$ [19, Sec. 5.1], implying that there are only finitely many reduced ideals in $\mathbb{Q}(\sqrt{D})$. Combining this with the fact that every ideal is equivalent to a reduced ideal yields the finiteness of the ideal class group.

Ideal reduction also leads to a method for computing in the class group. The idea is to use a reduced ideal to represent its entire equivalence class. Multiplying elements in the class group then consists of multiplying a reduced representative from each class and reducing the product. This operation is efficient, with bit complexity polynomial in $\log D$ [19, Sec. 5.4].

The main problem with this method is testing whether two ideal classes are equal. Although the number of reduced representatives of an equivalence class is finite, it could still be too large for an exhaustive comparison to be feasible. In particular, as we will see shortly, the *regulator* $R_D = \log \varepsilon_D$ provides an estimate of the number of reduced ideals in an equivalence class. It is known that $h_D R_D \approx \sqrt{D}$ [19, Eqn. 7.1]. So if h_D is small, one would have to exhaustively test $O(\sqrt{D})$ reduced equivalent ideals.

In order to speed up this process, it would be helpful if elements of the class group had their own internal structure. This is precisely what Shanks discovered in 1972 and termed *infrastructure* [26]. Although each ideal equivalence class has its own infrastructure, we confine our discussion to the class of principal ideals—the identity class of Cl_D—and refer to "the" infrastructure of $\mathbb{Q}(\sqrt{D})$ as the set of all reduced principal $\mathcal{O}_D$-ideals.

The ideal $\mathcal{O}_D = [r, r-1]$ is principal and reduced. Computing the continued fraction expansion of $(P_0 + \sqrt{D})/Q_0$ with $Q_0 = r$ and $P_0 = r - 1$, using the formulas from the previous section, yields a sequence of ideals $\mathfrak{a}_j = [Q_{j-1}, P_{j-1}]$ for $j = 1, 2, \ldots$. One can show that all these ideals are reduced and equivalent, satisfying

$$\mathfrak{a}_{j+1} = (\psi_j)\mathfrak{a}_j \tag{0.2}$$

where $\psi_j = (P_j + \sqrt{D})/Q_{j-1}$ [19, Sec. 5.3].

Notice the similarity of this process to that of the previous section for solving the Pell equation using continued fractions. In fact, the sequence of reduced principal ideals obtained in this fashion consists precisely of the ideals (θ_j) with $\theta_j = (G_{j-2} + B_{j-2}\sqrt{D})/r$; i.e., $\mathfrak{a}_j = (\theta_j)$. Thus

$$\theta_j = \prod_{k=1}^{j-1} \psi_k$$

which can be derived from (0.2).

One way to describe the structure of the infrastructure is to draw an analogy to a more familiar algebraic object, namely, a finite multiplicative cyclic group $G = \langle g \rangle$ of order n. To generate all the elements in this group, one begins with the identity element $1 = g^0$ and successively multiplies by g until finding $g^n = 1$. The elements of G can be visualized on a circle of circumference n, as depicted in Figure 1. Notice that the elements of G are regularly spaced around the circle. The *distance* of an element g^i is defined to be i and measures how far around the circle g^i is from 1

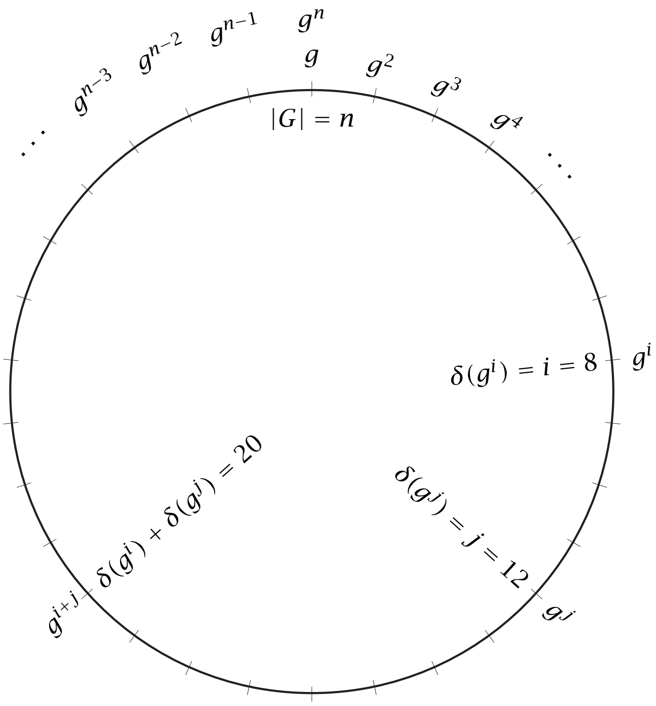

Figure 1. Cyclic group G of order n generated by g, depicted on a circle. Note that consecutive elements (powers of g) are distance 1 apart on the circle. The circumference of the circle is equal to n, the order of G.

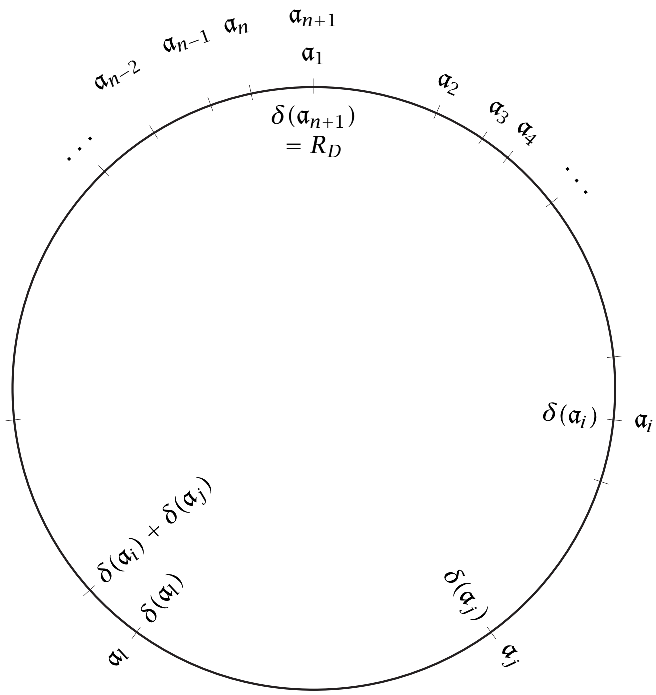

Figure 2. The infrastructure of the principal class of $\mathcal{O}_D$, depicted on a circle. Note that the distance between consecutive elements varies, but is generally close to 1. The circumference of the circle is equal to the regulator R_D.

(in clockwise direction). Using this definition, the distance between two consecutive group elements on the circle is 1, and the distance around the entire circle is n, the order of G.

The infrastructure, as we have seen, also describes a finite cyclic structure. To generate all the elements in the infrastructure, one starts with the ideal $\mathfrak{a}_1 = \mathcal{O}_D = [Q_0, P_0]$ (recall that $Q_0 = r$ and $P_0 = r - 1$) and computes the ideals $\mathfrak{a}_i = [Q_i, P_i]$, $i = 2, 3, \ldots$, by computing the continued fraction expansion of $(P_0 + \sqrt{D})/Q_0$, as described above, until arriving at $\mathcal{O}_D$ again. Thus one step in this continued fraction expansion plays a similar role to multiplication by g in G. The ideals $\mathfrak{a}_i$ can also be visualized on a circle, as depicted in Figure 2 (see [19, Fig. 7.1]; figure used with kind permission of Springer Science+Business Media). Here the *distance* of $\mathfrak{a}_i = (\theta_i)$ is defined to be $\delta(\mathfrak{a}_i) = \log \theta_i$. This notion of distance measures how far $\mathfrak{a}_i$ is around the circle from $\mathcal{O}_D$. Using this definition, the distance between two consecutive ideals $\mathfrak{a}_{i+1}$ and $\mathfrak{a}_i$ is $\delta(\mathfrak{a}_{i+1}, \mathfrak{a}_i) = \log(\theta_{i+1}/\theta_i) = \log \psi_i$. Although the distance between consecutive infrastructure ideals is therefore not constant, the Khintchine-Lévy Theorem implies that $\log \psi_i$ is on average about 1.18 [19, Theorem 3.17]. In other words, even though the ideals are not evenly spaced around the circle—unlike the cyclic group scenario—the distance between any two neighbors is, nevertheless, generally close to one. The distance around the entire circle, analogous to the order of G, is $R = \log \varepsilon_D$, the regulator of $\mathbb{Q}(\sqrt{D})$.

As an example, the infrastructure of $\mathbb{Q}(\sqrt{193})$ is listed in Table 1 (reproduced from [19, Table 7.1], table used with kind permission of Springer Science+Business Media). The coefficients of the ideals are taken from the continued fraction expansion of $(1 + \sqrt{193})/2$ from the previous section. The distances are computed using the corresponding G_j and B_j values. Notice that $\mathfrak{a}_{16} = \mathfrak{a}_1$, so $\delta(\mathfrak{a}_{16}) \approx 15.07631652$ is the regulator of $\mathbb{Q}(\sqrt{193})$. Notice also that $\delta(\mathfrak{a}_j) \approx j$ in all cases, as one would expect from the Khintchine-Lévy Theorem.

The analogy comparing the infrastructure to a cyclic group shows that the idea of computing the regulator[2] of $\mathcal{O}_D$ using the continued fraction expansion of $\sqrt{D}$ is similar in spirit to computing the order of a cyclic group by enumerating the entire group. The complexity of the latter is exactly n operations in G, and, similarly, the Khintchine-Lévy law implies that the complexity of the former is $O(R)$, which is $O(D^{1/2+\epsilon})$ in

[2]*Here, "computing" the regulator R_D means computing a sufficiently accurate floating point approximation.*

Table 1. Principal Cycle for $D = 193$

j	$\mathfrak{a}_j = [Q_{j-1}, P_{j-1}]$	$\delta(\mathfrak{a}_j)$
1	$[2,1]$	0
2	$[12,13]$	2.59869817
3	$[6,11]$	3.32835583
4	$[4,13]$	4.82844171
5	$[18,11]$	6.65671165
6	$[8,7]$	6.80572746
7	$[14,9]$	7.85709282
8	$[12,5]$	8.15679754
9	$[12,7]$	8.71127845
10	$[14,5]$	9.16513386
11	$[8,9]$	9.65688343
12	$[18,7]$	10.61682945
13	$[4,11]$	10.94102199
14	$[6,13]$	12.84657298
15	$[12,11]$	14.26937782
16	$[2,13]$	15.07631652

the worst case (when $h_D = 1$). However, Shanks's discovery of the infrastructure revealed even more structural similarities to G, which led to significant improvements in regulator computation.

In the cyclic group setting, one well-known improvement to computing the group order is to apply a time-memory tradeoff and use the *baby-step giant-step strategy*. The idea is that, in addition to moving through the group using successive multiplications by g (baby steps), one can take larger steps through the group by multiplying by g^i for some positive integer i (giant steps). As depicted in Figure 1, the distance between two successive baby steps is only 1, but each giant step advances by a distance of i, and in particular the distance of $g^j g^i$ is precisely $j + i$, the sum of the distances of the inputs. To compute the order n of G, one begins by computing a list of baby steps $\mathcal{L} = \{1, g, g^2, \ldots, g^l\}$ where $l \approx \sqrt{H}$ for a suitable bound $H \geq n$. Then the giant steps $g^{2l}, g^{3l}, \ldots$ are computed until an element is found in the list $\mathcal{L}$ of baby step elements, say $g^{il} = g^j$. This yields $g^{il} g^{-j} = 1$, and we have $n = il - j$ if H is a reasonably tight bound on n. The complexity of this algorithm is $O(\sqrt{H}) = O(\sqrt{n})$ operations in G. Modifications exist that achieve complexity $O(\sqrt{n})$ even without knowing a bound on n [5, 29].

Shanks's breakthrough was the discovery that there exists a similar notion of a giant step in the infrastructure. Consider the effect of multiplying an infrastructure ideal $\mathfrak{a}_j$ by $\mathfrak{a}_i$. The resulting infrastructure ideal $\mathfrak{a}$ is obtained by multiplying $\mathfrak{a}_j$ and $\mathfrak{a}_i$ and reducing their product, yielding $(\gamma)\mathfrak{a} = \mathfrak{a}_j\mathfrak{a}_i$ for some $\gamma \in \mathbb{Q}(\sqrt{D})$. The principal ideal factor (γ) arises from the fact that, in general, the reduced ideal $\mathfrak{a}$ is not equal to $\mathfrak{a}_j\mathfrak{a}_i$ but merely equivalent to it.

Denote by $\mathfrak{a}_j * \mathfrak{a}_i$ the first reduced ideal $\mathfrak{a} = (\gamma^{-1})\mathfrak{a}_j\mathfrak{a}_i$ obtained by applying reduction to the product ideal $\mathfrak{a}_j\mathfrak{a}_i$. As $\mathfrak{a}_j$ and $\mathfrak{a}_i$ are both principal, the ideal $\mathfrak{a}_j * \mathfrak{a}_i$ is also principal, and as it is reduced, it lies somewhere on the infrastructure. The question is, what is its distance? As $\mathfrak{a}_j = (\theta_j)$ and $\mathfrak{a}_i = (\theta_i)$, we obtain

$$\begin{aligned}\delta(\mathfrak{a}_j * \mathfrak{a}_i) &= \log(\theta_j\theta_i/\gamma) \\ &= \log\theta_j + \log\theta_i - \log\gamma \\ &= \delta(\mathfrak{a}_j) + \delta(\mathfrak{a}_i) - \log(\gamma).\end{aligned}$$

It can be shown that $-\log 2 \leq \log\gamma \leq \log\Delta$ [19, Eqn. 5.38], and hence $\delta(\mathfrak{a}_j * \mathfrak{a}_i) \approx \delta(\mathfrak{a}_j) + \delta(\mathfrak{a}_i)$ as the "error" $\log\gamma$ in distance is small in comparison to the circumference $R_D \in O(\sqrt{D})$ and, in general, the ideal distances $\delta(\mathfrak{a}_j)$ and $\delta(\mathfrak{a}_i)$. Thus, as illustrated in Figure 2, multiplication by $\mathfrak{a}_i$ advances the distance by approximately $\delta(\mathfrak{a}_i)$ and has a similar effect as a giant step in the cyclic group setting. The main difference is that in G a giant step using g^i advances the distance by exactly i, i.e., distances in G are exactly additive, whereas in the infrastructure case the advance in distance by applying a giant step with ideal $\mathfrak{a}_i$ is only approximately i.

Shanks's discovery imposes a fascinating structure on the cycle of reduced principal ideals that gives the infrastructure its name and is tantalizingly close to a cyclic group structure. Interestingly, among all the requirements on an abelian group, the infrastructure violates merely associativity—and only just barely. Closure holds (the result of the multiplication/reduction of two infrastructure ideals is another infrastructure ideal), and the operation is easily seen to be commutative. There exists an identity element (namely $\mathcal{O}_D$, as $\mathfrak{a} * \mathcal{O}_D = \mathfrak{a}\mathcal{O}_D = \mathfrak{a}$), and it is possible to define inverses (roughly, the inverse of the infrastructure ideal $\mathfrak{a} = [Q, P]$ is the conjugate ideal $\overline{\mathfrak{a}}$ of distance $R_D - \delta(\mathfrak{a}) + \log\mathcal{N}(\mathfrak{a})$ with $\mathcal{N}(\mathfrak{a}) = Q/r$). The fact that distances are not exactly additive in the infrastructure prevents it from being associative: the infrastructure ideals $(\mathfrak{a} * \mathfrak{b}) * \mathfrak{c}$ and $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c})$ both have distance close to $\delta(\mathfrak{a}) + \delta(\mathfrak{b}) + \delta(\mathfrak{c})$ but need not be equal.

Nevertheless, using this "giant step" operation in the infrastructure leads to a straightforward adaptation of baby-step giant-step to compute the regulator R_D. The baby-step list consists of the first l ideals obtained by applying baby steps (using the continued fraction algorithm), and the giant steps consist of successively multiplying by an ideal $\mathfrak{a}_l$ with distance close to l and reducing the product. After some adjustments to account for the fact that distances are not exactly additive [19, Sec. 5.3], $R_D = \delta(\mathfrak{a}_i) - \delta(\mathfrak{a}_j)$ is obtained once a giant-step ideal $\mathfrak{a}_i$ is found that is equal to one of the baby-step ideals $\mathfrak{a}_j$. Using the bound of $O(D^{1/2+\epsilon})$ for R_D, this algorithm has complexity $O(D^{1/4+\epsilon})$. As with order computation, there exists a variation that does not depend on an upper bound on R_D and requires only $O(\sqrt{R_D})$ infrastructure operations [7, 8]. The overall bit complexity of this algorithm is $O(\sqrt{R_D}D^{\epsilon})$, as each of the infrastructure operations has complexity polynomial in $\log D$. This is currently the fastest deterministic algorithm that unconditionally computes R_D.

To see how this algorithm works in practice, consider the previous example of $D = 193$ ([19, Example 7.15]). By expanding the continued fraction of $(1+\sqrt{193})/2$, essentially enumerating the entire infrastructure by baby steps, we found that $\varepsilon_{193} = 1764132 + 126985\sqrt{D}$, and thus $R_{193} = \log \varepsilon_{193} \approx 15.07631652$. Instead, suppose that the first six infrastructure ideals, taken from Table 1, form our baby-step list,

$$\mathcal{L} = \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_6\},$$

and that the ideal $\mathfrak{a}_4 = [4, 13]$ with distance $\delta(\mathfrak{a}_4) \approx 4.82844171$ is used for the giant steps. Successively multiply $\mathfrak{a}_4$ by itself and apply subsequent reduction, thereby obtaining the sequence of giant-step ideals:

$$\begin{aligned}
\mathfrak{b}_2 &= \mathfrak{a}_4 * \mathfrak{a}_4 = [8, 9] = \mathfrak{a}_{11}, & \delta(\mathfrak{b}_2) &\approx 9.65688343\ , \\
\mathfrak{b}_3 &= \mathfrak{b}_2 * \mathfrak{a}_4 = [12, 11] = \mathfrak{a}_{15}, & \delta(\mathfrak{b}_3) &\approx 14.26937782\ , \\
\mathfrak{b}_4 &= \mathfrak{b}_3 * \mathfrak{a}_4 = [6, 11] = \mathfrak{a}_3, & \delta(\mathfrak{b}_4) &\approx 18.40467235\ .
\end{aligned}$$

We find that $\mathfrak{b}_4 \in \mathcal{L}$ with $\mathfrak{b}_4 = \mathfrak{a}_3$, so

$$\begin{aligned}
&R_{193} \\
&= \delta(\mathfrak{b}_4) - \delta(\mathfrak{a}_3) \approx 18.40467235 - 3.32835583 \\
&= 15.07631652,
\end{aligned}$$

as required.

What Else Can (and Can't!) You Do with Infrastructure?

The analogy between a finite cyclic group and the infrastructure can be extended further, in the sense that a number of additional useful algorithms and techniques from the group setting can be adapted to solving problems in the infrastructure. For example, a more general version of the problem of computing the order of g is the *discrete logarithm problem*, computing the smallest power of g equal to a given group element a. The result represents the "distance" from the identity element to a. The infrastructure analogy is the *distance problem*: given an infrastructure ideal $\mathfrak{a}$, find its distance $\delta(\mathfrak{a})$ from $\mathcal{O}_D$. Via compact representations, which are discussed in more detail at the end of this section, this problem can be shown to be polynomially equivalent to the *principal ideal problem*: finding a generator of $\mathfrak{a}$. As in the cyclic group case, the distance problem can be solved using the baby-step giant-step algorithm in $O(\sqrt{\delta(\mathfrak{a})})$ operations in the infrastructure. Testing whether two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent can also be done with this method by testing whether $\mathfrak{a}\bar{\mathfrak{b}}$ is principal.

A fundamental algorithm in the cyclic group setting is fast exponentiation, by which g^n, the group element of "distance" n from 1, can be computed rapidly (in $O(\log n)$ group operations) by combining squarings and multiplications by g according to the binary expansion of n. In the infrastructure, the analogue of this algorithm allows one to compute efficiently the ideal whose distance is closest to n. By combining giant steps (in this case, ideal squaring followed by reduction) with baby steps, such an ideal can be computed in $O(\log n)$ infrastructure operations.

As in the cyclic group setting, the basic operation of finding an element with distance close to n enables several additional applications and algorithms. One such application, common to both settings, is public-key cryptography. For example, in a cyclic group, Diffie and Hellman [11] showed how two parties can agree on a shared secret over an insecure communication channel by exchanging g^a and g^b (where a and b are generated randomly by the respective parties and kept secret) and each independently computing the common group element $g^{ab} = (g^a)^b = (g^b)^a$. An analogous infrastructure key exchange protocol was proposed in 1989 by Buchmann and Williams [9] using the computation of closest ideals in the infrastructure in place of exponentiation. Roughly speaking, the protocol participants exchange infrastructure ideals close to a and b and independently compute the infrastructure ideal close to ab. The result is a protocol whose security is related to the principal ideal problem in a similar way to how the Diffie-Hellman scheme is related to the discrete logarithm problem. This protocol is noteworthy in that it was the first public-key cryptosystem proposed whose security was based on a structure that is not a group. Other protocols, as well as improvements to the original cryptosystem, have also been proposed; a survey can be found in [19, Ch. 14].

The ability to compute closest ideals also leads to a simple method for testing whether a given real number S is a good numerical approximation of an integer multiple of R_D. Once again, the idea stems from the cyclic group setting, where S is a multiple of the order of a generator g if and only if g^S is the group identity. Similarly, in the infrastructure scenario, if S is a multiple of the regulator R_D, then the closest ideal to S has to be $\mathcal{O}_D$, because R_D is the distance around the entire infrastructure, i.e., from $\mathcal{O}_D$ to itself. Thus, one simply computes the ideal closest to S and checks whether it is $\mathcal{O}_D$. This observation also enables accurate approximations of R_D to be computed efficiently given only a coarse approximation: one simply computes the ideal closest to the approximation and computes the resulting distance to the required precision.

An application of testing for multiples of the regulator using closest ideals is an improvement, due to Lenstra [20], of the baby-step giant-step algorithm for computing the regulator R_D. After computing an approximation H to $h_D R_D$, which can be done analytically [19, Sec. 10.2], the ideal $\mathfrak{a}$ with distance closest to H is computed. Baby-step giant-step is then used to search for the ideal closest to $\mathfrak{a}$ whose distance is a multiple of the regulator; the number of infrastructure operations required is thus asymptotic to $\sqrt{|H - h_D R_D|}$, the square root of the error in the approximation. Given this multiple of the regulator, a combination of baby-step giant-step (to check whether the regulator is larger than a given bound) and closest ideal computations (to find smaller multiples of R_D) is used to find R_D. By balancing the time required for the various parts of the algorithm, an overall complexity of $O(D^{1/5+\epsilon})$ is achieved. This algorithm is unconditionally correct, but the runtime applies only if the Extended Riemann Hypothesis (ERH) holds, as it is required to guarantee the accuracy of the approximation H of $h_D R_D$.

There are some techniques from the cyclic group setting that unfortunately do not translate well to the infrastructure. One such technique is computing the order of the group given a multiple of the group order. This can be done in polynomial time given the prime factorization of the order multiple m by finding, for each prime divisor p, the largest integer i for which m/p^i is also a multiple of the order. This technique does not generalize to the infrastructure, as there is no known way to "factor" multiples of the regulator, which are real numbers.

The best-known method to solve the problem of finding R_D given an integer multiple S of R_D is the technique used in Lenstra's $O(D^{1/5+\epsilon})$ algorithm, namely, using baby-step giant-step to check whether the regulator is larger than a certain bound and checking whether S/p is also a multiple of R_D by testing whether the ideal closest to S/p is equal to $\mathcal{O}_D$ for all primes p less than another suitable bound. When used in the context of Lenstra's algorithm, where a regulator multiple must also be computed, optimization results in the $O(D^{1/5+\epsilon})$ runtime. When the regulator multiple S is given as input, reoptimization results in an algorithm with complexity $O(S^{1/3}D^{\epsilon})$ for unconditionally computing R_D from S.

An interesting consequence of this algorithm is that it leads to an efficient practical algorithm for unconditionally computing R_D. Buchmann's index-calculus algorithm [4] computes R_D in time subexponential in $\log D$, but both the runtime and correctness of this algorithm depend on the ERH. Using the previous algorithm to verify unconditionally that this output is in fact the regulator results in an overall algorithm [15, 10] that computes R_D unconditionally in expected time $O(D^{1/6+\varepsilon})$, as the index-calculus algorithm computes R_D in expected time $O(D^{\epsilon})$ and $R_D \in O(D^{1/2+\epsilon})$. Although this algorithm is not deterministic, due to its dependence on the index-calculus algorithm to produce a multiple of R_D, it is the fastest known algorithm that computes an unconditionally correct approximation of R_D.

Two more techniques from the cyclic group setting that do not work well in the infrastructure are the Pollard-rho algorithm [24] for computing the order n of a group element g and the Pohlig-Hellman algorithm [23] for discrete logarithms. Pollard's algorithm uses a random walk in the group and a method to detect cycles in this walk. The result is a probabilistic algorithm with expected runtime $O(\sqrt{n})$ that requires only a constant amount of storage, unlike baby-step giant-step, which requires $O(\sqrt{n})$ storage. The random walk and cycle finding algorithm can certainly be applied in the infrastructure. The problem is that, by virtue of the probabilistic nature of the algorithm, the results produced are not minimal; that is, the algorithm produces an integer multiple of R_D which need not be equal to R_D. In the case of a cyclic group, the output can be factored and minimized as described above. However, as there is no appropriate notion of factorization for distances (which are real numbers), one would again be forced to apply a minimization strategy similar to that described above for multiples of the regulator. In addition, as the distances of the ideals involved will grow to be much larger than R_D, the precision requirements to ensure numerical accuracy are almost certainly too unwieldy to make this approach practical.

The Pohlig-Hellman algorithm solves the discrete logarithm problem by reducing it to discrete logarithm problems in order p subgroups for all primes p that divide the order n of G. Thus this algorithm is very effective when n is smooth, i.e., has only small prime factors. Adapting this algorithm to the infrastructure seems fruitless, as there is no corresponding notion of smoothness for the regulator.

One additional computational issue that arises in the infrastructure is the representation of the fundamental unit itself, as well as principal ideal generators. When computing only the regulator or ideal distances, the sizes of operands encountered are generally manageable, although, as discussed in the next section, the numerical accuracy or precision of their floating point approximations becomes problematic. The alternative, explicitly working with the principal ideal generators as opposed to their logarithms, is tempting in that problems with approximations are eliminated, but the size of the operands is too large. For example, the size of the fundamental unit (i.e., the regulator) is $O(D^{1/2+\epsilon})$, so it cannot even be written in time polynomial in $\log D$.

Fortunately, this problem can also be solved using arithmetic in the infrastructure. The idea is to use a type of binary exponentiation to come up with a representation of principal ideal generators, including the fundamental unit, in the form of a power-product of elements of $\mathbb{Q}(\sqrt{D})$. An expression of this form is called a *compact representation* [6], [19, Ch. 12]. The compact representation of an element $\theta \in \mathbb{Q}(\sqrt{D})$ can be computed in time polynomial in $\log\log|\theta|\log D$ given an approximation of $\log|\theta|$. The basic arithmetic operations (including testing equality, multiplication, division, norm, computing $x \mod m$ and $y \mod m$ where $\theta = x + y\sqrt{D}$) can be performed on compact representations in polynomial time [18]. Most importantly, compact representations have *size* polynomial in $\log\log|\theta|\log D$, which motivates their name. The result is that the algorithms for computing R_D mentioned above can also compute a compact representation of the fundamental unit, as well as the fundamental solution of Pell's equation, without changing the asymptotic runtime. In addition, computations requiring explicit manipulation of fundamental units and solutions of Pell's equation, for example, as part of the resolution of other Diophantine equations [18], can be performed even for large values of D.

All of the applications mentioned in this section can, in principle, be implemented relatively easily. However, there are a number of practical concerns which are described in the next section.

Practical Issues

One of the difficulties arising in infrastructure arithmetic stems from the fact that distances are real numbers and thus need to be approximated to sufficient accuracy in any computer implementation. To that end, any (potentially unknown) infrastructure ideal is represented by a known infrastructure ideal and an approximation of the relative distance between the two ideals. More explicitly, let $p \in \mathbb{N}$ be a prespecified precision parameter. For any $f \in \mathbb{N}$ with $f < 2^p$, a *near reduced (f,p)-representation* of a reduced ideal $\mathfrak{a}$ is a triple $(\mathfrak{b}, d, k)$ consisting of another reduced ideal $\mathfrak{b}$ and two integers $d > 0$ and $k < 0$ [16]. Here $\mathfrak{b} = (\theta)\mathfrak{a}$ for some unknown relative generator $\theta \in \mathbb{Q}(\sqrt{D})$ that is of magnitude about 2^k, and d is a $(p+1)$-bit integer that approximates the most significant $p+1$ bits of θ with relative error $2^{-p}f$. In practice, it is possible to keep the relative errors significantly below one. Moreover, $|k|$ and θ tend to be small, and, in fact, the ideals in any two near-reduced (f,p)-representations of the same reduced ideal can be shown to be at most five baby steps apart in the infrastructure.

Ideal arithmetic is now conducted on (f,p)-representations. If $\mathfrak{a}$ is taken to be $\mathcal{O}_D$, then each (f,p)-representation consists of an infrastructure ideal and an approximation of its principal ideal generator. Arithmetic in the infrastructure can then be performed using arithmetic on these representations. However, such arithmetic introduces error propagation, so care must be taken in choosing the precision p large enough to keep the relative errors sufficiently small. For example, in the key agreement protocol mentioned in the section "What Else Can (and Can't!) You Do with Infrastructure?", suppose that the two communicants use secret exponents of bit size n. In order to guarantee that the two parties arrive at the same key ideal after execution of the protocol, i.e., after their respective double exponentiations, one needs to ensure that $p \geq 2n + \log_2(50n)$ [16].

The computation of giant steps presents another computational nuisance. As mentioned earlier, the standard giant-step algorithm first multiplies two reduced ideals and subsequently reduces the primitive part of the product ideal. This technique has two disadvantages. First, in the reduction procedure, it is very costly to compute all the basis coefficients Q_j, P_j of the (unnecessary) intermediate ideals via the continued fraction algorithm. Second, multiplying two reduced ideals results in an ideal whose basis coefficients are roughly twice as large as those of the two input ideals; the reduction process gradually shrinks the coefficients back down to original size.

In the 1980s Shanks began to compute class numbers of imaginary quadratic fields with his programmable hand-held calculator. Frustrated by the fact that the large size of the intermediate results arising in ideal multiplication exceeded the display capacity of his calculator, he developed a significantly more efficient ideal multiplication and reduction algorithm to which he assigned the Fortran designator NUCOMP (short for "new composition")[3] [27].

NUCOMP, in essence, stops the multiplication process before completion and applies a type of intermediate reduction before generating the reduced ideal product. Recall that traditional giant-step computation applies the continued-fraction algorithm to the quadratic irrational

[3] *Shanks described his work using the language of quadratic forms, in which composition of forms is the analogue of ideal multiplication.*

$(P + \sqrt{D})/Q$ arising from the product ideal $\mathfrak{a} = [Q, P]$ whose coefficients are of order D until a reduced ideal is obtained. Shanks's idea was to avoid computing the nonreduced product ideal $\mathfrak{a}$ altogether and replace the quadratic irrational $\alpha = (P + \sqrt{D})/Q$ by a suitable *rational* approximation of α whose numerator and denominator are of magnitude $\sqrt{D}$, i.e., significantly smaller. Then the continued-fraction algorithm is replaced by the more efficient extended Euclidean algorithm, and the partial quotients obtained in both methods will be identical, up to a certain point in the continued fraction expansion. Since the basis coefficients of any of the intermediate ideals can be recovered through simple formulas involving the convergents of the rational approximation, the costly computation of all the intermediate ideals is avoided.

But at which point in the Euclidean algorithm is the reduced target ideal—or at least an "almost" reduced ideal—reached? In collaboration with Atkin, Shanks answered this question as well. Once the remainders have decreased to approximately $\sqrt[4]{D}$, the corresponding ideal is recovered. This ideal is frequently reduced, and is at most two continued fraction steps away from being reduced. Moreover, this computation only involves integers of magnitude $\sqrt{D}$ or less (with a few exceptions that may be as large as $D^{3/4}$).

Shanks originally formulated his NUCOMP method for imaginary quadratic fields, but it was extended to real quadratic fields by van der Poorten [30]. Numerical experiments [17], [16] showed that NUCOMP can be up to 50 percent faster than ordinary ideal multiplication and reduction.

Beyond Infrastructure in Real Quadratic Fields

While the ideal class group of an imaginary quadratic field is a very large finite abelian group, its counterpart in real quadratic fields tends to be a rather dull object, namely, a very small group that is frequently trivial. Instead, surprisingly, the principal class—which is, after all, merely the identity element of this group—exhibits an interesting, almost grouplike structure. As described above, the cycle of reduced principal ideals constitutes the infrastructure. This raises the natural question of whether infrastructures occur in other settings. Indeed, this is the case.

The closest analogue to real quadratic number fields is real quadratic (or *hyperelliptic*) *function fields*. Informally speaking, to obtain such a field, $\mathbb{Q}$ is simply replaced by a rational function field $\mathbb{F}_q(x)$ of odd characteristic.[4] Then the square root of a polynomial $D(x) \in \mathbb{F}_q[x]$ of even degree and square leading coefficient is adjoined to $\mathbb{F}_q(x)$. Ideals are now represented by a pair of polynomials in $\mathbb{F}_q[x]$ instead of a pair of integers, and for reduced ideals, these polynomials have degree less than $\deg(D)/2$. Arithmetic on ideals is completely analogous to real quadratic field arithmetic, and, once again, every ideal class has an associated infrastructure [28]. The main difference between real quadratic function fields and number fields is that, in the former, distances are *integers*. Loosely speaking, distances are degrees of quadratic irrational functions, as opposed to logarithms of quadratic irrational numbers. In particular, ideals are discretely and relatively evenly spaced around the infrastructure, and a baby step always produces an advance in distance of at least one (and almost always exactly one for large base fields $\mathbb{F}_q$). The result is a cleaner and simpler ideal arithmetic, avoiding the need for numerical approximations as discussed in the previous section. Moreover, interestingly, integer distances make it possible to provide an explicit and effective embedding of the principal infrastructure into a finite cyclic group whose order is the regulator, as was discovered independently by Fontein [12] and Mireles Morales [22]. Such an embedding into a finite cyclic group does not exist in the number field setting, although embedding the infrastructure into an *infinite* cyclic group is possible (see [20]).

Beyond quadratic extensions, any *global field*, i.e., any number field or function field over a finite field, of unit rank one exhibits an infrastructure analogous to the one described above. Number fields of unit rank one can have degree at most four over $\mathbb{Q}$; specifically, they include real quadratic, complex cubic, and totally complex quartic fields. In contrast, function fields of unit rank one can have arbitrarily high extension degree over $\mathbb{F}_q(x)$. Moreover, unit groups of higher rank result in infrastructures that are essentially higher-dimensional tori. In his 1987 *Habilitationsschrift* [3], Buchmann established that in fact every number field has an infrastructure and presented the first generic algorithm for computing the class group and regulator of an arbitrary number field. Twenty-one years later, Schoof in his excellent 2008 treatise [25] provided a modern treatment of the general number field setting, using *Arakelov divisor* theory—the number field analogue to divisor theory of function fields—as a natural setting for the infrastructure phenomenon and Buchmann's algorithm. Most recently, the number field and function field scenarios were finally combined by Fontein [13], who gave a unified and completely

[4] *Hyperelliptic function fields can also be defined over finite fields of characteristic 2, but as their representation is somewhat more complicated, we will not consider them here.*

general description of the infrastructure of any global field, including baby steps and giant steps, the relationship of infrastructure arithmetic to arithmetic in the divisor class group, and (for the function field setting) an algorithm for computing a system of fundamental units and the regulator of the field.

Open Problems

Although Shanks's discovery of the infrastructure of a real quadratic field has already led to new applications and improved algorithms in that setting, research into this rich field is by no means exhausted. There are still a number of interesting open problems.

A noticeable gap persists between what is possible for computing the regulator R_D of a quadratic number field unconditionally and deterministically and what seems possible in practice. The fastest deterministic, unconditional algorithm has complexity $O(\sqrt{R_D}D^{\epsilon})$, or $O(D^{1/4+\epsilon})$ using the fact that $R_D \in O(D^{1/2+\epsilon})$. Assuming the ERH only for the complexity analysis reduces this to $O(D^{1/5+\epsilon})$. Allowing nondeterminism (i.e., randomization) yields $O(D^{1/6+\epsilon})$. Finally, allowing nondeterminism and the assumption of the ERH for the correctness of the output reduces the complexity to subexponential time $O(\exp(\sqrt{\log D \log\log D}))$. Are further improvements possible?

Work continues to improve the performance of arithmetic in the infrastructure. The adaptation of Shanks's NUCOMP technique from imaginary to real quadratic fields represents notable progress in this direction, and further improvements are certainly possible. The development of (f, p)-representations to maintain accurate approximations of distances has helped by reducing the precision requirements; perhaps this can be improved further.

Finally, arithmetic in general infrastructures is still in its infancy. In particular, very little work—algorithmic and especially computational—has been done for infrastructures of global fields with degree larger than 3. The works of Schoof and Fontein provide two different settings with which to conceptualize these infrastructures. Which one will turn out to be more advantageous in terms of concrete algorithms and implementations?

It is the authors' hope that this article has given the reader a glimpse into the infrastructure and its many interesting applications and generalizations. There is still much ongoing work in this area and even more left to be done. Readers are encouraged to join the research into this interesting area.

Further Reading

As this article was written in an expository manner, many details and proofs were deliberately omitted. For a thorough treatment of infrastructure in real quadratic fields, including many of the associated algorithms and applications, we recommend [19]. The book of Buchmann and Vollmer [8] treats much of this material in the language of binary quadratic forms and is also highly recommended. For more information about infrastructure in functions fields and higher-dimensional analogues, we refer the reader to the references in the section "Beyond Infrastructure in Real Quadratic Fields".

Acknowledgment

The authors thank Sarah Chisholm, Monireh Rezai Rad and Peter Zvengrowski for their careful proofreading and suggestions for improvement of this article.

References

1. A. Amthor, Das Problema Bovinum des Archimedes, *Zeitschrift für Math. u. Physik* (Hist. Litt. Abtheilung) **25** (1880), 153–171.
2. Archimedes, *The Cattle Problem*, in English verse by S. J. P. Hillion and H. W. Lenstra Jr., Mercator, Santpoort, 1999.
3. J. Buchmann, *Zur Komplexität der Berechnung von Einheiten und Klassenzahlen algebraischer Zahlkörper*, Habilitationsschrift, Düsseldorf, 1987.
4. ______, A subexponential algorithm for the determination of class groups and regulators of algebraic number fields, *Séminaire de Théorie des Nombres* (Paris), 1988–89, pp. 27–41, Prazr. Math., vol. 91, Birkhäuser Boston, 1990.
5. J. Buchmann, M. J. Jacobson Jr., and E. Teske, On some computational problems in finite abelian groups, *Mathematics of Computation* **66** (1997), no. 220, 1663–1687.
6. J. Buchmann, C. Thiel, and H. C. Williams, Short representation of quadratic integers, *Computational Algebra and Number Theory, Mathematics and its Applications* vol. 325, Kluwer, Dordrecht, 1995, pp. 159–185.
7. J. Buchmann and U. Vollmer, A Terr algorithm for computations in the infrastructure of real-quadratic number fields, *Journal de Théorie des Nombres de Bordeaux* **18** (2006), no. 3, 559–572.
8. ______, *Binary Quadratic Forms: An Algorithmic Approach*, Algorithms and Computation in Mathematics, vol. 20, Springer-Verlag, Berlin, 2007.
9. J. Buchmann and H. C. Williams, A key-exchange system based on real quadratic fields, CRYPTO '89, Lecture Notes in Computer Science, vol. 435, Springer, New York, 1989, pp. 335–343.
10. R. de Haan, M. J. Jacobson Jr., and H. C. Williams, A fast, rigorous technique for computing the regulator of a real quadratic field, *Mathematics of Computation* **76** (2007), no. 260, 2139–2160.
11. W. Diffie and M. Hellman, New directions in cryptography, *IEEE Transactions on Information Theory* **22** (1976), 472–492.
12. F. Fontein, Groups from cyclic infrastructures and Pohlig-Hellman in certain infrastructures, *Advances in*

Mathematics of Communications **2** (2008), no. 3, 293–307.

13. ———, The infrastructure of a global field of arbitrary unit rank, *Mathematics of Computation* **80** (2011), no. 276, 2325–2357.
14. C. F. Gauss, *Disquisitiones Arithmeticae*, Springer Verlag, 1986, English ed.: translated by A. A. Clark.
15. M. J. Jacobson Jr., Á. Pintér, and P. G. Walsh, A computational approach for solving $y^2 = 1^k+2^k+\cdots+x^k$, *Mathematics of Computation* **72** (2003), no. 244, 2099–2110.
16. M. J. Jacobson Jr., R. Scheidler, and H. C. Williams, An improved real quadratic field based key exchange procedure, *Journal of Cryptology* **19** (2006), 211–239.
17. M. J. Jacobson Jr. and A. J. van der Poorten, *Computational aspects of NUCOMP*, Algorithmic Number Theory—ANTS-V (Sydney, Australia), Lecture Notes in Computer Science, vol. 2369, Springer-Verlag, Berlin, 2002, pp. 120–133.
18. M. J. Jacobson Jr. and H. C. Williams, Modular arithmetic on elements of small norm in quadratic fields, *Designs, Codes and Cryptography* **27** (2002), no. 1–2, 93–110.
19. ———, *Solving the Pell Equation*, CMS Books in Mathematics, Springer, New York, 2009.
20. H. W. Lenstra Jr., On the calculation of regulators and class numbers of quadratic fields, London Mathematical Society Lecture Note Series, vol. 56, Cambridge University Press, 1982, pp. 123–150.
21. ———, Solving the Pell equation, *Notices of the AMS* **49** (2002), no. 2, 182–192.
22. D. J. Mireles Morales, *An analysis of the infrastructure in real function fields*, E-print archive no. 2008/299, 2008.
23. S. C. Pohlig and M. E. Hellman, An improved algorithm for computing logarithms over $GF(p)$ and its cryptographic significance, *IEEE Transactions on Information Theory* **24** (1978), 106–110.
24. J. M. Pollard, Theorems on factorization and primality testing, *Proceedings of the Cambridge Philosophical Society* **76** (1974), 521–528.
25. R. J. Schoof, Computing Arakelov class groups. *Algorithmic Number Theory: Lattices, Number Fields, Curves and Cryptography*, Math. Sci. Res. Inst. Publ., vol. 44, Cambridge Univ. Press, Cambridge, 2008, pp. 447–495.
26. D. Shanks, The infrastructure of real quadratic fields and its applications, *Proceedings of the 1972 Number Theory Conference*, Univ. Colorado, Boulder, Colorado, 1972, pp. 217–224.
27. ———, On Gauss and composition. I, II, *Proceedings of the NATO ASI on Number Theory and Applications* (R. A. Mollin, ed.), Kluwer Academic Press, 1989, pp. 163–179.
28. A. Stein, Explicit infrastructure for real quadratic function fields and real hyperelliptic curves, *Glasnik Matematicki* **44** (2009), no. 1, 89–126.
29. D. C. Terr, A modification of Shanks' baby-step giant-step algorithm, *Mathematics of Computation* **69** (2000), no. 230, 767–773.
30. A. J. van der Poorten, A note on NUCOMP, *Mathematics of Computation* **72** (2003), no. 244, 1935–1946.
31. H. C. Williams, Solving the Pell equation, *Proceedings of the Millennial Conference on Number Theory*, A K Peters, Natick, MA, 2002, pp. 397–435.

Hypergeometric Functions, How Special Are They?

Frits Beukers

Section 1. Introduction

In the world of standard functions, the hypergeometric functions take a prominent position in mathematics, both pure and applied, and in many branches of science. They were introduced by Euler as power series expansions of the form

$$1+\frac{a\cdot b}{c\cdot 1}z+\frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2+\cdots,$$

where a,b,c are rational parameters. By specialization of the parameters, Euler obtained the various classical functions that were around at that time. For example, taking $b=c=1$ gives us Newton's binomial series for $(1-z)^{-a}$ and taking $a=b=1/2, c=3/2$ gives us $\arcsin(\sqrt{z})/\sqrt{z}$. Finally, taking all parameters equal to 1 recovers the ordinary geometric series, which more or less explains the name hypergeometric series that was given by Euler to his series. Hypergeometric functions also include functions that were entirely new in Euler's time. For example, taking $a=b=1/2, c=1$, one obtains the function

$$\frac{2}{\pi}\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-zx^2)}},$$

a so-called elliptic integral of the first kind. It is a period of the family of elliptic curves $y^2=(1-x^2)(1-zx^2)$ parameterized by z and is one of the most often quoted functions in algebraic geometry. Euler also found the hypergeometric equation, which is the second-order linear differential equation that is satisfied by hypergeometric series. It reads

$$z(z-1)f''+((a+b+1)z-c)f'+abf=0.$$

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DOI: http://dx.doi.org/10.1090/noti1065

This is a differential equation that occurs in a multitude of branches of mathematics, mathematical physics, and applied sciences.

Years later, Gauss studied hypergeometric functions not only as values of Euler's hypergeometric series but also as solutions of the hypergeometric equation throughout the complex plane, an approach that was entirely new at that time. In this way Gauss very soon became aware of the problem of their multivaluedness, known nowadays as the monodromy problem. He published only part of his work in 1812 [11, p. 123]. A sequel that describes the behavior in the complex plane was found in Gauss's *Nachlass*; see [11, p. 207]. Because of Gauss's work, the functions given by Euler's hypergeometric series are now often called Gauss hypergeometric functions.

The next major contribution came from Riemann. In the article [19] from 1857 he gave a complete description of the monodromy group for Gauss's hypergeometric function. The monodromy group of a linear differential equation in the complex plane characterizes the behavior of the analytic continuation of its solutions. In this way hypergeometric functions became an important testing ground for Riemann's fundamentally new ideas on analytic continuation. Riemann's work was taken up by H. A. Schwarz, Felix Klein, and others, and hypergeometric functions made their appearance in the early days of algebraic geometry and modular forms. In his 1893 lectures, Klein [17] gave an extensive exposition of Riemann's ideas and their consequences. For a more concise overview see [2].

Simultaneously, by the end of the nineteenth century, people had introduced many generalizations of Gauss hypergeometric functions by increasing the number of parameters or the number of variables or both. With a few exceptions, the ensuing study of them was for a large part of a descriptive nature, meaning: for each new class the system of corresponding (partial) differential equations had to be determined, a basis of solutions around various special points was to be found, together with their domain of convergence. In this process, the conceptual understanding of these generalizations lagged far behind the well-developed ideas around Gauss's hypergeometric function. Thus a large part of the subject of generalized hypergeometric functions became relegated to the domain of special functions, which to many people means many formulas but no mathematical depth. Fortunately, developments by the end of the twentieth century turned this image around somewhat, and it is my hope that in this century the turnaround will be complete. It is the purpose of this article to introduce generalized hypergeometric functions in one and several variables and hint at some simple, almost combinatorial, structures that underlie them. We do this by looking at hypergeometric functions that are at the same time algebraic. The structure of this article is as follows:

- Section 2: A number of random-looking but relevant examples;
- Sections 3, 4, 5: Gauss hypergeometric functions and examples of their generalizations;
- Section 6: A-hypergeometric functions, a unified way of looking at all the previous examples;
- Section 7: An example of a result that holds for general A-hypergeometric systems;
- Section 8: A short discussion on monodromy.

Section 2. Some Peculiar Examples

The solution of the general fifth-degree equation has a notorious history, and it is known that it cannot be achieved by the repeated use of taking radicals. However, it does not mean that the equation is hard to solve. It is well known that, after some transformations involving radicals, the fifth-degree equation can be reduced to a three-term equation of the form

$$zx^5 - x + 1 = 0,$$

where z is a parameter and x the unknown. Such forms are called Bring-Jerrard forms. An exercise in Lagrange inversion shows that a solution is given by the power series

$$\sum_{n\geq 0} \binom{5n}{n} \frac{z^n}{4n+1}.$$

> Lagrange inversion
>
> Let $f(x)$ be a power series in x with $f(0) = 0$, $f'(0) \neq 0$. Then Lagrange inversion tells us that a solution to $f(x) = z$ in x is given by the power series
>
> $$x = \sum_{k\geq 1} \frac{z^k}{k!} \left(\frac{d}{dx}\right)^{k-1} \left(\frac{x}{f(x)}\right)^k \Bigg|_{x=0}$$
>
> in powers of z. Change the fifth-degree equation to $z(x+1)^5 - x = 0$ and apply the inversion with $f(x) = x/(x+1)^5$ to obtain our solution of the fifth-degree equation.
>
> Another exercise with a surprising answer is when we take $f(x) = xe^{-x}$.

This turns out to be a hypergeometric function. To see this, let us introduce the symbol $(x)_n = x(x+1)\cdots(x+n-1)$, the so-called *Pochhammer symbol* or *shifted factorial*. In particular, $(1)_n = n!$. Note that we can rewrite the coefficients of our fifth-degree solution as

$$\binom{5n}{n}\frac{1}{4n+1} = \frac{5^{5n}}{4^{4n}} \times \frac{(1/5)_n(2/5)_n(3/5)_n(4/5)_n}{(1/2)_n(3/4)_n(5/4)_n n!}.$$

Denote

$$\phi(z) = \sum_{n\geq 0} \frac{(1/5)_n(2/5)_n(3/5)_n(4/5)_n}{(1/2)_n(3/4)_n(5/4)_n n!} z^n.$$

Then the solution to our fifth-degree equation can be written as $\phi(5^5 z/4^4)$. The function ϕ is an example of a generalized hypergeometric function.

Our second example comes from a beautiful observation by Fernando Rodriguez-Villegas [20]. In his work on estimates for the prime counting function $\pi(x)$, Chebyshev used arguments that amount to studying prime factors of the numbers

$$u_n = \frac{(30n)!n!}{(15n)!(10n)!(6n)!}.$$

Rodriguez-Villegas observed that the numbers u_n are integers and that the generating function $u(z) = \sum_{n\geq 0} u_n z^n$ is an algebraic function in z. This means that $u(z)$ satisfies a nontrivial polynomial equation with polynomials in z as coefficients. In this example, the minimal degree of the equation turns out to be 483840. The proof of this observation is to rewrite $u(z)$ as $\psi(2^{14}3^9 5^5 z)$ (see Figure 1) and then use a result of Heckman and the author in [6] on generalized hypergeometric functions, which implies that $\psi(z)$ is an algebraic function.

A third example involves two-variable functions. Consider the polynomial

$$\Delta = 1 + 4x + 4y + 18xy - 27x^2y^2$$

and the algebraic function $g(x, y)$ defined by the cubic equation

$$g^3 - g^2 - (3xy - x - y)g - xy(x + y + 1) = 0$$

$$\psi(z) = \sum_{n\geq 0} \frac{(1/30)_n(7/30)_n(11/30)_n(13/30)_n(17/30)_n(19/30)_n(23/30)_n(29/30)_n}{(1/5)_n(1/3)_n(2/5)_n(1/2)_n(3/5)_n(2/3)_n(4/5)_n n!} z^n,$$

Figure 1.

and $g(0,0) = 1$. Then

$$\sqrt{(g-3xy)/\Delta} = \sum_{m,n\geq 0} \frac{(1/2)_{2m-n}(1/2)_{2n-m}}{m!n!} x^m y^n.$$

The two-variable series is a so-called Horn series of type G_3. Horn series have the property that the Pochhammer symbols that occur may have negative indices. In that case one must use the more general definition $(a)_k = \Gamma(a+k)/\Gamma(a)$. For negative values of k, this amounts to $(a)_k = 1/(a-1)(a-2)\cdots(a-|k|)$.

These three examples may have reaffirmed your impression that we are working with special functions indeed. However, the goal of this article is to explain Theorem 5, which describes a general combinatorial criterion to detect instances such as the ones above. In fact, the two-variable example was found that way. I believe that this is only part of a much wider circle of ideas that underlie the domain of hypergeometric functions in one and more variables. In the next section I will be more systematic and introduce hypergeometric functions, together with a number of their properties.

Section 3. Gauss Hypergeometric Function

Choose three parameters a, b, c, usually rational numbers, and consider the power series

$${}_2F_1(a,b,c|z) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n n!} z^n$$

in the complex variable z. It is well defined as long as c is not an integer ≤ 0. This is Euler's series from the introduction of this article. It is a simple exercise to show that its radius of convergence is equal to 1 whenever $a, b \notin \mathbb{Z}_{\leq 0}$. In the case when a or b is in $\mathbb{Z}_{\leq 0}$, the infinite series becomes a polynomial.

Let us abbreviate the notation ${}_2F_1(a,b,c|z)$ by $f(z)$ for the moment, and let θ be the differential operator $z\frac{d}{dz}$. A simple calculation using the factorial structure of the coefficients of $f(z)$ shows that

$$z(\theta+a)(\theta+b)f = \theta(\theta+c-1)f.$$

After replacing θ by $z\frac{d}{dz}$ again and an expansion, we find the second-order differential equation

$$z(z-1)f'' + ((a+b+1)z-c)f' + abf = 0,$$

known as the *hypergeometric differential equation*. A second solution around $z = 0$, when $c \notin \mathbb{Z}$, is given by the series expansion

$$z^{1-c}\,{}_2F_1(a+1-c, b+1-c, 2-c|z).$$

Together with the above series they form a basis of the vector space of solutions of the hypergeometric equation around the origin of the complex plane.

Beginning with Gauss, mathematicians such as Kummer and Goursat discovered many relations that exist between hypergeometric functions with their argument z replaced by certain rational functions in z. An example is

$$\begin{aligned}{}_2F_1(a,b,a+b+1/2|4z-4z^2) \\ = {}_2F_1(2a,2b,a+b+1/2|z),\end{aligned}$$

and for an account I refer to my overview [2]. It was through Riemann's work that many such identities found a conceptual basis with proofs that are only a few lines long.

As an application of Riemann's work on analytic continuation in geometrical form, H. A. Schwarz in 1873 gave a classification of all hypergeometric functions which are at the same time algebraic; see [21]. This resulted in the famous Schwarz list. For example, it turns out that ${}_2F_1(19/60, 49/60, 4/5|z)$ is an algebraic function of degree 720. Its Galois group—and at the same time monodromy group—G has the property that G modulo its center is the alternating group A_5. For the full list I refer to Schwarz's original paper and, for a more recent account, [7] or [2]. In these lists one has to realize that only irreducible hypergeometric equations are considered, i.e., equations whose differential operator does not factor in the ring $\mathbb{C}(z)[d/dz]$ of differential operators. Here is a simple criterion.

Theorem 1. *The hypergeometric equation is irreducible if and only if the sets $\{a,b\}$ and $\{0,c\}$ are disjoint when considered modulo $\mathbb{Z}$.*

From now on we will be interested only in irreducible (systems of) differential equations.

Section 4. Higher-Order Hypergeometric Functions

There is a generalization of the hypergeometric equation to higher-order equations for which one could also compute the analytic continuation, albeit in a preliminary way. This was realized notably through the work of Clausen and later in the article [26] by J. Thomae from 1870.

Let $k \in \mathbb{Z}_{\geq 2}$ and take two sets of parameters $\mathbf{a} = \{a_1, \ldots, a_k\}$ and $\mathbf{b} = \{b_1, \ldots, b_k\}$, where we take $b_k = 1$ by default. The *hypergeometric function of order k* is defined by

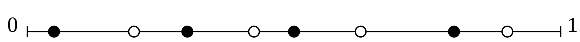
$$_kF_{k-1}(\mathbf{a}, \mathbf{b}|z) = \sum_{n\geq 0} \frac{(a_1)_n \cdots (a_k)_n}{(b_1)_n \cdots (b_{k-1})_n n!} z^n.$$

Sometimes it is also referred to as the Clausen-Thomae function. Again, the radius of convergence is 1, and it satisfies a differential equation of order k given by

$$\begin{aligned} z(\theta + a_1) \cdots (\theta + a_k)f \\ = (\theta + b_1 - 1) \cdots (\theta + b_{k-1} - 1)\theta f. \end{aligned}$$

The derivation goes along in the same way as in the Gauss case. This time it is better to leave it in this form. Again, the equation is irreducible if and only if the sets $\mathbf{a}$ and $\mathbf{b}$ are disjoint modulo $\mathbb{Z}$.

In 1989 Heckman and I [6] succeeded in extending Schwarz's list of algebraic Gauss hypergeometric functions to the case of higher-order hypergeometric functions. As a byproduct of [6] there is a simple criterion to decide whether or not a given hypergeometric function is algebraic. It is called the *interlacing criterion*. We say that two sets of k real numbers $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ in the interval $[0, 1)$ *interlac* if they are disjoint and their elements occur alternately in increasing order. For example, the white set and black set pictured below are interlacing.



More generally, two sets of real numbers $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are said to interlace modulo $\mathbb{Z}$ if the sets $\{a_1 - \lfloor a_1 \rfloor, \ldots, a_k - \lfloor a_k \rfloor\}$ and $\{b_1 - \lfloor b_1 \rfloor, \ldots, b_k - \lfloor b_k \rfloor\}$ interlace on $[0, 1)$. The interlacing criterion reads as follows.

Theorem 2 (Beukers, Heckman (1989))**.** *Let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Q}$ be the parameters of an irreducible generalized equation (we have $b_k = 1$ by default). Let D be their common denominator. Then the solutions of the hypergeometric equation are algebraic functions in z if and only if the sets*

$$\{ra_1, \ldots, ra_k\} \quad \text{and} \quad \{rb_1, \ldots, rb_k\}$$

interlace modulo $\mathbb{Z}$ for all integers r with $\gcd(r, D) = 1$ *and* $1 \leq r < D$.

As an application recall the function $\psi(z)$ from the introduction. We recognize it as a generalized hypergeometric function with parameters

$$\mathbf{a} = (1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30)$$

and

$$\mathbf{b} = (1/5, 1/3, 2/5, 1/2, 3/5, 2/3, 4/5, 1).$$

Notice that these two sets interlace modulo $\mathbb{Z}$. The interlacing criterion tells us that we should also look at the sets $r\mathbf{a}$ and $r\mathbf{b}$ for all integers r relatively prime to 120. However, the numerators of the elements of $\mathbf{a}$ form a complete set of numbers between 1 and 30 that are relatively prime to 30. Consequently, $r\mathbf{a} \equiv \mathbf{a} (\mathrm{mod}\ \mathbb{Z})$ for all r relatively prime to 30. Similarly, $r\mathbf{b} \equiv \mathbf{b} (\mathrm{mod}\ \mathbb{Z})$ for all r relatively prime to 30. Therefore, interlacing $r\mathbf{a}$ and $r\mathbf{b}$ modulo $\mathbb{Z}$ is equivalent to interlacing of $\mathbf{a}$ and $\mathbf{b}$, which we have already verified. The conclusion is that $\psi(z)$ is an algebraic function. Unfortunately the interlacing criterion gives no information about the degree of this function over the field $\mathbb{C}(z)$. For this we must invoke the other parts of [6], which tell us that the Galois group in this case is $W(E_8)$, the Weyl group of the root system E_8. This group has order 696729600, and a careful analysis shows that the degree of $\psi(z)$ is 483840, a divisor of $|W(E_8)|$.

Section 5. Several Variables

By the end of the nineteenth century several authors had introduced several-variable versions of hypergeometric functions. The best known are the Appell functions F_1, F_2, F_3, F_4. Their basic properties are described at length in the book [1] by Appell and Kampé de Fériet from 1926. We will not go into these functions extensively, but recall the definition of Appell's F_1:

$$F_1(a, b, b', c|x, y) = \sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n} m! n!} x^m y^n.$$

Of course a, b, b', c are parameters. The other Appell functions are variations on this type of series. They all satisfy systems of partial differential equations of order 2. Since these equations look rather unappealing, I will not quote them here. Another example is the Horn G_3 function, alluded to in the beginning:

$$G_3(a, b|x, y) = \sum_{m,n\geq 0} \frac{(a)_{2m-n}(b)_{2n-m}}{m! n!} x^m y^n.$$

Horn G_3 is one among a list of fourteen Horn series, which includes Appell's series. They arise from the philosophy that a two-variable power series $\sum_{m,n\geq 0} A(m, n)x^m y^n$ can be considered hypergeometric if the ratios $P(m, n) = A(m, n)/A(m, n-1)$ and $Q(m, n) = A(m, n)/A(m-1, n)$ are rational functions in m, n, together with the compatibility conditions $P(m, n)Q(m-1, n) = P(m, n-1)Q(m, n)$. If, in addition, one assumes that the numerator and denominator of P, Q have degree 2, one arrives at the ten Horn series and the four Appell series. The degree 2 condition corresponds to the requirement that the partial differential equations for these functions have order 2.

Of course one can increase the number of variables. We then get the so-called Lauricella functions F_A, F_B, F_C, F_D. We give the series expansion for the three-variable version of Lauricella F_D introduced by G. Lauricella in 1893 [18]:

$$F_D(a, b, b', b'', \gamma | x, y, z)$$
$$= \sum_{m,n,p\geq 0} \frac{(a)_{m+n+p}(b)_m(b')_n(b'')_p}{(c)_{m+n+p}m!n!p!} x^m y^n z^p.$$

It extends Appell's F_1 to three variables, and it is probably not hard for the reader to figure out what the n-variable version is. I mention this class in particular, since the structure of its monodromy group has been the subject of an extensive study that started with E. Picard in the nineteenth century and was completed by Deligne and Mostow around 1980; see [9]. In this paper Deligne and Mostow study discrete and arithmetic symmetry groups of the complex hyperbolic ball. Using Lauricella F_D functions, they discovered a number of groups that were hitherto unknown.

Apart from increasing the number of variables, one could also increase the order of the partial differential equations that define these functions. For example, in the book by Appell and Kampé de Fériet [1, Part I, Ch. IX], we find a section on two-variable functions that are the analogue of the one-variable higher-order hypergeometric functions. Then one is again obliged to study the dimension of the solution space, systems of differential equations, etc. This is what I refer to as a descriptive activity, and one might wonder where the end is.

Section 6. A-hypergeometric Functions

Fortunately there is a way to treat all previously mentioned examples of hypergeometric functions in a unified way. It was discovered by Gel'fand, Kapranov, and Zelevinsky by the end of the 1980s in a series of papers [12], [13], [14], [15] and is known under the name of A-hypergeometric functions. In honor of their discoverers, they are also referred to as GKZ-hypergeometric functions. My preference is the name A-hypergeometric function. Around the same time, B. Dwork developed a general theory of hypergeometric functions that has many parallels with A-hypergeometric functions and that culminated in his book [10]. Unfortunately, the book is hard to read because of its cumbersome notation and the interference of p-adic considerations in which Dwork was mostly interested.

In what follows I give a crash course on A-hypergeometric functions, concentrating only on those aspects that are immediately required for this article. Many other important issues will not be touched upon. For a more complete account, one might consult the overviews [25], [3], or the book [22], which is very computationally flavored.

The A-hypergeometric approach starts with a finite set A (hence the name) of N lattice points $\mathbf{a}_1, \ldots, \mathbf{a}_N$ in $\mathbb{Z}^r$ such that the rth coordinate of each point is 1 and the $\mathbb{Z}$-span of the $\mathbf{a}_i$ equals $\mathbb{Z}^r$. The $r \times N$-matrix with $\mathbf{a}_1, \ldots, \mathbf{a}_N$ as columns will also be denoted by A. We call it the A-matrix. There is also a vector $\boldsymbol{\alpha} \in \mathbb{Q}^r$ of parameters. The data $A, \boldsymbol{\alpha}$ are the combinatorial data that determine a system of hypergeometric differential equations. We arrive at them in the following way. Let $L \in \mathbb{Z}^N$ be the lattice of relations,

$$(l_1, \ldots, l_N) \in L \iff l_1\mathbf{a}_1 + \cdots + l_N\mathbf{a}_N = \mathbf{0}.$$

Its rank equals $d := N - r$. Choose a basis of L and let B be the $d \times N$ matrix with these basis vectors as rows. We call B a B-matrix. In the literature the transpose of B is usually introduced under the name of Gale dual. Notice that $A.B^t = O$, the $r \times d$ zero matrix. Denote the columns of B by $\mathbf{b}_1, \ldots, \mathbf{b}_N$. Our differential equations and their solutions live in a space of N complex variables $v_1, \ldots, v_N$. Choose $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N)$ such that $\gamma_1\mathbf{a}_1 + \cdots + \gamma_N\mathbf{a}_N = \boldsymbol{\alpha}$ and consider the formal Laurent series

$$\Psi(B, \boldsymbol{\gamma}|\mathbf{v}) = \sum_{\mathbf{m}\in\mathbb{Z}^d} \prod_{j=1}^{N} \frac{v_j^{\mathbf{b}_j\cdot\mathbf{m}+\gamma_j}}{\Gamma(\mathbf{b}_j \cdot \mathbf{m} + \gamma_j + 1)}$$

in the variables $v_1, \ldots, v_N$. Note that, given A and $\boldsymbol{\alpha}$, it is independent of the particular choice of B, but it does depend on the choice of $\boldsymbol{\gamma}$. The system of A-hypergeometric functions is the system of all partial differential equations in the derivations $\partial/\partial v_i$ and coefficients in $\mathbb{C}(v_1, \ldots, v_N)$ that annihilate this formal series. There is a very standard way to write down these operators, but I refrain from writing them here. It turns out that we hardly need them. What is important is that there is a freedom of choice in the solutions $\boldsymbol{\gamma}$ to $\gamma_1\mathbf{a}_1 + \cdots + \gamma_N\mathbf{a}_N = \boldsymbol{\alpha}$ up till shifts in the vector space generated by L. However, the system of differential equations is in general independent of this choice. We denote the system of A-hypergeometric equations by $H_A(\boldsymbol{\alpha})$.

Let us see how this works in the case of the Gauss hypergeometric function. Consider

$$f_{\text{Gauss}} = \sum_{n\geq 0} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)\Gamma(n+1)} z^n,$$

which equals the Gauss hypergeometric series up to a constant factor. We use the Γ-function identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to rewrite this as a series proportional to

$$\sum_{n\geq 0} \frac{z^n}{\Gamma(-n-a+1)\Gamma(-n-b+1)\Gamma(n+c)\Gamma(n+1)}.$$

This is beginning to look like our series Ψ. First we note that we might as well sum over all $n \in \mathbb{Z}$, simply because $1/\Gamma(n+1)$ is zero whenever n is a negative integer. As a last step, replace z^n by $v_1^{-n-a}v_2^{-n-b}v_3^{n+c-1}v_4^n$ to get

$$f_A = \sum_{n\in\mathbb{Z}} \frac{v_1^{-n-a}v_2^{-n-b}v_3^{n+c-1}v_4^n}{\Gamma(-n-a+1)\Gamma(-n-b+1)\Gamma(n+c)\Gamma(n+1)},$$

which has the shape of Ψ defined above with $B = (-1,-1,1,1)$ and $\boldsymbol{\gamma} = (-a,-b,c-1,0)$. The interpretation is that f_A is a series in four variables v_i whose restriction to $v_1 = v_2 = v_3 = 1, v_4 = z$ reproduces the Gauss hypergeometric series. Similarly, the restriction of the A-hypergeometric equations to this line recovers the classical hypergeometric equation. One of the innovations of this approach is that in Ψ we have freedom of choice in the parameter $\boldsymbol{\gamma}$. We can shift it by rational multiples of $(-1,-1,1,1)$ and still have a formal series satisfying the same differential equations. As an exercise one can consider $\boldsymbol{\gamma}' = (c-a-1, c-b-1, 0, 1-c)$, which equals $\boldsymbol{\gamma} + (1-c)(-1,-1,1,1)$. Then write the corresponding $\Psi(B,\boldsymbol{\gamma}'|\mathbf{v})$, set $v_1 = v_2 = v_3 = 1, v_4 = z$, and carry out the above procedure in reverse direction. We get the second series solution

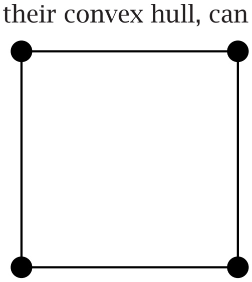
$$z^{1-c}\,{}_2F_1(a-c+1, b-c+1, 2-c|z)$$

of the Gauss hypergeometric equation. Similarly, shifting $\boldsymbol{\gamma}$ such that the first or second component becomes zero reproduces the two basic solutions of the Gauss hypergeometric equation around $z = \infty$.

Thus we can identify the Gauss hypergeometric equation with an A-hypergeometric system with parameters $N = 4, d = 1$ and a B-matrix $(-1,-1,1,1)$. Clearly, $r = N - d = 3$, and an $r \times N$ A-matrix which has B as a kernel can be given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The columns are vectors in $\mathbb{R}^3$ but with the last coordinate $x_3 = 1$. The corresponding points, together with their convex hull, can be pictured as



2F1

in the plane given by $x_3 = 1$.

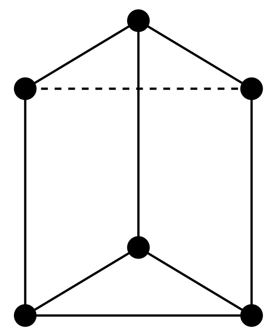
The same point of view works for every classical hypergeometric series we have seen so far. As an example take the series expansion of Appell's F_1 given earlier. There one easily sees that $d = 2$ (the number of variables), $N = 6$ (the number of Γ-factors), and the B-matrix is

$$\begin{pmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

From $r = N - d = 4$ and this B-matrix one deduces the following possible A-matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here is a picture of the corresponding points in the hyperplane $x_4 = 1$:



F1

By playing with the parameters $\boldsymbol{\gamma}$ in the corresponding formal series $\Psi(B,\boldsymbol{\gamma}|\mathbf{v})$, one easily obtains all the local series expansions for solutions of the Appell F_1-system, which are obtained with far more effort in the classical literature.

We now present some of the striking results on A-hypergeometric equations. Given the set $A \subset \mathbb{Z}^r$, we can construct its convex hull, which we denote by $Q(A)$. This is a polytope of dimension $r-1$ which lies in the affine hyperplane $x_r = 1$. We also introduce the positive cone $C(A)$, which is the $\mathbb{R}_{\geq 0}$-span of the vectors from A. We say that the A-hypergeometric system is *resonant* if $\boldsymbol{\alpha} + \mathbb{Z}^r$ and the boundary of $C(A)$ have a nontrivial intersection. We say that the system is *nonresonant* if the intersection is empty.

Theorem 3 (GKZ)**.** *A nonresonant A-hypergeometric system is irreducible.*

The converse of the statement is almost true; see for example [24] or [4]. This irreducibility result, together with its converse, implies all the irreducibility statements that have been compiled over the years for individual systems of hypergeometric equations. Another striking theorem is the following.

Theorem 4 (GKZ)**.** *Around any nonsingular point the analytic solution space of an A-hypergeometric system has finite dimension. Moreover, if the system* *is nonresonant, the dimension equals* $(r-1)!$ *times the Euclidean volume of* $Q(A)$.

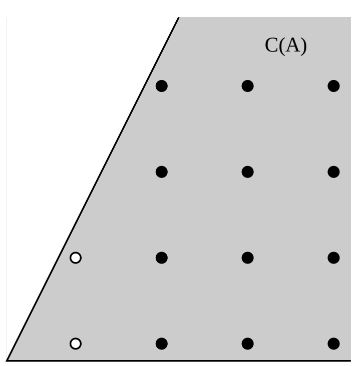
The normalization for the volume of $Q(A)$ is that which assigns volume 1 to the simplex with vertices $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{r-1}$ in $\mathbb{R}^{r-1}$ (the $\mathbf{e}_i$ are the standard basis vectors). The dimension of the solution space is also called the holonomic rank of the system. The theorem, together with the pictures above, tells us immediately that the holonomic rank of the Gauss hypergeometric equation is 2 and, for Appell F_1, it is 3.

Another classic result of GKZ gives a one-to-one correspondence between local series expansions of certain bases of solutions with the regular triangulations of A. Since we do not need it here, I refer instead to the overviews [25], [3].

Section 7. Algebraic Solutions

In this section we extend the interlacing criterion for one-variable equations to the case of A-hypergeometric systems. We make the additional assumption that the sets A are assumed to be *normal*. That is, $C(A) \cap \mathbb{Z}^r$ is equal to the $\mathbb{Z}_{\geq 0}$-span of A. All classical hypergeometric systems satisfy this condition. We will also assume that we are in the *nonresonant* situation.

Let us now consider the set $K(A, \boldsymbol{\alpha}) = C(A) \cap (\boldsymbol{\alpha} + \mathbb{Z}^r)$. Recall that $C(A)$ is the positive cone spanned by the elements of A. We shall call a point $\mathbf{p}$ in this set an *apex-point* if $\mathbf{p} - \mathbf{q}$ does not lie in $C(A)$ for any other $\mathbf{q} \in K(A, \boldsymbol{\alpha})$. Loosely speaking, a point $\mathbf{p}$ is an apex point if it cannot be seen from any other point in $K(A, \boldsymbol{\alpha})$ in a direction in $C(A)$. Below is a two-dimensional sketch. The grey area represents the cone $C(A)$, and the dots form the shifted lattice $\boldsymbol{\alpha} + \mathbb{Z}^2$. The two white dots are the apex points in this example.

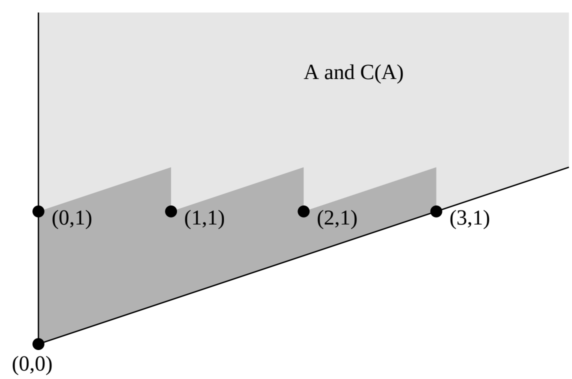


It is an easy lemma to show that the number of apex points is at most the rank of the hypergeometric system. We say that the set of apex points is *maximal* if their number equals the rank. We are now in a position to state the main theorem of this article, which we call the *apex point criterion*.

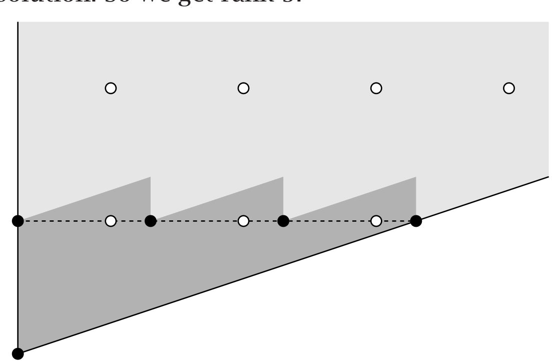
Theorem 5 (Beukers, 2010). *Consider a nonresonant A-hypergeometric system* $H_A(\boldsymbol{\alpha})$ *and assume that A is normal. Suppose the parameters* $\alpha_1, \dots, \alpha_r$ *are rational with common denominator* D. *Then the solution space of the system* $H_A(\boldsymbol{\alpha})$ *contains a nontrivial algebraic function if and only if the apex sets of* $C(A) \cap (r\boldsymbol{\alpha} + \mathbb{Z}^r)$ *are maximal for all integers* r *with* $0 < r < D$ *and* $\gcd(r, D) = 1$.

Because of the irreducibility of $H_A(\boldsymbol{\alpha})$, the occurrence of *one* nontrivial algebraic solution is equivalent to *all* solutions being algebraic. It is a nontrivial exercise to show directly that this condition is equivalent to the interlacing condition we had earlier in the one-variable case.

To illustrate the apex point criterion, we go to the Horn G_3-system, which has the advantage that $A \subset \mathbb{Z}^2$, so we can draw pictures. Recall the series $G_3(a, b|x, y)$. A possible set A consists of the points $(0,1), (1,1), (2,1), (3,1)$. Below is a picture, together with the positive cone $C(A)$. The darker grey area represents the location of the apex points.

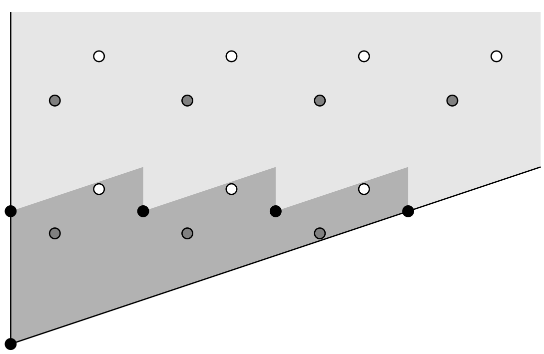


The set $Q(A)$ is the line segment between $(0,1)$ and $(3,1)$. It is the union of three unit intervals, so its volume is 3, which is the rank of the G_3-system. As a fine point, I mention that in the classical literature the rank is 4. This is because there is a classical spurious solution $x^\rho y^\sigma$ with $\rho = -(2a+b)/3, \sigma = -(a+2b)/3$. However, the system in the A-hypergeometric version of G_3 contains more equations and does not allow this solution. So we get rank 3.



The parameters a, b are represented in this picture by the vector $(-2a-b, -a-b)$. Let us first take the case where a, b are rational and $a+b$ is an integer. Thus the second coordinate of $(-2a-b, -a-b)$ is an integer. All points with integral second coordinate not on the boundary of $C(A)$ and in the domain of apex points are pictured below by the dashed line. In addition, the white dots represent an example of a shifted lattice $(-2a-b, -a-b) + \mathbb{Z}^2$.
So one easily sees that the number of apex points is maximal in this case. Of course, the apex points of $r(-2a-b, -a-b) + \mathbb{Z}^2$ are also on this line for any integer r. Hence the conditions of Theorem 5 are fulfilled, and we conclude that $G_3(a, b|x, y)$ is an algebraic function in x, y if a, b are nonintegral rationals with $a + b \in \mathbb{Z}$. Our initial example $a = b = 1/2$ is a particular case of this.

There is another choice that satisfies the apex point criterion. Take $a = 1/2, b = 1/3$ and $a = 1/2, b = 2/3$. The corresponding shifted lattices are depicted by the white and grey dots.



In both cases we see that the number of apex points is maximal, and we can conclude that $G_3(1/2, 1/3|x, y)$ and $G_3(1/2, 2/3|x, y)$ are algebraic. Essentially, up to shifts by integers and interchange of a, b, one can show that these are the only possibilities for $G_3(a, b|x, y)$ to be algebraic. An analysis such as this has been carried out by Esther Bod in her Ph.D. thesis [7]. She extended Schwarz's list of algebraic hypergeometric functions to all two-variable Appell and Horn functions and the several-variable Lauricella systems. Before her work, other authors had already made such an extension to Appell's functions and Lauricella F_D but with entirely different methods.

Section 8. Monodromy

I will finish this article with a few remarks on analytic continuation and monodromy. Consider an A-hypergeometric system $H_A(\boldsymbol{\alpha})$ and the vector space of solutions around a point $P \in \mathbb{C}^N$. Take a closed loop **c** beginning and ending in P and continue the solutions analytically along **c**. After returning in P, the solutions may not have returned to their original value. Instead the solution space underwent a linear transformation. The group generated by these transformations is called the monodromy group of $H_A(\boldsymbol{\alpha})$. Of course, if all solutions of the system are algebraic, the monodromy group is finite. With a bit more effort one can show that the converse is also true.

Unfortunately, determination of the monodromy group for general A-hypergeometric systems is still an open problem. In particular, Theorem 5 cannot be proven by using monodromy calculations. Thus, in order to prove Theorem 5, we need to resort to other methods. Fortunately there is a key in the form of a conjecture.

Conjecture 6 (Grothendieck)**.** *Consider a finite system $\mathcal{L}$ of linear partial differential equations in the independent variables $z_1, \ldots, z_N$ and with coefficients in $\mathbb{Z}[z_1, \ldots, z_N]$. Suppose that $\mathcal{L}$ has finite rank R. Then the solution space of $\mathcal{L}$ consists of algebraic functions if and only if, for almost all primes p, the system L reduced modulo p has R polynomial solutions which are linearly independent over $\mathbb{F}_p[z_1^p, \ldots, z_N^p]$.*

Some remarks are in order. By almost all primes we mean all primes minus a finite set of them. The notation $\mathbb{F}_p$ stands for the integers modulo p, which is a finite field. In characteristic p the partial derivates of elements in $\mathbb{F}_p[z_1^p, \ldots, z_N^p]$ are zero and hence should be considered as constants. Roughly speaking, Grothendieck's conjecture comes down to saying that all solutions of a linear system of partial differential equations are algebraic if and only if almost every reduction mod p has a basis consisting of polynomial solutions modulo p. It would be great if this theorem could be proven, but it has not—not even in the case of ordinary ($N = 1$) differential equations.

However, one particular instance has been proven. In a beautiful paper, Nick Katz [16] has shown that the conjecture is true for systems that are part of a Gauss-Manin system, that is, a system of differential equations associated with periods of a family of algebraic varieties. For $\boldsymbol{\alpha} \in \mathbb{Q}^r$ the A-hypergeometric system $H_A(\boldsymbol{\alpha})$ can be shown to be of the required form, and thus Grothendieck's conjecture is true in this case. The proof of Theorem 5 now consists of showing that the apex point criterion is equivalent to the statement that the solution set of $H_A(\boldsymbol{\alpha})$ modulo p contains the maximal number of polynomial solutions for almost all p.

This proof sketch may also explain why we cannot say anything about the degrees of the algebraic functions involved or the order of the monodromy group. It is clear that at some point one would like to be able to determine monodromy for general A-hypergeometric systems. Of course, in many papers that deal with special instances of several-variable hypergeometric functions, the determination of the monodromy group does play a crucial role. The drawback of this approach is that, for every new set A, one must perform a nontrivial amount of work to determine the singular locus of the system and the fundamental group of its complement and then find some way to determine analytic continuation of so-called Euler integrals defining the A-hypergeometric functions. It is my hope that eventually one can circumvent these intricate considerations and instead use the conceptual simplicity of A-hypergeometric systems to describe their monodromy, for example, through constructions in combinatorial algebra. If this could be possible, the turnaround announced in the introduction of this article might be complete.

Acknowledgments

Many thanks to Tom Archibald, Alicia Dickenstein, Ferdinand Verhulst, and the anonymous referee reader for their critical reading of a preliminary version of this article.

References

[1] P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques: Polynomes d'Hermite*, Paris: Gauthier-Villars, 1926.

[2] F. Beukers, Gauss' hypergeometric function, in: *Arithmetic and Geometry around Hypergeometric Functions*, Progr. Math., 260, Birkhäuser, Basel, 2007, pp. 23–42. Also accessible at `http://www.staff.science.uu.nl/~beuke106/GaussHF.pdf`

[3] ______, Notes on A-hypergeometric functions, in: *Arithmetic and Galois Theories of Differential Equations*, Séminaires et Congrès, Société mathématique de France, vol. 23; 2011, pp. 25–61. Also accessible at `http://www.staff.science.uu.nl/~beuke106/AHGcourse.pdf`

[4] ______, Irreducibility of A-hypergeometric systems, *Indag. Math (N.S.)* **21** (2011), 30–39.

[5] ______, Algebraic A-hypergeometric functions, *Invent. Math.* **180** (2010), 589–610.

[6] F. Beukers and G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$, *Invent. Math.* **95** (1989), 325–354.

[7] E. Bod, Algebraicity of the Appell-Lauricella and Horn hypergeometric functions, *J. Differential Equations* **252** (2012), 541–566.

[8] W. Couwenberg, G. Heckman, and E. Looijenga, Geometric structures on the complement of a projective arrangement, *Publ. Math. Inst. Hautes Études Sci.* **101** (2005), 69–161.

[9] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and nonlattice integral monodromy, *Publ. Math. Inst. Hautes Études Sci.* **63** (1986), 5–89.

[10] B. Dwork, *Generalized Hypergeometric Functions*. Oxford Mathematical Monographs, Oxford University Press, New York, 1990.

[11] C. F. Gauss, Gesammelte Werke, Bd. III, 1866.

[12] I. M. Gelfand, M. I. Graev, and A. V. Zelevinsky, Holonomic systems of equations and series of hypergeometric type, *Doklady Akad. Nauk SSSR* **295** (1987), 14–19 (in Russian).

[13] I. M. Gelfand, A. V. Zelevinsky, and M. M. Kapranov, Equations of hypergeometric type and Newton polytopes, *Doklady Akad. Nauk SSSR* **300** (1988), 529–534 (in Russian).

[14] ______, Hypergeometric functions and toral manifolds, *Functional Analysis and its Applications* **23** (1989), 94–106. Correction in ibid. **27** (1993), 295.

[15] ______, Generalized Euler integrals and A-hypergeometric functions, *Adv. in Math.* **84** (1990), 255–271.

[16] N. M. Katz, A conjecture in the arithmetic theory of differential equations, *Bull. Soc. Math. France* **110** (1982), 203–239; corrections ibid., 347–348.

[17] F. Klein, *Vorlesungen über die hypergeometrische Funktion*, Grundlehren der Mathematischen Wissenschaften, vol. 34, Springer Verlag, Berlin, 1933.

[18] G. Lauricella, Sulla funzioni ipergeometriche a più variabili, *Rend. Circ. Math. Palermo* **7** (1893), 111–158.

[19] B. Riemann, Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha,\beta,\gamma,x)$ darstellbare Funktionen, *Mathematische Werke*, Artikel IV.

[20] F. Rodriguez-Villegas, Integral ratios of factorials and algebraic hypergeometric functions, *Oberwolfach Reports*, July 2005, math/0701362. Also accessible at `http://www.math.utexas.edu/users/villegas/publications/oberwolfach-05.pdf`.

[21] H. A. Schwarz, Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elementes darstellt, *J. Reine Ange. Math.* **75** (1873), 292–335.

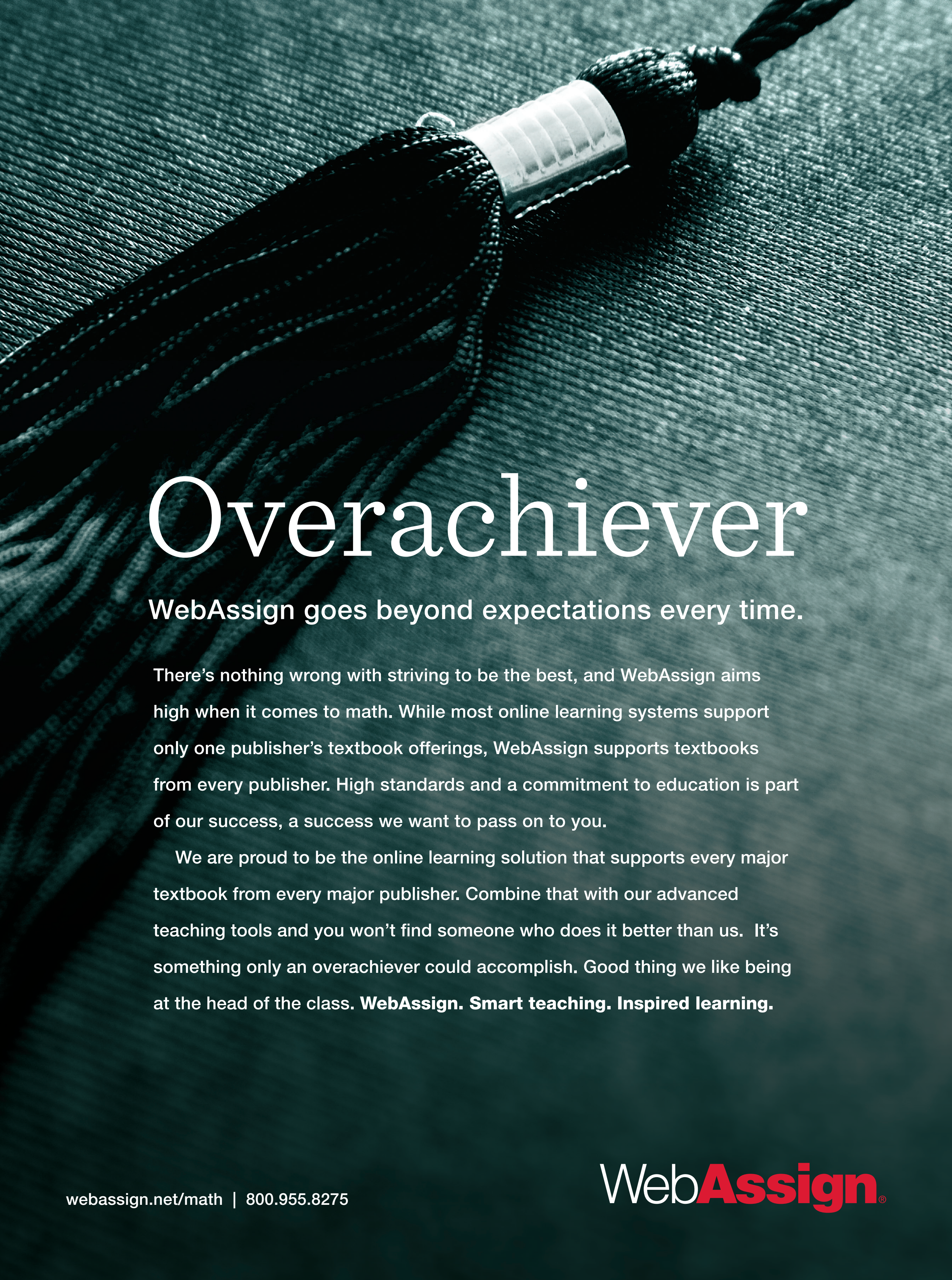
[22] M. Saito, B. Sturmfels, and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer, New York, 2000.

[23] G. C. Shepard and J. A. Todd, Finite unitary reflection groups, *Canadian Math. J.* **6** (1954), 274–304.

[24] M. Schulze and U. Walther, Resonance equals reducibility for A-hypergeometric systems, *Algebra Number Theory* **6** (2012), 527–537.

[25] J. Stienstra, GKZ hypergeometric structures, in: *Arithmetic and Geometry around Hypergeometric Functions*, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 313–371.

[26] J. Thomae, Ueber die höheren hypergeometrischen Reihen, insbesondere über die Reihe: $1+\frac{a_0a_1a_2}{1\cdot b_1b_2}x+\frac{a_0(a_0+1)a_1(a_1+1)a_2(a_2+1)}{1\cdot 2\cdot b_1(b_1+1)b_2(b_2+1)}x^2+\cdots$, *Math. Ann.* **2** (1870), 427–441.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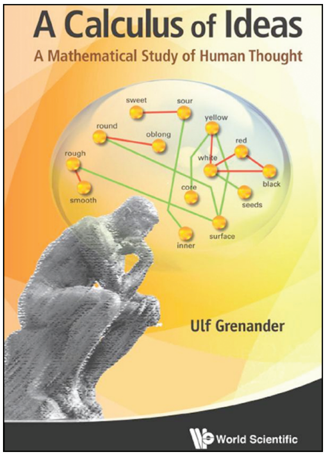
Book Review



A Calculus of Ideas: A Mathematical Study of Human Thought

Reviewed by Andrew I. Dale

A Calculus of Ideas: A Mathematical Study of Human Thought
Ulf Grenander
World Scientific Publishing Co., September 2012
US$55.39, 236 pages
ISBN-13: 978-9814383189

In 1963 Ulf Grenander published his *Probabilities on Algebraic Structures*. That book's introduction describes it as an attempt "to present a unified and coherent theory of the calculus of probability" on things like topological semigroups, topological vector spaces, and algebras. Continuing with some of the ideas emerging from this work (see [6, p. xix]), he presented "a mathematical model intended for the description and analysis of patterns" [4, p. 79] at a conference in Loutraki, Greece, in 1966. This signalled the early days of a study in pattern theory that has lasted for almost fifty years, the latest contribution (in book form) being *A Calculus of Ideas: A Mathematical Study of Human Thought*.

In the preface to this work Grenander states, "This monograph reports a thought experiment with a mathematical structure intended to illustrate the workings of a mind" [p. ix].[1] But his intent is not to present a *general* pattern of human thought. Rather, "We will try to present only a shell, a scheme only, of human thought that will have to be filled with content different for each individual, setting different values to the (many) mind parameters" [p. 4]. The book is thus written from a very personal point of view—even the dog Rufsan, whose name appears frequently, was Grenander's own dog.

Whether the human mind can be understood, and if so in what sense of "understood", is a difficult matter, and Grenander's aim here is to explore one facet of this problem. That aim is summarized in the abstract as follows:

> We shall reduce mind activities to elementary ideas and mental operations on them. Connecting the ideas according to certain rules we will be led to an algebraic structure, one for each cultural sphere. Within such a sphere individuals are characterized by the usage they make of the mental concepts as expressed by a probability measure resulting in a calculus of ideas. [p. xi]

This is a bold and ambitious goal, but, unperturbed, Grenander pursues and reaches it.

By the way, a similar connection between algebra and mental activity was made in the late nineteenth century by Thomas Huxley, who, in his study of David Hume, wrote, "The nervous system is an apparatus for supplying us with a sort of algebra of fact, based on those symbols" [8, p. 96] (by "symbols" here, Huxley meant sensations).

In his Loutraki conference paper Grenander considered the basic "building blocks": signs, configurations and images, as related by similarity, synonymity, and deformation transformations. Using these blocks he discussed the breaking down of patterns into elementary parts, thus obtaining "a grammatical description of the image presented to the observer" [4, p. 79]. This enabled him "to express many notions we need in terms of simple and well-known concepts in elementary algebra" [4, p. 81]. In *A Calculus of Ideas* this earlier analysis is replaced by a synthesis: "Our goal is to build a model of the mind in pattern theoretic terms"

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[1] *References to page numbers without a citation are to the book being reviewed. All words in italics in quotations from this book are given that way in the original.*

DOI: http://dx.doi.org/10.1090/noti1069

[p. 15]. However, the constructions of Patterns of Thought (PoT) in the present work are based on the principles of General Pattern Theory (GPT—Grenander regarded his 1993 book of this name as his chef d'œuvre [7, p. 416]), and Grenander presents a useful summary of the needful.

Grenander's original terms have been changed slightly over the years; thus in the book under review the objects studied are graphs whose vertices are called *generators* and with the vertices joined by *bonds*. The generators, the building blocks whose combination is to be effected, constitute the substance of GPT, the possibility of change in these generators being governed by appropriate operations (e.g., elements of a similarity group).

Of course one also thinks in pictures. Francis Galton [2], for instance, described most carefully the ways (colors, position on a printed page, different fonts, etc.) in which some people "see" numbers in their minds. Plainly, then, ideas need not be presented in words. It is, however, important to note that one should try not to think of the generators as words or the graphs as sentences: "We shall deal with *concepts—not words*" [p. 21]. Furthermore, "*Thinking comes before language, it is the primary mental activity*" [p. 22]. Or as Huxley expressed it more than a century ago, "The elements of consciousness and the operations of the mental faculties, under discussion, exist independently of and antecedent to, the existence of language" [8, p. 121].

Grenander's idea that *ideas* and *words* are distinct is, however, somewhat at variance with the conclusions of other philosophers. For example, Bertrand Russell wrote:

> Words and ideas are, in fact, interchangeable; both have meaning, and both have the same kind of causal relations to what they mean. The difference is that, in the case of words, the relation to what is meant is in the nature of a social convention, and is learnt by hearing speech, whereas in the case of ideas the relation is "natural", i.e., it does not depend upon the behaviour of other people, but upon intrinsic similarity and (one must suppose) upon physiological processes existing in all human beings, and to a lesser extent in the higher animals. [10, p. 111]

Finding a complete theory of the brain to be absent, Grenander bases his work here "on introspection and on what has been learned over the centuries in a less formal setting about the working of the mind by clinicians and what can be found in novels, poetry and plays…Expressed differently, our approach could perhaps be stated as studying the software of the mind rather than the hardware" [pp. 4–5].

In "An Architecture for the Mind", the first of the four parts into which the main text is divided, Grenander considers an Algebra of Thinking, pointing out that his intention is to search for answers to the following questions: "*What are the mental objects that make up the mind? What are the mental operations that act upon these objects? How do these objects combine to form thoughts?*" [p. 16]. Subsequently, he outlines his Algebra of Human Thought with an axiomatic description of the algebra, with thoughts being framed as compositions of generators (primitive ideas). These generators are perhaps best represented by an organizational tree. Thus, for instance, the sentence "John asks what is the cat's name" is represented by a tree with "question3" at the top (the "3" indicating that three arrows emerge from this generator), with arrows leading to "John", "cat", and "name". As further illustration, Grenander presents some trees drawn from passages in Goethe's *Die Wahlverwandtschaften* (Elective affinities).

More specifically, we have a group G of generators, a collection M of modalities (bond values), and a set of regular thoughts (definition omitted in this review) of regularity R (and how many of us, given the generators "agent" and "James", wouldn't feel an almost irresistible urge to choose a bond value of 007). In this setting the set of all such regular thoughts is called $\{MIND(R), P\}$, where P is a probability on $MIND(R)$. $MIND$ then "represents all the thoughts that are possible currently, whether likely to occur or not" [p. 41].

In part two, "Personality of a Mind", Grenander introduces an *idea function* Q on G taking on positive values. A large value of $Q(g)$ indicates that the primitive idea g is likely. A positive *acceptor* or *association* function $A(g_1, g_2)$ is defined which measures the likeliness of direct association between ideas g_1 and g_2 in the thinking of $MIND$. In terms of these two functions and a normalizing function Z (the partition function of statistical mechanics), the *mind equation* is set up for the prior probability $p(thought)$. Note that, in [5, p. 2] Grenander wrote, in connection with a very similar formula, "Z is notoriously difficult to calculate except for the simplest cases." Note too that "the Q's and A's determine the *character of an individual mind*" [p. 45].

"Two Personalities" is the title of the third part. Here Grenander explores the building of a thinking machine called a GOLEM. (In Jewish folklore the Golem is an animated being created from inanimate matter and having human form. Many stories about such creatures exist, but in general Golems are seen as rather unintelligent "beings" which, when given instructions, carry them out literally.) The GOLEM code (using MATLAB as the programming language) is given in an appendix. While the best way to understand GOLEM is to work through the program, Grenander describes in the main text some features of the functioning of MIND in this case: free associations, inferential thinking, associations driven by themes and continuous thought are all investigated. Grenander is refreshingly honest when, in considering the behavior of GOLEM, he writes, "The code is working but it does not work well" [p. 99], but he nevertheless concludes that "*The human mind can be understood*" [p. 100].

Some consideration is given to the selection of the mind parameters G, Q, A, … used in the programming, and Grenander points out that the representation he uses is that of his own mind. Here he introduces more software, called LEGACY.

In his fourth and final part Grenander considers the relation of GOLEM and LEGACY to actual human thinking (or, if you prefer, between MIND and brain). We build our way from simple ideas and their connections through thoughts, weight functions Q, and acceptor functions A to joint probability measures for thoughts. On the way, the conclusion is reached that "*MIND achieves conceptual inference via conditional probabilities*" [p. 128].

In Chapter 7 Grenander indulges himself, becoming "less systematic, more free-wheeling" [p. 153] than before. He considers matters such as introspection, as well as the thinking (better: logical) machine developed in the thirteenth century by Raymond Lully (Raymundus Lullus, Ramon Llull). This device comprised a number of concentric circular discs that, on rotation, yielded theological statements.

The final chapter contains some doubts and certainties and also lists "14 basic rules for thinking about thinking" [p. 165], of which we mention the following: "[1.] Thoughts are made up of discrete entities: ideas. [10.] Thoughts are created probabilistically by the mind equation. [14.] The high level study of thinking should take place in mind space, not physical space" [pp. 165–6].

Grenander suggests that "we feel that we have proposed a cohesive theory with some credibility of how the human mind works" [p. 165]. He might take some consolation from the words of Alan Turing: "We can only see a short distance ahead, but we can see plenty there that needs to be done" [11, p. 460].

A number of papers on matters of conjecture on physical and mathematical topics were published in European journals in the seventeenth and eighteenth centuries in sections entitled *Classe de Philosophie spéculative*. In some respects *A Calculus of Ideas* can also be seen as "speculative philosophy". Indeed, speaking in general of his work in pattern theory, Grenander once said, "My present approach is completely speculative, based on introspection rather than hard data" [7, p. 417].

In several instances Grenander illustrates his investigations here with literary references, and if we go away after having read this book with nothing more than an awakened interest in ourselves, let me suggest one further such reference, this time from the conclusion of J. B. Priestley's essay "All the News", as given in [9, p. 231]: "I am also wondering what, after all, *is* the pattern of my mind."

Readers of the book will encounter ideas from a wide range of areas, including mathematics, artificial intelligence, psychology, neural networks, computer science, and philosophy. But the book is not really a text in any of these disciplines. In his interview with Nitis Mukhopadhyay, Grenander said, "I refer to pattern theory as the intellectual adventure of my life" [7, p. 416]. The adventure is clearly by no means over, and the reader of *A Calculus of Ideas* will fully enjoy the excitement of joining Grenander in his enthralling enterprise.

References

[1] A. Dold and B. Eckmann (eds.), *Symposium on Probability Methods in Analysis*, Lecture Notes in Mathematics, vol. 31. Berlin: Springer-Verlag, 1967.

[2] F. Galton, *Inquiries into Human Faculty and Its Development* (reprint of 1907 edition), London: J. M. Dent & Sons, 1928.

[3] U. Grenander, *Probabilities on Algebraic Structures*, New York: John Wiley & Sons, 1963.

[4] ______, *Toward a theory of patterns*, [1], pp. 79–111, 1967.

[5] ______, Advances in pattern theory, *The Annals of Statistics* **17** (1989), no. 1, 1–30.

[6] ______, *General Pattern Theory: A Mathematical Study of Regular Structures*, Oxford: Clarendon Press, 1993.

[7] U. Grenander and N. Mukhopadhyay, A conversation with Ulf Grenander, *Statistical Science* **21**, (2006), no. 3, 404–426.

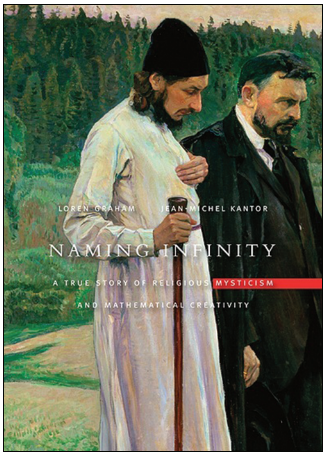
[8] T. H. Huxley, *Hume with Helps to the Study of Berkeley*, London: Macmillan and Co., 1908.

[9] J. B. Priestley, *Apes and Angels. A Book of Essays*, London: Methuen & Co., 1928.

[10] B. Russell, *Human Knowledge, Its Scope and Limits*, London: George Allen and Unwin, 1948.

[11] A. M. Turing, Computing machinery and intelligence, *Mind* **59** (1950), no. 236, 433–460.

Book Review



Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity

Reviewed by Alexey Glutsyuk

Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity
Loren Graham and Jean-Michel Kantor,
Belknap Press of Harvard University Press, 2009.
ISBN-13: 978-06740-329-34.

During the nineteenth century, a foundational crisis in mathematics led to signal events of fundamental importance. The first was the creation of set theory by Georg Cantor at the end of the nineteenth century. The second was the creation of the theory of functions and of measure theory and integration theory by the French trio Emile Borel, René Baire, and Henri Lebesgue. Their works relied heavily on Cantor's set theory. A major contribution to the further development of set theory, function theory, and topology was made by Russian mathematicians: Dmitry Egorov, Nikolai Luzin, and their school, the famous Lusitania.

The book of Jean-Michel Kantor and Loren Graham presents the history of this important period of mathematics through vivid portraits that bring to life the personalities of the mathematicians. The main heros of the book are Cantor, the above-mentioned French trio, and a Russian trio consisting of Egorov, Luzin, and their close friend Pavel Florensky, an extremely talented scientist and engineer and a priest of the Russian Orthodox Church. The book intertwines and links their mathematical research with their cultural and religious backgrounds.

The authors describe the continuous development of mathematics from Cantor to the Russians. Cantor, who was the first to compare different kinds of infinities and prove key results about them, was a Protestant Christian believer and a philosopher of "free mathematics". With Cantor's set theory as a basis, the young French mathematicians Borel, Baire, and Lebesgue created modern measure theory and function theory. But when some difficulties and paradoxes were discovered in the foundations of set theory, they retreated from research in the subject. After that, further research was carried out by the Russians Egorov and Luzin, who were Orthodox Christian believers,[1] and by their students. They were men of great spirit and courage, inspired by their Christian faith, and they attacked difficult classical problems directly. The authors of the book claim that the mathematical research of the Russian trio was inspired by Name Worshipping, which was a heretical current in the Russian Orthodox Church at the beginning of the twentieth century.

More than half of the book is devoted to the Russian mathematicians, who worked during a dramatic period of Russian history: the Revolutions of 1905 and 1917, the Civil War, the Bolsheviks' rise to power, Stalin's terror, …. The book shows how, in these very difficult conditions, Egorov and Luzin managed not only to obtain their famous

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DOI: http://dx.doi.org/10.1090/noti1071

[1] *As shown by the lives of Cantor and the Russian trio—and by the book under review—a religion does not contradict science; the two can complement each other in a harmonious way. The same point of view is expressed in the book* Science and Religion *(Moscow, Obraz, 2007) by Archbishop Luka (Voyno-Yasenetsky) of Crimea (1877–1961), a famous Russian and Soviet surgeon. He shows (with detailed historical analysis and citations) that a majority of the most famous scientists were believers. (Archbishop Luka was persecuted by the Bolsheviks for his Christian faith, before he was awarded the Stalin Prize for his achievements in surgery. He was recently canonized by the Russian Orthodox Church.)*

results, but also to create the outstanding Moscow mathematical school "Lusitania". This school put Moscow on the mathematical map of the world and made it one of the world centers with a maximal concentration of outstanding mathematicians. The majority of famous Moscow mathematicians are descendants of Lusitania.

The authors describe the dramatic personal fates of the Russian trio and the Lusitania students after the Revolution of 1917. During Stalin's terror, Egorov, Florensky, and Luzin were persecuted for their Christian faith. All three are highly admirable, especially Egorov and Florensky, who showed great personal bravery. When the Bolsheviks cruelly persecuted Christian believers, these two men remained believers and did not change their habitudes at all. In fact, Florensky's courage only increased: he caused sensations by always wearing his priest's robe at scientific and engineering meetings. Egorov and Florensky were arrested, and their lives ended tragically: Egorov died in detention, and Florensky was executed. Luzin, a believer and a professor of the old generation, barely escaped a similar destiny after he was accused, publicly and wrongly, of being a traitor. He was more productive scientifically than Egorov and Florensky, though more unstable and less brave. The book describes in an honest way how some of the famous Lusitania students were contradictory people, with good and bad aspects.

I liked very much the authors' choice of the picture on the book's cover, a reproduction of the painting *Philosophers* by the famous Russian painter Mikhail Nesterov. The painting represents two great Russian philosophers and priests, Pavel Florensky and Sergei Bulgakov, whose fates were completely different. Bulgakov was expelled from Russia by the Bolsheviks on the Philosophers' Ship, along with many other philosophers. After his expulsion, he remained extremely active as a philosopher and theologist and published a tremendous number of works. He was one of the key creators of the famous Saint Serge Orthodox Institute in Paris. Many other scientists and philosophers decided to leave Russia after the Revolution of 1917. Florensky was one of the very few theologists and philosophers who decided to stay.[2] He was quite aware of the new political situation in Russia and of what his fate would be. Nevertheless, he decided to stay and to serve Russia with all his energy.

It is very impressive that the authors, not being of Russian origin, know the history of Russia, its mathematics, and its church so deeply. They have done an unimaginable amount of work, including many trips to and across Russia, many interviews, and extensive reading in many archives. I would like to add that the author Jean-Michel Kantor greatly helped young mathematicians from the Former Soviet Union during a very difficult period in the 1990s. At that time, in order to have something to eat, many mathematicians from the Former Soviet Union chose either to leave the country or to leave mathematics in order to earn money. Jean-Michel miraculously organized financial assistance from the French government, thereby saving many young mathematicians, including myself, by allowing them to do only mathematics while staying in their country. I wish to take this occasion to thank him a lot once more.

Returning to the book, I would like to make a remark related to my own preferences. I would have been glad if the book had put less emphasis on details about personal lives and philosophical reasonings about inspiration, and more emphasis on mathematics (explained in a way understandable by nonmathematicians) and on history. The reasoning behind my preference is that, while one can check whether a scientist was inspired by some other scientific work, one cannot check in a rational way whether the inspiration for a person's scientific creativity came from outside of science.

I will now describe in more detail the content of the book. The first chapter presents the origin and history of Name Worshipping, which, according to the authors, was a source of inspiration for the Russian mathematical trio. The main part of the chapter is devoted to a dramatic event of February 1913: the storming of St. Pantaleimon Monastery at Mount Athos by the army of Russian Tsar Nikolai II and the cruel expulsion of the Name Worshipping monks from the monastery.

The second chapter describes the life and mathematical achievements of Georg Cantor and the reception of his theory by other famous mathematicians of his time, including a detailed history of the development of Cantor's set theory and of his famous continuum hypothesis (CH), with links to his philosophy.

The third chapter is devoted to the reception of Cantor's theory in France and the French trio of Borel, Baire, and Lebesgue. It starts with an important event in the history of mathematics, the International Congress in Paris in 1900. At the congress, Hilbert made clear that Cantor's theory would play a major role in the future development

[2] *While in prison after his second arrest, Florensky had the opportunity to emigrate to the Czech Republic together with his family. He refused. Florensky's grandson and biographer, Igumen (Father Superior) Andronik (Trubachev) says that he does not know of any other case in which a Gulag camp prisoner refused to leave the camp. The biography, published in Moscow in 2007, would be interesting to those who would like to learn more about Pavel Florensky.*

of mathematics and placed the continuum hypothesis at the top of his famous problem list. The book describes some of the French trio's contributions that were heavily based on Cantor's set theory: the Heine–Borel theorem, the basis of the future "Borel measure"; the introduction of Borelian and measurable sets; the introduction by Baire of the notion of semicontinuity and his classification of discontinuous limits of continuous functions; and the construction of the "Lebesgue integral". The authors intertwine the development of mathematics in France with descriptions of the cultural spirit and important historical events in France at the beginning of the twentieth century. One is the tragic Dreyfus Affair, in which leading French mathematicians, including Henri Poincaré, actively defended Dreyfus. The authors also describe the lives of the members of the French trio, such as the extremely rich and intense life of Borel, who, besides being a mathematician, played many other roles: Navy minister, mayor of his home town, and participant in the Résistance.

The rest of the chapter focuses on contradictions and paradoxes that appeared in the foundations of Cantor's set theory at the beginning of the twentieth century, such as the difficulties found by Cantor[3] himself in 1895 and various paradoxes, including that of Russell. There is also a discussion of Zermelo's Axiom of Choice and the famous exchange of five letters about it by Borel, Baire, Lebesgue, and Hadamard. This exchange confirmed the critical state of the foundations of mathematics and raised important problems that were partially solved later, including famous incompleteness results by Gödel and Cohen. Even now, not all is resolved.

Chapter four is devoted to the Russian trio: Dmitry Egorov, Nikolai Luzin, and Pavel Florensky. At the end of the nineteenth and the beginning of the twentieth century, Russian mathematics was closely related to philosophy and religion, and the chapter describes the spirit of this time in a remarkable way. The authors discuss the creation of Markov chains, which appeared as a result of a philosophical debate between P. A. Nekrasov and A. A. Markov. Nekrasov, who was a Christian believer and a supporter of the Tsar's power, drew motivation from philosophy related to the question of free will and was thereby led to make overly strong claims about probabilities. Markov, an atheist and a critic of both Tsarist power and the Russian church, constructed his famous chains as a counterexample to Nekrasov's statement. Nikolai Bugaev, the teacher of the three members of the Russian trio and the president of the Moscow Mathematical Society, defended free will and connected it to mathematics. While many mathematicians were frightened by discontinuous functions and called them "monsters", Bugaev called them beautiful and morally strengthening because they freed the human being from "fatalism". The opinion of his student Florensky was that the nineteenth century was intellectually a disaster and that one of its main origins was the "governing principle of continuity", which "was cementing everything in one gigantic monolith".

The book describes the lives of the Russian trio before the Revolution of 1917, their mathematical works, and their personal qualities. It briefly discusses Egorov's first famous achievement in differential geometry, after which "Egorov surfaces" appeared. Egorov is described as a deep Christian believer whose modesty mixed in a remarkable way with his courage to express his disagreement on matters of principle. For example, he signed a petition protesting the 1903 pogrom against Jews in Kishinev even though he had not been politically active. The authors describe Luzin's mental crisis and depression after he saw bloody events in the Revolution of 1905, and they discuss how correspondence with Florensky helped Luzin to recover, become a Christian believer, and return to mathematics. This shows that, for Luzin and for the whole Russian trio, Christian belief was the Pillar and Ground of the Truth (to use words from the title of Florensky's book). Florensky converted to the Christian orthodox faith at the age of seventeen. Eventually, after successfully graduating from Moscow University, he left mathematics, studied at the Theological Academy at Sergiev Posad, and became a priest. Florensky protested the execution of Peter Schmidt, a revolutionary lieutenant of the Tsar's army. He did not share Schmidt's political opinions; he simply opposed capital punishment. After that, Florensky was arrested and held in jail for a week, where he wrote one of his mathematical works.

Chapter five describes the relations between Russian mathematics and "mysticism". Henri Lebesgue spoke of "naming a set". Luzin emphasized the significance of naming in his mathematical work. As already mentioned above, the authors of the book relate the creativity of the Russian trio to Name Worshipping. This was a heretical current in the Russian church. Its supporters practiced the Jesus Prayer and claimed that, after repeating it correctly many times, a person achieves a unity with God: roughly speaking, the name of God is God himself. The authors explain the influence of Name Worshipping on the mathematical creativity of the Russian trio in set theory, basically, by noting the importance of naming in both of them.

[3] *As is mentioned in the book, Cantor escaped from contradictions by naming the objects "too big to be sets" as "Absolute".*

Remark. This is the point of view of the authors of the book under review. From my own point of view, a claim that Name Worshipping was a major inspiration for the Russian trio would seem a bit too strong.

Chapter six gives an impressive description of the spirit and life of the mathematical school founded by Egorov and Luzin, the famous Lusitania. The professors created an atmosphere of openness and closeness. Sometimes Luzin's classes finished in his apartment, with discussions about mathematics, culture, arts, religion, etc., that would continue into the night. Most of the students were young, having joined the Lusitania when they were around seventeen years old. The book describes two key achievements of Lusitanians, namely, the proof of the continuum hypothesis for Borelian sets by Pavel Alexandrov (1915) and the creation of descriptive set theory (1916) by Mikhail Suslin and Nikolai Luzin, after Suslin found a fundamental mistake in Lebesgue's seminal paper of 1905.

Chapter seven describes the dramatic fates of the members of the Russian trio after the Revolution of 1917. Egorov and Florensky were persecuted for having courageously confessed their Christian belief. The authors describe the attacks against Egorov, his arrest and imprisonment, his hunger strike in detention, his hospitalization, and finally his death. A highly admirable person appearing in the book is Nikolai Chebatorev, a famous mathematician though not a Lusitanian. Chebatorev was an atheist and a former Red Army soldier. He and his wife tried, at huge risk to themselves, to save the believer Egorov. The authors discuss the fate of Florensky, who was first arrested in 1928 and sent into exile. After his second arrest in 1933, he never came back. It is absolutely remarkable that, even while in detention, Florensky remained very active and made many important scientific and engineering achievements. He spent the last period of his life as a prisoner at the infamous Solovetsky Gulag camp, where he created a famous iodine enterprise. Much later, after Stalin's death, it was found that he had been executed in 1937.

Luzin was much more cautious than Egorov and Florensky: he became a "secret believer". At some point, he even stopped going to church and restarted only at the end of the Second World War. However, his caution did not save him, as the authorities knew he was a believer and a professor of the old generation. The authors of the book describe the "Luzin Affair", initiated by a communist mathematician, Ernst Kolman. Tragically, many of Luzin's former students and friends, including some famous mathematicians, were against him in the Luzin Affair and agreed that Luzin was a traitor. Luckily, Luzin was saved from imprisonment and death by a letter of support from the famous physicist Peter Kapitsa to Stalin.

Chapter 8 describes the fates of the best-known members of the Lusitania school. It starts with the impressive genealogical tree of Luzin's school, his students, grandstudents, etc., which includes the most famous Russian mathematicians. Traditionally, in Soviet times, the Moscow mathematical school was called the "Luzin school". The name of Egorov as one of its founding fathers was not mentioned at all, because of his arrest and subsequent death. I wish to thank the authors for mentioning this fact and for noting that, even after

not given the credit he deserved. Moscow mathematicians have an obligation to correct this. The authors also present portraits of some of Luzin's famous former students, with an emphasis on Andrei Kolmogorov, Pavel Alexandrov, and Pavel Urysohn. Descriptions of some of their mathematical works are intertwined with information about their personal lives. The friendship of Alexandrov and Urysohn included a very productive collaboration in topology as well as swimming, trips abroad, etc. Urysohn wrote one of his famous mathematical papers on the beach at Batz-sur-mer, just a few days before he drowned while swimming. Alexandrov and Kolmogorov were also friends and collaborators, and they both were among the accusers of their former teacher Luzin in the Luzin Affair. Both of them were asked by the police to write a condemnation of Alexander Solzhenitsyn, calling him a traitor, and both did so. Shortly before his death, Kolmogorov confessed that he would fear the secret police to his last day.

Chapter 9 presents the authors' conclusions about, in particular, the relationship between scientific creativity and religion. There is also a discussion of the history of the further development of the descriptive set theory that Luzin and Suslin created.

This book under review weaves mathematics, history, religion, philosophy, and human drama in a remarkable story that will appeal to a wide audience. It is accessible to nonmathematicians and is also well structured, so that readers interested in specific topics can read parts of the book independently of the rest. I highly recommend this unusual and compelling book.

Meeting of Dreams

Dana Mackenzie

"If you build it, they will come." This quote, which has entered the lexicon thanks to the baseball-themed movie *Field of Dreams*, could also have been the motto for the Heidelberg Laureate Forum (HLF), which was held in the last week of September 2013. It was a "meeting of dreams" that brought the legends of computer science and mathematics together in one place for one week to hobnob with a younger generation of scientists who are just beginning their careers. Fortunately, unlike in the movie, the legendary greats did not have to come back from the dead.

The HLF was the brainchild of Klaus Tschira, the founder of the German software firm SAP and, through his Klaus Tschira Foundation, one of Europe's leading philanthropists. Like the movie's protagonist, Tschira organized his meeting of dreams in his metaphorical backyard: Heidelberg, home to one of the oldest universities in the world, whose cobblestoned beauty was relatively untouched by the wars of the twentieth century. (The wars of the seventeenth century are another matter. Heidelberg Castle, site of the closing banquet, was blown up by the French in 1689 and never fully rebuilt; it remains a picturesque ruin.)

Tschira conceived of the HLF as a counterpart to the Lindau Nobel Laureate Meeting, which has been held every year since 1951. Mathematics and computer science do not, of course, have a Nobel Prize, but Tschira thought that should not be an obstacle to getting the younger and older generations together. "I saw in Lindau how young physicists and chemists and biologists gained inspiration from meeting the Nobel laureates face to face, so I thought, why not do this for mathematics and computer science?" says Tschira.

In 2012 Tschira and Andreas Reuter, the Scientific Chairperson of the HLF, proposed the idea to the organizations that award the top prizes in mathematics and computer science: the Association for Computing Machinery (ACM), which administers the Turing Prize; the Norwegian Academy of Sciences and Letters, which awards the Abel Prize in mathematics; and the International Mathematics Union (IMU), which awards the Fields Medal in mathematics and the Nevanlinna Prize in information sciences. Without the enthusiastic support of all these organizations, "this forum would not have happened," said Tschira at the closing banquet.

The HLF committee contacted every living laureate of the four prizes and sent photographer Peter Badge around the world to take new studio-quality portraits of more than ninety of them. Poster-sized versions of Badge's portraits were displayed prominently in a public square of Heidelberg throughout the week of the meeting, giving mathematicians the rare illusion of being as important as politicians. (The first day of the HLF coincided with Germany's national elections, and the faces of the scientists competed with those of the candidates for public attention.)

So the stage was set; but would the laureates come? The answer from the computer scientists was a resounding yes, as thirty superstars of the subject made the pilgrimage to Heidelberg. They included Vint Cerf, the "chief Internet evangelist" at Google and designer of TCP/IP, the Internet communication protocol; Ronald Rivest, coinventor of the famous RSA public-key encryption system; Butler Lampson, who launched the personal computer from his famous laboratory at Xerox Palo Alto Research Center, and Alan Kay, who helped

Dana Mackenzie is a freelance mathematics and science writer who was invited to attend the Heidelberg Laureate Forum as an "official blogger". His posts as well as those of the other bloggers can be found at `www.scilogs.com/hlf`.

DOI: http://dx.doi.org/10.1090/noti1072

him do it; and Silvio Micali and Shafi Goldwasser, the lone female laureate, who developed a new cryptosystem based on zero-knowledge proofs.

The turnout on the mathematical side was much sparser. Only nine laureates of the Fields Medal and Abel Prize appeared in Heidelberg. The brave nine (just enough for a baseball team!) were Sir Michael Atiyah, Gerd Faltings, Curtis McMullen, Stephen Smale, Endre Szemeredi, Srinivasa Varadhan, Cedric Villani, Vladimir Voevodsky, and Efim Zelmanov. While the dividing line between mathematicians and computer scientists can be fuzzy (the Nevanlinna Prize winners and even some Turing Award winners could be counted in either group), Tschira did not mince words when expressing his disappointment. "Some may feel that there are not many mathematicians present here," he said during the opening ceremony. "Personally, I feel very sorry about this. It is even more deplorable because the jury admitted the same number of young researchers in each discipline." (One hundred young mathematicians and one hundred young computer scientists were selected to attend, out of approximately six hundred applicants.) If the purpose of the meeting was dialogue between the generations, the younger generation of mathematicians could be forgiven for inferring that the older generation did not want to talk with them.

Ingrid Daubechies, the current president of the IMU, agreed with Tschira that the disappearing act of the mathematics laureates was a great opportunity lost for mathematics. Not only did they miss a chance to inspire a new generation of mathematicians, they also missed the chance to reforge the links between mathematics and computer science. "For example, there is so much interaction today between mathematics and machine learning," Daubechies said. "People from geometry, topology, and a lot of different fields are working on ways to find structure in large data sets." The problem of "big data" cuts across all the sciences, and this would have been a chance for top-level mathematicians to engage themselves with it.

One can only speculate on the reasons why the mathematics laureates did not attend. The author of this piece believes that the forum organizers did not fully grasp the cultural differences between computer science and mathematics. The computer science laureates who attended the forum are more than mere colleagues—they are the people who invented the subject, from software to hardware to theory. As such, they are a much closer-knit group than the mathematics laureates. Juris Hartmanis, 1993 Turing Award laureate, pointed out that many of the computer scientists who came to the forum also attended the ACM meeting earlier this summer. By contrast, there is no one meeting that attracts a large proportion of the leading mathematicians. The International Congress of Mathematicians is the closest thing, but it occurs only once every four years. I believe the HLF Foundation needs to work closely with national or regional mathematics organizations as well as with the IMU. Fortunately, 2014 happens to be an "on" year for the International Congress of Mathematicians. The HLF should, of course, take advantage of that quadrennial opportunity to get its message out.

The small turnout of mathematics laureates was only a small blot on an otherwise spectacularly successful meeting. Atiyah's "Advice to a Young Mathematician" address contained a number of points that the young researchers were talking about all week. For example, "Good ideas come from bad lectures," Atiyah said. "If the speaker's proof is terrible, you should start asking yourself, is there a better way?" Voevodsky delivered an inspiring explanation of his research on "univalent foundations of mathematics"—a subject that should be of interest to computer scientists, as it relates to computer verification of proofs. In many lectures, the laureates showed that they are not afraid to venture beyond their original expertise. Witness Voevodsky moving from topology to type theory, or Smale moving from dynamical systems to the protein folding problem.

The formal lectures were only part, and arguably not even the most important part, of the forum. The week offered many chances for social interaction: banquets at the Schwetzingen Castle and Heidelberg Castle, a free biergarten, a long combination boat ride and dinner on the Neckar River. The boat ride seemed to be particularly successful for social interaction, as the laureates and young researchers were sitting literally elbow to elbow for several hours, with no way off the boat and no choice but to mingle!

I'll leave the final word on the forum to the participants, some of whom I contacted by email after returning home.

> The students present were the most impressive bunch I have seen assembled anywhere. [They] had a wide perspective with depth, of science in general. I certainly enjoyed my exchanges with them, from Brazil, from Pakistan, from Iran, etc.
> *—Stephen Smale, Fields Medal laureate*

> I had many wonderful conversations that included common problems we face as a young researcher, such as how do you balance family life, social life, dating and work? How do you gauge progress? Why do you want to be a researcher?... The few researchers I met that didn't enjoy the HLF seemed to approach it the wrong way. They expected it to be a research conference and wanted very specialized, technical

talks by the laureates in their field.
—Kevin Hughes, young researcher, University of Edinburgh

The Heidelberg Laureate Forum was definitely the experience of a lifetime for a student like me….[P]articularly inspiring was the workshop on work-life balance. I had expected it to only be student researchers and was surprised when Avi Wigderson came in and sat down next to me.
—Irene Rae, young researcher, University of Wisconsin

What I liked most about HLF was the opportunity to talk to the laureates and find out a bit more about their personal journeys, to have them listen to me and treat me like a person…. Giving young researchers the opportunity to speak as well would have been great."
—Philip Asare, young researcher, University of Virginia

Although no official announcement has been made at the time of this writing, all signs are that the meeting of dreams will convene again in 2014. Klaus Tschira confirmed that he definitely wants to hold another one next year and to invite both mathematicians and computer scientists again. Says Tschira, "We'll make some improvements next year, because we're still on a learning curve, but for the first try, we did pretty well."

Musings on MOOCs

MOOCs (massive open online courses) are causing a revolution in higher education today. What will be the impact of this revolution on mathematics teaching in colleges and universities? The *Notices* is hosting a discussion of MOOCs, which began in the November 2013 issue with the Opinion column "MOOCs and the future of mathematics" by Robert Ghrist of the University of Pennsylvania. The following three articles continue the discussion. With the aim of continuing the discussion over the coming several months, the *Notices* invites readers to submit short pieces (eight hundred words or less) on the subject of MOOCs in mathematics. Please send contributions to `notices-mooc@ams.org`.

Petra Bonfert-Taylor

When I was invited to teach a MOOC through Wesleyan's contract with Coursera, I was excited because I love exploring technology. But my excitement was soon mixed with questions. Although I have been enthusiastically experimenting with technologically driven curricular pedagogies such as flipped classrooms and blended learning, this was to be a whole different order of public endeavor. Also, for the first time in my experience as a faculty member, I felt that the foundations on which I construct my classes were in question. Is the MOOC model educationally effective, or, more accurately, for which cohort of students is the model effective? Will MOOCs, or their evolutionary spawn, destroy the time-tested and successful educational model in which we work (and believe)? More prosaically, if I agree to teach a MOOC, what should it be about and how would I construct it?

I claim no special creative insight into the problems and opportunities that MOOCs present: indeed, each of the above thoughts, and much, much more, have been hashed over ad infinitum in essays, blogs, papers, and over numerous cups of coffee in department lounges. I find that opinions are strong and firmly held, but it is unfortunate that the amount of practical hands-on experience that informs many of these opinions is so slight. I quickly became fascinated at the possibility of discovering for myself something of the potential of MOOCs through the actual experience of producing a MOOC, and for me that was the decisive motivation for going forward.

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DOI: http://dx.doi.org/10.1090/noti1074

We are, regardless of the ultimate fate of MOOCs, at a moment where hands-on engagement, experimentation, and assessment are required of the professoriate. A central tenet of the Academy is that knowledge should be both free and accessible and that it should be diffused widely. Whether or not MOOCs can or should replace the more traditional modes of teaching has come to be nearly moot in my mind: it is clear that MOOCs have a fabulous potential for providing a platform to spread knowledge widely and inexpensively.

My MOOC, "Analysis of a Complex Kind", has an ambitious goal to take students from the beginnings of complex numbers, via important concepts such as Cauchy theory and an excursion to fractals, all the way to the Riemann Hypothesis and its relation to prime numbers. Squeezed into six weeks, the course cannot offer full rigor 100 percent of the time. My goal in this course is not to simply fill a bucket with knowledge, but also to spark a lasting interest and curiosity in a beautiful corner of mathematics. We can easily make the argument that a greater appreciation for mathematics would have many positive and practical outcomes for K–12 education in the United States, for the economy, for political decision making, etc. Easy access to compelling presentations of mathematics can only help.

Now, a course on complex analysis won't be the one that convinces vast numbers of people of the beauty and utility of mathematics, but I have hopes that it's one (of many) steps in the right direction. It would be selfish, however, to insist that the major impact of MOOCs be felt mainly at our own front door. Statistics show, for example, that Coursera students come from 195 nations, with forty percent living in the developing world. MOOCs have a tremendous potential to provide a service to people around the globe.

I will end on a practical note. Preparing my MOOC has been an all-encompassing adventure. Since the course does not follow a typical undergraduate class on complex analysis, I had to develop the curriculum, write all of the lectures, typeset all of the lectures (this is the part you don't have to do when lecturing with chalk), record, and finally edit the lectures. I was fortunate in that I already had some recording experience from previous experiments with flipped classrooms and blended learning; nonetheless, hundreds of hours went into this process. Next was the creation of in-video quizzes that help students remain engaged with the material and check in with their understanding, and finally, weekly homework assignments and exams. It was a lot of work, and I was continually questioning the tradeoffs I was incurring for doing this. The extra amount of work, in addition to my regular faculty, research, and teaching duties, is overwhelming at times. On one of those days when I felt this acutely I received an email from a prospective student in South Africa. In it, the student wrote: "I can't begin to express just how much it means to me (and to others like me, I'm sure) that people such as yourself are willing to put in so much effort to enable people who haven't got the opportunity to really pursue a mathematics education properly. I look forward to the day, hopefully in the near future, when one will be able to get an, at least approximate, education in maths online as you would if you attended a full time university."

I was moved and humbled by this email. It reminded me of all of the advantages I have had throughout my life and career, advantages in part due to happy accidents such as when, where, and to whom I was born, and the opportunities that were right on my doorstep for the taking. In short, the knowledge on which my career was based was easily accessible to me. Also, it reminded me of how very fortunate I am to be given the opportunity to reach out to people all over the world and offer to them a small portion of a beautiful topic in mathematics—for free.

David M. Bressoud

MOOCs are not the issue. They are simply one manifestation of the proliferation of web-based resources that range from complete lectures to demonstration videos to Q&A sites to online homework. The real issue is how the availability of all of these facilities will affect the future of mathematics and its instruction.

These tools can be enormously freeing. When I do not have to grade routine homework or work through multiple examples of a particular technique, I can make better use of my time with my students to probe their understanding and focus on their questions and difficulties.

MOOCs will be most useful when they provide a range of carefully vetted and coherent materials from which faculty can pick and choose. We are already at the point where a faculty member can say, "After I have introduced the next topic to you in class, watch this lecture, online. Read this material, online or in print. Try these problems, online. Come to class where I will help you through the difficulties you have encountered. Before moving on, check your understanding against these problems, online. See if you can draw from all we have learned to tackle this challenging problem, presented online. Get feedback on your ideas and attempts from your peers, online. Come to class where you will talk through your solution and the entire class will critique it." This approach to education—the instructor managing a rich and immersive experience for the students—is still unusual. I believe that the day is coming when it will be expected. As the future will demand more of teachers, so teachers will be unable to meet these expectations without the aid of web-based resources.

The appeal of this scenario is that it concentrates the human component of education on what humans are best at: drawing on one's own experience to recognize the difficulties that others are encountering and tailoring guidance to help each individual progress. There may come a time when computers can emulate all of the complexities of human interactions of which we are capable. We are still a long way from that day. However, the availability of such web-based agents does mean that there is less need for impersonal instruction delivered by live people. The business model that has supported so many departments of mathematics—a large research faculty with low teaching responsibilities supported by meeting large numbers of students in settings that require minimal interaction—will no longer be sustainable.

There is one other observation I wish to make. Ghrist ["MOOCs and the future of mathematics", *Notices*, November 2013] points out that low-cost platforms can liberate us from the publishing bureaucracy. The problem is that few of us will have the time to develop our own materials, and anyone who searches for such resources online is quickly inundated with options. In an era of overwhelming choices, it is the reputable bundlers who will dominate. MOOCs will continue to play a role in satisfying the needs of the curious and the highly self-motivated, especially those without access to particular courses, but they will be most useful as bundles of high-quality coordinated tools. If they are ever to be self-supporting, this is what will fund them. The big publishers understand this. They no longer produce just fat, full-color books. Big publishers now assemble multifaceted packages that do everything from keeping track of grades to providing videos by leading educators. They know that what sells such packages is their usefulness to the human instructor. This is the direction in which MOOCs will need to evolve.

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Harvey Diamond

Math is hard. It's labor intensive. It requires engagement. The ideas are important. The symbology we use to express those ideas is dense and abstract and takes getting used to. That's why every mathematics text advises the reader, if they want to learn, to grab paper and pencil. And it's why graduate students stay up till the wee hours discussing mathematics with each other. The public knows math is hard, though it has only a hazy view of why. *And the public is tired of math being hard.* College administrators are tired of math being hard. We've gotten the message: You will teach them what they need to know. We'll give you just these resources and you'll get it done.

And now *we're* tired of math being hard.

Class sizes have crept up, or exploded in some cases. Administrators understand $1, 2, 3, \ldots, n$, and after you've got 50 students, why not 100, or 300, or 1,000, or 10,000. Large sections, or large multisection courses, are now managed as much as taught, with uniform pacing, monitored attendance, the handouts handed out, the worksheets worked, the computer assignments clicked through, the sample problems reviewed, and no one reads the textbook anymore. No one has to. Exams are circumscribed by questions the students have already practiced; they think that's only fair—that's how it's always been for them. The exams are graded by teams of harried GTAs and adjuncts, or they're multiple choice and computer graded. Post the grades, compute the averages and distribution, write it all up for your teaching portfolio or for the annual report to the dean. See, math really isn't that hard; it just has to be managed and then assessed. To be sure, this view is something of an exaggeration and overly broad. It does not recognize the many individuals positively affecting outcomes, making extensive and innovative efforts to make this system work as well as it can. But the gradient is clearly uphill, and the negative trends are unmistakable.

Into *that* environment come the MOOCs. I don't mean the MOOC model per se, but rather the idea of learning organized around a comprehensive suite of online materials. Instead of a lecture in a classroom with 50 students, or 300 students, with students copying notes in silence, we have, say, an online video, carefully produced by experts in the field, featuring the best teachers, that can be stopped and reviewed, at any time, at any place, even on your phone. Instead of handouts there are links to online textbooks, supplementary information, deeper analysis and discussion, applications to physical situations, dynamically worked out example problems with commentary, little questions that help you know if you understood the material, challenging questions, and you can take as much time as you need. You can watch it with friends, with group mentoring, with online queries possible, with teaching assistants, with a professor. Anything is possible. Before, you could get away with memorizing problem templates. Now you can be guided and prodded into listening and understanding and articulating because there are unlimited resources and plenty of time—your time instead of limited class time. So, why would all this not be an improvement if done well? Why could this not bring about not just savings in personnel costs and instructional time, but a return to deeper learning, more intellectually accessible, and more widely available, than ever before?

Of course we know what's missing: human feedback, the judgment of written work, which is of particular importance in mathematics, perhaps the necessary discipline of having to be somewhere at a certain time, the sharing of ideas. But there is no reason all that cannot be supplied. If we don't do it, you might find commercial tutoring firms (Kaplan, Princeton Review, etc.) moving in to play such assistive roles. There are many positive possibilities for education here and an ability to accommodate a wide range of student backgrounds, abilities, and learning preferences.

So bring it on, but do it right. There are people and technology companies behind these recent efforts who know how to get things done and who know quality.

There is not space here for other important issues: Making sure that standards are maintained in content and testing; how credit will be granted and recognized; what the longer term effect will be on colleges and universities and our entire higher education system; and, for us in particular, what the mathematics faculty will look like in ten or twenty years if universities start to think they can get along without us.

I must briefly address one more issue: using MOOC platforms in K–12 education. If we're going to open the knowledge of mathematics to everyone in the world, how about teachers and children, who labor with textbooks and materials of limited quality and depth? There are laudable learning aids online, such as Khan Academy, but why not a comprehensive curriculum? A relatively small investment by the National Science Foundation or the Department of Education or even private philanthropy could get that set up, free for any student or school system to use. Think what that might do to open up opportunity to everyone.

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Persistent Learning, Critical Teaching: Intelligence Beliefs and Active Learning in Mathematics Courses

Benjamin Braun

Over the past several decades, social psychologists have built a substantial body of evidence showing that beliefs about the nature of intelligence have a major impact on motivation and achievement. Such beliefs generally fall into one of two types: The belief that each individual has a fixed ability in mathematics is referred to as a *fixed intelligence* belief. The belief that individuals are capable of continually developing their mathematical abilities through persistence and effort is referred to as a *malleable intelligence* belief. This article surveys recent research regarding the positive role malleable intelligence beliefs play for mathematics students and highlights connections between such beliefs and active learning techniques.

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The author dedicates this article to his parents, Jim and Pam Braun, for their unwavering belief in the potential of at-risk youth and children with disabilities.

Thanks to Jeff Blanchard, Laura Braun, David Bressoud, Steve Klee, Carl Lee, Katy Ott, and the anonymous reviewers for their thoughtful and extensive comments and suggestions.

DOI: http://dx.doi.org/10.1090/noti1066

Beliefs about Intelligence

Carol Dweck, currently a professor of psychology at Stanford, was one of the first researchers to emphasize the impact of intelligence beliefs on student learning. Much of Dweck's research has studied the fascinating effects of different types of praise on children [1, Chapter 3]. For example, her work shows that praise focused on developing malleable intelligence beliefs positively affects subsequent student achievement, while praise that cultivates fixed intelligence beliefs has the opposite effect. Dweck and her coauthors have recently turned their attention to the positive role malleable intelligence beliefs play in mathematics education, particularly for girls and women. Their research is interesting, because fixed intelligence beliefs are commonly expressed in mathematics; most of us regularly hear student comments such as "I'm bad at math" or hear teachers say their students are "hopeless at math."

In a thought-provoking survey article [4], Dweck describes how the math performance gap between male and female eighth-grade students is almost eliminated when one restricts comparison to those students holding malleable intelligence beliefs. She describes a similar phenomenon occurring with undergraduates in a pre-med chemistry course. In a study of undergraduate students in calculus courses [5], Good, Rattan, and Dweck find that female students who perceive malleable intelligence beliefs in their environment have a stronger sense of belonging in mathematics than female students who perceive an environment dominated by fixed intelligence beliefs. Further, students who themselves hold malleable intelligence beliefs are more likely to perceive the same in their learning environments. This study strongly suggests that the resulting sense of belonging has a positive impact on students' desire to pursue mathematics further.

In another recent work, Rattan, Good, and Dweck [10] provide evidence that ,associated withwhen college-level mathematics teachers hold a fixed intelligence belief regarding their students and when this belief is reflected in student interactions, there is a negative impact on student self-beliefs and motivation. This belief is often conveyed by teachers through the use of well-intentioned "comforting" language, e.g., telling students that some people are not as good at math as others or that it is okay to not do well in math or telling a student that they will not be called on in class because the teacher knows it is stressful for them. Previous evidence of the effect of teacher feedback on student beliefs is found in work by Meyer et al. [8]. They find that praise given to students on easy tasks as a form of encouragement is interpreted by students as an evaluation of low competence, while constructive criticism given to students on difficult tasks is interpreted as an evaluation of high competence. In other words, students are adept at detecting when they are being patronized or implicitly put down; when this happens, what is intended to be supportive or comforting becomes instead demoralizing and insulting. Meanwhile, teachers convey a sense of confidence in their students when they provide appropriate challenges followed by critical analysis of student work.

Active Learning

One way to create a classroom environment that cultivates malleable intelligence beliefs, supporting students through sequences of challenges and critical responses, is the use of active learning techniques. These include many well-known methods: e.g., cooperative learning, peer-based instruction, guided discovery, and inquiry-based learning. While active learning techniques are not all identically effective and while they require persistence by teachers to be successfully applied, a growing body of evidence suggests that such methods generally have a positive effect on student learning and attitudes in mathematics [7], engineering [9], and other STEM disciplines [11, Chapter 6]. Active learning has also been studied extensively at the K–12 level; hence these methods deserve attention from mathematicians teaching courses aimed at future K–12 teachers.

One of the reasons that active learning methods are beneficial is that they create classroom environments focused on the intellectual and emotional growth of students, as opposed to teachers' transmission of various truths to an audience. Bain [2, Chapter 3] observes that the cultivation of such a classroom environment is a characteristic of highly effective college teachers across disciplines. Consider for example the familiar situation where a student can provide a formally correct statement of a definition or theorem that has been presented to him, yet the same student does not have a rich context in which to understand this formal statement: e.g., examples, nonexamples, related problems, standard misconceptions, etc. In this case, despite the successful transmission of a formal mathematical statement by a teacher, genuine student intellectual growth does not occur.

A useful language with which to describe this situation is provided by Tall and Vinner [12], who introduce the notions of *concept definition* and *concept image* to distinguish between the formal definition of a mathematical concept and the total individual cognitive structure associated with the concept, respectively. The important point of their work is that people tend to do creative mathematical thinking using their concept images rather than formal concept definitions; this underlies the typical mathematician's habit of generating small examples to get a feel for a problem. Mason [3, Chapter 20] expands the idea of concept image into a threefold framework consisting of awareness, emotion, and behavior that teachers can use when preparing to teach a concept. This extended framework supports the idea that teachers should seek to develop students' mathematical understanding in a multifaceted manner extending beyond the presentation of formally correct mathematics.

These ideas apply to mathematical practices as well as mathematical concepts. For example, individual understanding of "proof" changes dramatically over time [6, Chapter 2], and our shared standard and style for proof are a learned social construct. Selden [6, Chapter 17] surveys a broad range of research on expectations for student performance regarding proofs in K–12, undergraduate, and graduate courses, along with associated challenges for students in transition between these environments. In Seldon's article one finds multiple examples of active learning methods, from universities in almost a dozen different countries, focused on the teaching of mathematical reasoning and proof.

Active learning methods encourage a broad and gradual mathematical development on the part of students, including both mathematical content and mathematical practice, while implicitly emphasizing the importance of persistence and effort. It is immediate that such methods go hand-in-hand with malleable intelligence beliefs. Where active learning techniques underscore the malleable nature of learning, malleable intelligence beliefs provide an educational worldview in which students expect setbacks and failures as an ordinary part of the process of understanding.

Conclusion

It is important to remember that there is no "silver bullet" in teaching and that successful use of unfamiliar teaching methods requires patience and occasional failure. Also, as in all human endeavors, the exceptions break the rule: a lecture can sometimes help a student past his confusion; fixed intelligence beliefs can be beneficial in certain circumstances; and, obviously, students with malleable intelligence beliefs are not guaranteed success in a given mathematics course, since success in a course depends on many factors.

Despite these exceptions, the evidence given here supports cultivating malleable intelligence beliefs in our students and using active learning methods in our classrooms. It suggests that these activities have the potential to engage our students more fully, motivating them to seek deep understanding. These activities can help teachers develop a more nuanced vision of their students' experiences than traditional teaching methods. At their best, these activities can serve as catalysts to develop teachers and learners who value small failures as a step toward big success, who recognize learning as a complex process without end, and who believe in the potential of every student.

References

[1] J. Aronson (ed.), *Improving Academic Achievement: Impact of Psychological Factors on Education*, Academic Press, 2002.

[2] K. Bain, *What the Best College Teachers Do*, Harvard University Press, 2004.

[3] M. Carlson and C. Rasmussen (eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, MAA Notes #73, Mathematical Association of America, 2008.

[4] C. S. Dweck, Is math a gift? Beliefs that put females at risk. In S. J. Ceci and W. M. Williams (eds.), *Why Aren't More Women in Science? Top Researchers Debate the Evidence,* American Psychological Association, 2006.

[5] C. Good, A. Rattan, and C. S. Dweck, Why do women opt out? Sense of belonging and women's representation in mathematics, *Journal of Personality and Social Psychology* **102** (2012), 700–717.

[6] G. Hanna and M. de Villiers (eds.), *Proof and Proving in Mathematics Education*, Springer, 2012.

[7] S. Laursen, M. Hassi, M. Kogan, A. Hunter, and T. Weston, *Evaluation of the IBL Mathematics Project: Student and Instructor Outcomes of Inquiry-Based Learning in College Mathematics*, Assessment & Evaluation Center for Inquiry-Based Learning in Mathematics (report to the Educational Advancement Foundation and the IBL Mathematics Centers), University of Colorado, Ethnography & Evaluation Research, 2011. `http://www.colorado.edu/eer/research/steminquiry.html`

[8] W. Meyer, M. Bachmann, U. Biermann, M. Hempelmann, F. Ploger, and H. Spiller, The informational value of evaluative behavior: Influences of praise and blame on perceptions of ability, *Journal of Educational Psychology* **71**, no. 2 (1979), 259–268.

[9] M. Prince, Does active learning work? A review of the research, *J. Engr. Education* **93**, no. 3, (2004), 223–231.

[10] A. Rattan, C. Good, and C. S. Dweck, "It's ok, not everyone can be good at math": Instructors with an entity theory comfort (and demotivate) students, *Journal of Experimental Social Psychology* **48** (2012), 731–737.

[11] S. R. Singer, N. R. Nielsen, and H. A. Schweingruber (eds.), *Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering*, National Research Council, The National Academies Press, 2012.

[12] D. Tall and S. Vinner, Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics* **12** (1981), 151–169.

Mathematics People

James and Marilyn Simons Awarded Carnegie Medal

James H. and Marilyn H. Simons, chairman and president, respectively, of the Simons Foundation, have been awarded the Andrew Carnegie Medal for Philanthropy for 2013.

The Simons Foundation is dedicated to advancing the frontiers of research in mathematics and the basic sciences. Its philanthropic activities include a major research initiative on the causes of autism and the establishment of an institute for research in mathematics and theoretical physics. The Foundation is particularly interested in the growing interaction between the physical and life sciences and has established and endowed several such research programs at universities and institutions both in the United States and abroad.

James Simons, former chair of the mathematics department at the State University of New York at Stony Brook, is the founder and chair of Math for America, a nonprofit organization with a mission to significantly improve mathematics education in U.S. public schools. He is the founder and retired CEO of Renaissance Technologies LLC, a quantitative investment firm. He is a trustee of Brookhaven National Laboratory, the Institute for Advanced Study, Rockefeller University, and the Mathematical Sciences Research Institute in Berkeley and is also a member of the board of the MIT Corporation and chair emeritus of the Stony Brook Foundation.

Marilyn Simons, an economist, has actively championed and worked on behalf of many nonprofit organizations in New York City and her native Long Island. She is currently vice chairman of the board of Cold Spring Harbor Laboratory. In addition, she has been involved in improving academic options for children with special needs and youth in underserved communities. She is treasurer and former president of the Learning Spring School, a New York City school for children with diagnoses on the autism spectrum. She is also a member of the board of trustees at the East Harlem Tutorial Program, an after-school program in New York City.

The Carnegie Medal was established in 2001 to mark the centennial of Andrew Carnegie's retirement from business and the beginning of his efforts to reinvest his wealth in a way that would "do real and permanent good in this world."

—From a Carnegie Foundation announcement

Scholze Awarded 2013 SASTRA Ramanujan Prize

Peter Scholze of the University of Bonn has been awarded the 2013 SASTRA Ramanujan Prize, given annually for outstanding contributions by young mathematicians to areas influenced by the work of Srinivasa Ramanujan. The age limit for the prize has been set at thirty-two because Ramanujan achieved so much in his brief life of thirty-two years. The prize was awarded at the International Conference on Number Theory and Galois Representations at SASTRA University in Kumbakonam (Ramanujan's hometown), where the prize has been given annually.

The prize citation reads as follows: "Peter Scholze is awarded the 2013 SASTRA Ramanujan Prize for his pathbreaking contributions that lie at the interface of arithmetic algebraic geometry and the theory of automorphic forms, and especially in the area of Galois representations. The prize recognizes his unusually original and remarkable master's thesis at the University of Bonn in which he gave a new proof of the local Langlands conjecture for p-adic local fields that was published in two papers in *Inventiones Mathematicae* in 2013, and generalizations of these methods to l-adic representations of Shimura varieties in collaboration with Sug-Woo Shin that appeared in two papers in the *Journal of the American Mathematical Society* in 2013. The prize recognizes his revolutionary Ph.D. thesis at the University of Bonn in which he initiates the theory of perfectoid spaces, to prove an important case of the weight monodromy conjecture of Fields Medalist Pierre Deligne, and discusses far-reaching ramifications of the study of perfectoid spaces in Hodge theory, and spectral sequences, as described in his monumental paper in *Publications Mathématiques de l'IHES*, and his 2013 paper in *Forum of Mathematics, Pi*. The prize also recognizes his important collaboration with Jared Weinstein extending earlier work of Rapoport-Zink on moduli spaces of p-divisible groups. Finally the prize recognizes his most recent work on the existence of Galois representations for mod p cohomology of locally symmetric spaces for linear groups over a totally real or CM fields that has surprising implications on the Betti cohomology of locally symmetric spaces."

Scholze was born in Dresden, Germany, in 1987 and received his Ph.D. from the University of Bonn in 2012. As a student he was a winner of three gold medals and one silver medal at the International Mathematics Olympiads. He was appointed a five-year Clay Research Fellow in 2011.

The members of the 2013 SASTRA Ramanujan Prize Committee were Krishnaswami Alladi (Chair, University of Florida), Kathrin Bringmann (University of Cologne), Roger Heath-Brown (Oxford University), David Masser (University of Basel), Barry Mazur (Harvard University), Ken Ribet (University of California, Berkeley), and Ole Warnaar (University of Queensland). Previous winners of the SASTRA Ramanujan Prize are Manjul Bhargava and Kannan Soundararajan (2005, two full prizes), Terence Tao (2006), Ben Green (2007), Akshay Venkatesh (2008), Kathrin Bringmann (2009), Wei Zhang (2010), Roman Holowinsky (2011), and Zhiwei Yun (2012).

—Krishnaswami Alladi,
University of Florida

Aistleitner Awarded 2013 Information-Based Complexity Young Researcher Award

Christoph Aistleitner of Technische Universität Graz, Austria, has been awarded the 2013 Information-Based Complexity Young Researcher Award. The award is given for significant contributions to information-based complexity by a young researcher who has not reached his or her thirty-fifth birthday by September 30th of the year of the award. The award consists of US$1,000 and a plaque.

—Joseph Traub,
Columbia University

Benkart Awarded Noether Lectureship

Georgia Benkart of the University of Wisconsin, Madison, has been named the 2014 Noether Lecturer by the Association for Women in Mathematics (AWM). She was honored "for her prominence as an international leader in the structure and representation theory of Lie algebras and related algebraic structures." She received her Ph.D. from Yale University and has published more than 100 journal articles in the fields of modular Lie algebras, combinatorics of Lie algebra representations, graded algebras and superalgebras, and quantum groups and related structures. She is a former president of AWM and is currently an associate secretary of the AMS. She will deliver the Noether Lecture at the 2014 Joint Mathematics Meetings in Baltimore, Maryland. The lectureship, named in honor of Emmy Noether, honors women who have made fundamental and sustained contributions to the mathematical sciences.

—From an AWM announcement

Miller Awarded Knuth Prize

Gary L. Miller of Carnegie-Mellon University has been awarded the 2013 Donald E. Knuth Prize for his "algorithmic contributions to theoretical computer science. His innovations have had a major impact on cryptography as well as number theory, parallel computing, graph theory, mesh generation for scientific computing, and linear system solving." According to the prize citation, "Miller introduced the first efficient algorithm to test whether a number is prime (divisible evenly only by 1 and itself). The generation of large primes is an essential part of the RSA public key cryptosystem, on which much of today's Internet commerce depends." He has also made significant contributions to the theory of isomorphism testing and set up the theoretical foundations for mesh generation, and he was the first to design meshing algorithms with near-optimal runtime guarantees. He received his Ph.D. from the University of California, Berkeley.

The Knuth Prize, named in honor of Donald Knuth of Stanford University, is given every eighteen months by the Association for Computing Machinery (ACM) Special Interest Group on Algorithms and Computation Theory (SIGACT) and the Institute of Electrical and Electronics Engineers (IEEE) Technical Committee on the Mathematical Foundations of Computing. It carries a cash award of US$5,000.

—From an ACM announcement

Campbell Awarded 2013 CMS Graham Wright Award

Harold Edward Alexander (Eddy) Campbell of the University of New Brunswick has been named the recipient of the 2013 Graham Wright Award for Distinguished Service by the Canadian Mathematical Society (CMS). The award is given annually to an individual who has made significant contributions to the Canadian mathematical community and in particular the CMS.

—From a CMS announcement

Prizes of the Australian Mathematical Society

The Australian Mathematical Society has awarded two major prizes for 2013. Craig Westerland of the University of Melbourne was awarded the Australian Mathematical Society Medal for his research in algebraic topology. The prize is awarded to a member of the Society under the age of forty years for distinguished research in the mathematical sciences. A significant portion of the research work should have been carried out in Australia. Terry O'Kane of Commonwealth Scientific and Industrial Research Organisation (CSIRO) Marine and Atmospheric Research was awarded the J. H. Michell Medal for outstanding original contributions to difficult and important problems in applied mathematics. The medal honors outstanding new researchers.

—From an Australian Mathematical Society announcement

2013 Golden Goose Awards Announced

Mathematicians Lloyd Shapley and the late David Gale, along with economist Alvin Roth, have been selected as recipients of the 2013 Golden Goose Award, given to researchers "whose federally funded research may not have seemed to have significant practical applications at the time it was conducted but has resulted in tremendous societal and economic benefit." They were recognized for work that led to the national kidney exchange and other programs, such as the national matching program for new medical residents and hospitals. The kidney donor matching program grew out of federally funded research on matching pairs by Gale and Shapley. Their work, known as the Gale-Shapley deferred acceptance algorithm, used marriage as a model and provided the foundation for economist Roth to develop numerous practical market applications. Their subsequent research on matching was the basis for today's kidney matching program that has saved so many lives. Shapley and Roth shared the 2012 Nobel Prize in Economics for their work. The AMS is a financial supporter of the Golden Goose Award.

—From the Golden Goose Award website

Mathematics Opportunities

AMS-Simons Travel Grants Program

Starting February 1, 2014, the AMS will begin accepting applications for the AMS-Simons Travel Grants program, with support from the Simons Foundation. Each grant provides an early-career mathematician with US$2,000 per year for two years to reimburse travel expenses related to research. Sixty new awards will be made in 2014.

Applications will be accepted starting February 1, 2014, through `www.mathprograms.org`. The deadline for 2014 applications is **March 31, 2014**.

Applicants must be located in the United States or be U.S. citizens to apply. For complete details of eligibility and application instructions, visit `www.ams.org/programs/travel-grants/AMS-SimonsTG` or contact Steven Ferrucci, email `ams-simons@ams.org`, telephone 800-321-4267, ext. 4113.

—AMS announcement

Proposal Due Dates at the DMS

The Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) has a number of programs in support of mathematical sciences research and education. Listed below are some of the programs and their proposal due dates for the year 2014. Please refer to the program announcement or contact the program director for more information.

January 6, 2014 (letter of intent): Industry/University Cooperative Research Centers Program

January 10, 2014 (full proposal): Algorithms for Threat Detection (ATD)

January 17, 2014 (full proposal): Secure and Trustworthy Cyberspace

January 22, 2014 (full proposal): Catalyzing New International Collaborations

January 23, 2014 (full proposal): Major Research Instrumentation Program

March 4, 2014 (full proposal): Industry/University Cooperative Research Centers Program

March 17, 2014 (full proposal): Innovation Corps Teams Program

April 22, 2014 (full proposal): Catalyzing New International Collaborations

April 25, 2014 (full proposal): NSF-CBMS Regional Research Conferences in the Mathematical Sciences

May 23, 2014 (full proposal): Research Experiences for Undergraduates (REU), Antarctica Program

June 3, 2014 (full proposal): Mentoring Through Critical Transition Points in the Mathematical Sciences (MCTP); Research Training Groups in the Mathematical Sciences

June 15, 2014 (full proposal): Workforce Program in the Mathematical Sciences

June 16, 2014 (full proposal): Innovation Corps Teams Program

July 7, 2014 (full proposal): Innovation Corps Sites Program

July 22, 2014 (full proposal): Catalyzing New International Collaborations

August 19, 2014 (full proposal): International Research Experiences for Students (IRES)

August 27, 2014 (full proposal): Research Experiences for Undergraduates (REU)

September 15, 2014 (full proposal): Innovation Corps Teams Program; Joint DMS/NIGMS Initiative to Support Research at the Interface of the Biological and Mathematical Sciences (DMS/NIGMS)

September 19, 2014 (full proposal): Focused Research Groups in the Mathematical Sciences (FRG); Secure and Trustworthy Cyberspace

September 26, 2014 (full proposal): Industry/University Cooperative Research Centers Program

September 30, 2014 (full proposal): Computational and Data-Enabled Science and Engineering (CDS&E)

October 7, 2014 (full proposal): Analysis; Combinatorics; Foundations

October 10, 2014 (full proposal): Algebra and Number Theory

October 15, 2014 (full proposal): Mathematical Sciences Postdoctoral Research Fellowships

For further information see the website `http://www.nsf.gov/funding/pgm_list.jsp?org=DMS&ord=date`. The mailing address is Division of Mathematical Sciences, National Science Foundation, Room 1025, 4201 Wilson Boulevard, Arlington, VA 22230. The telephone number is 703-292-5111.

—From the DMS website

EDGE for Women Summer Program

The EDGE Program (Enhancing Diversity in Graduate Education) was launched in 1998 with the goal of strengthening the ability of women students to successfully complete graduate programs in the mathematical sciences. The program has a special emphasis on inclusion of women from underrepresented groups.

EDGE sponsors a summer program consisting of two core workshops in analysis and algebra/linear algebra, as well as shorter workshops in vital areas of mathematical research in pure and applied mathematics. The program also promotes networking and community by facilitating collaborative problem solving, by including workshop facilitators from institutions across the country, guest speakers from academia and industry, and graduate student mentors. A follow-up mentoring program and support network is established with the participants' respective graduate programs.

Applicants to the program should be women who are either graduating seniors who have applied to Ph.D. programs in the mathematical sciences or recent recipients of undergraduate degrees who are now entering Ph.D. programs. All applicants should have completed standard junior-senior level undergraduate courses in analysis and abstract algebra. Women from groups underrepresented in the mathematical sciences are especially encouraged to apply. Final acceptance to the program is contingent upon acceptance to a Ph.D. program in the mathematical sciences.

Pending funding, the EDGE 2014 Summer Program will be held at Harvey Mudd College in Claremont, California, from June 2 through June 27. A stipend plus travel, room, and board will be awarded to participants. The application deadline for the program is **March 3, 2014**. Acceptance to the program will be announced in April.

For further information, visit the website `http://www.edgeforwomen.org/`.

—EDGE for Women announcement

NSF Major Research Instrumentation Program

The National Science Foundation (NSF) Major Research Instrumentation (MRI) program seeks to increase access to shared scientific and engineering instruments for research and research training in institutions of higher education, museums, science centers, and not-for-profit organizations in the United States. This program especially seeks to improve the quality and expand the scope of research and research training in science and engineering by providing shared instrumentation that fosters the integration of research and education in research-intensive learning environments. Proposals must be for either acquisition or development of a single instrument or for equipment that, when combined, serves as an integrated research instrument (physical or virtual).

Proposals may be submitted only by institutions of higher education in the United States or its territories or possessions or by nonprofit organizations such as museums, science centers, observatories, research laboratories, professional societies, and similar organizations involved in research or educational activities. The deadline for full proposals is **January 23, 2014**. For more information see `http://www.nsf.gov/pubs/2013/nsf13517/nsf13517.htm`.

—From an NSF announcement

NSF Algorithms for Threat Detection

The Division of Mathematical Sciences (DMS) at the National Science Foundation (NSF) has formed a partnership with the Defense Threat Reduction Agency (DTRA) to develop the next generation of mathematical and statistical algorithms for the detection of chemical agents, biological threats, and threats inferred from geospatial information. Proposals are solicited from the mathematical sciences community in two main areas: mathematical and statistical techniques for genomics and mathematical and statistical techniques for the analysis of data from sensor systems. The deadline for full proposals is **January 10, 2014**. For more details, see `http://www.nsf.gov/pubs/2012/nsf12502/nsf12502.htm`.

—From an NSF announcement

National Academies Research Associateship Programs

The Policy and Global Affairs Division of the National Academies is sponsoring the 2014 Postdoctoral and Senior Research Associateship Programs. The programs are meant to provide opportunities for Ph.D., Sc.D., or M.D. scientists and engineers of unusual promise and ability to perform research at more than 100 research laboratories throughout the United States and overseas.

Full-time associateships will be awarded for research in the fields of mathematics, chemistry, earth and atmospheric sciences, engineering, applied sciences, life sciences, space sciences, and physics. Most of the laboratories are open to both U.S. and non-U.S. nationals and to both recent doctoral recipients and senior investigators. Amounts of stipends depend on the sponsoring laboratory. Support is also provided for allowable relocation expenses and for limited professional travel during the period of the award.

Awards will be made four times during the year, in February, May, August, and November. The deadline for application materials to be postmarked or for electronic submissions for the February 2014 review is **February 1, 2014**. Materials for the May review are due **May 1, 2014**; for the August review, **August 1, 2014**; and for the November review, **November 1, 2014**. Note that not all sponsors participate in all four reviews. Applicants should refer to the specific information for the laboratory to which they are applying.

For further information and application materials, see the National Academies website at `http://sites.nationalacademies.org/PGA/RAP/PGA_050491` or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email `rap@nas.edu`.

—From an NRC announcement

CAIMS/PIMS Early Career Award

The Canadian Applied and Industrial Mathematics Society (CAIMS) and the Pacific Institute for Mathematical Sciences (PIMS) sponsor the Early Career Award in Applied Mathematics to recognize exceptional research in any branch of applied mathematics, interpreted broadly. The nominee's research should have been conducted primarily in Canada or in affiliation with a Canadian university. The prize is to be awarded every year to a researcher less than ten years past the date of Ph.D. at the time of nomination.

The award consists of a cash prize of C$1,000 and a commemorative plaque presented at the CAIMS Annual Meeting. The recipient will be invited to deliver a plenary lecture at the CAIMS annual meeting in the year of the award. A travel allowance will be provided. The deadline for nominations is **January 31, 2014**. For more information see `http://www.pims.math.ca/pims-glance/prizes-awards`.

—From a PIMS announcement

PIMS Education Prize

The Pacific Institute for the Mathematical Sciences (PIMS) awards an annual prize to a member of the PIMS community who has made a significant contribution to education in the mathematical sciences. This prize is intended to recognize individuals from the PIMS universities, or other educational institutions in Alberta, British Columbia, and Saskatchewan, who have played a major role in encouraging activities which have enhanced public awareness and appreciation of mathematics, as well as fostering communication among various groups and organizations concerned with mathematical training at all levels. The deadline for nominations is **March 15, 2014**. For more information see the website `http://www.pims.math.ca/pims-glance/prizes-awards`.

—From a PIMS announcement

Inside the AMS

Math in Moscow Scholarships Awarded

The AMS has made awards to five mathematics students to attend the Math in Moscow program in the spring of 2014. Following are the names of the undergraduate students and their institutions: Andrew D. Hanlon, Pennsylvania State University; Lisandra Hernandez-Vazquez, Florida International University; Vishesh Jain, Stanford University; Nicholas F. Marshall, Clarkson University; and Vy Nguyen, Southern Methodist University. Each received a cash award of US$9,000.

Math in Moscow is a program of the Independent University of Moscow that offers foreign students (undergraduate or graduate students specializing in mathematics and/or computer science) the opportunity to spend a semester in Moscow studying mathematics. All instruction is given in English. The fifteen-week program is similar to the Research Experiences for Undergraduates programs that are held each summer across the United States.

The AMS awards several scholarships for U.S. students to attend the Math in Moscow program. The scholarships are made possible through a grant from the National Science Foundation. For more information about Math in Moscow, consult `http://www.mccme.ru/mathinmoscow` and the article "Bringing Eastern European mathematical traditions to North American students", *Notices*, November 2003, pages 1250–1254.

—Elaine Kehoe

Erdős Memorial Lecture

The Erdős Memorial Lecture is an annual invited address named for the prolific mathematician Paul Erdős (1913–1996). The lectures are supported by a fund created by Andrew Beal, a Dallas banker and mathematics enthusiast. The Beal Prize Fund is being held by the AMS until it is awarded for a correct solution to the Beal Conjecture (see `www.math.unt.edu/~mauldin/beal.html`). At Beal's request, the interest from the fund is used to support the Erdős Memorial Lecture.

The 2013 Erdős Memorial Lecturers were Endre Szemerédi of Rutgers University and Barry Mazur of Harvard University. The 2014 Erdős Memorial Lecture will be held at the 2014 Spring Southeastern Section Meeting at the University of Tennessee, Knoxville, on March 22, 2014. The lecturer will be Maria Chudnovsky of Columbia University.

—AMS announcement

From the AMS Public Awareness Office

Bryna Kra

Bryna Kra Gives Arnold Ross Lecture. Bryna Kra (Northwestern University) gave the 2013 Fall Arnold Ross Lecture, "Patterns and Disorder: How Random Can Random Be?" at the Museum of Science and Industry in Chicago. Kra spoke to an audience of over 150 Chicago-area high school students and teachers about patterns and the difference between patterned sets and random ones. After her lecture, eight students played Who Wants to Be a Mathematician. Read about Kra's lecture and the game at `http://www.ams.org/programs/students/wwtbam/arl2013-fall`.

Mathematics at the 2013 SACNAS National Conference. The AMS hosted an exhibit and Who Wants to Be a Mathematician at the 2013 SACNAS National Conference held in San Antonio, Texas. See photo slideshows and a report on mathematics at the conference, including the Undergraduate Student Posters in Mathematics, Conversations with Scientists, Keynote talk by Robert Megginson, and sessions at `http://www.ams.org/meetings/sacnas2013-mtg`.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers
`paoffice@ams.org`

Reference and Book List

*The **Reference** section of the* Notices *is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.*

Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are `notices@math.wustl.edu` in the case of the editor and `smf@ams.org` in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Upcoming Deadlines

December 15, 2013: Applications for AMS Epsilon Fund. See `http://www.ams.org/programs/edu-support/epsilon/emp-epsilon` or contact the AMS Membership and Programs Department by email at `prof-serv@ams.org` or by telephone at 800-321-4267, ext. 4113.

December 15, 2013: Applications for Service, Teaching, and Research (STaR) Program. See `http://matheddb.missouri.edu/star/`.

December 20, 2013: Proposals for 2015 AMS Short Courses. Submit proposals to Ellen Maycock at `ejm@ams.org`.

December 20, 2013: Applications for National Defense Science and Engineering Graduate (NDSEG) Fellowships. See `http://ndseg.asee.org/`.

December 24, 2013: Registration for free workshop on "Writing a Competitive Grant Proposal to NSF-EHR" held prior to the Joint Mathematics Meetings. See `http://tinyurl.com/ka4fcdu`.

December 31, 2013: Nominations for Otto Neugebauer Prize. See the website `http://www.euro-math-soc.eu/otto_neugebauer_prize.html`.

January 1, 2014: Applications for Boston fellowships for Math for America (MfA). See `http://www.mathforamerica.org/`.

January 10, 2014: Full proposals for NSF Algorithms for Threat Detection program. See "Mathematics Opportunities" in this issue.

January 12, 2014: Applications for priority consideration for Los Angeles and New York City fellowships for Math for America (MfA). See `http://www.mathforamerica.org/`.

January 13, 2014: Applications for Jefferson Science Fellowships. See `http://sites.nationalacademies.org/PGA/Jefferson/PGA_046612`; email: `jsf@nas.edu`; telephone: 202-334-2643.

January 15, 2014: Applications for AMS-AAAS Mass Media Summer

Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

Fellowships. See the website http://www.aaas.org/programs/education/MassMedia; or contact Dione Rossiter, Manager, Mass Media Program, AAAS Mass Media Science and Engineering Fellows Program, 1200 New York Avenue, NW, Washington, DC 20005; telephone 202-326-6645; fax 202-371-9849; email drossite@aaas.org. Further information is also available at http://www.ams.org/programs/ams-fellowships/media-fellow/massmediafellow and through the AMS Washington Office, 1527 Eighteenth Street, NW, Washington, DC 20036; telephone 202-588-1100; fax 202-588-1853; email amsdc@ams.org.

January 23, 2014: Full proposals for NSF Major Research Instrumentation Program. See "Mathematics Opportunities" in this issue.

January 31, 2014: Nominations for CAIMS/PIMS Early Career Award. See "Mathematics Opportunities" in this issue.

January 31, 2014: Entries for AWM Essay Contest. Contact the contest organizer, Heather Lewis, at hlewis5@naz.edu, or see https://sites.google.com/site/awmmath/home

February 1, 2014: Applications for February review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

February 1, 2014: Applications for AWM Travel Grants, Mathematics Education Research Travel Grants, Mathematics Mentoring Travel Grants, and Mathematics Education Research Mentoring Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; telephone: 703-934-0163; or email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

February 9, 2014: Applications for Los Angeles, New York, Utah, and Washington, D.C., fellowships for Math for America (MfA). See http://www.mathforamerica.org/.

February 12, 2014: Applications for Research in Industrial Projects for Students (RIPS) of the Institute for Pure and Applied Mathematics (IPAM). See www.ipam.ucla.edu.

February 15, 2014: Applications for AMS Congressional Fellowship. See http://www.ams.org/programs/ams-fellowships/ams-aaas/ams-aaas-congressional-fellowship or contact the AMS Washington Office at 202-588-1100, email: amsdc@ams.org.

February 15, 2014: Nominations for AWM-Joan & Joseph Birman Prize in Topology and Geometry. See the website http://www.awm-math.org.

March 3, 2014: Applications for the EDGE for Women Summer Program. See "Mathematics Opportunities" in this issue.

March 15, 2014: Nominations for PIMS Education Prize. See "Mathematics Opportunities" in this issue.

March 31, 2014: Applications for AMS-Simons Travel Grants. See "Mathematics Opportunities" in this issue.

April 15, 2014: Applications for fall 2014 semester of Math in Moscow. See http://www.mccme.ru/mathinmoscow, or contact: Math in Moscow, P.O. Box 524, Wynnewood, PA 19096; fax: +7095-291-65-01; email: mim@mccme.ru. Information and application forms for the AMS scholarships are available on the AMS website at http://www.ams.org/programs/travel-grants/mimoscow, or contact: Math in Moscow Program, Membership and Programs Department, American Mathematical Society, 201 Charles Street, Providence RI 02904-2294; email student-serv@ams.org.

May 1, 2014: Applications for May review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

May 1, 2014: Applications for AWM Travel Grants and Mathematics Education Research Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; telephone: 703-934-0163; or email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

August 1, 2014: Applications for August review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

October 1, 2014: Applications for AWM Travel Grants and Mathematics Education Research Travel Grants. See https://sites.google.com/site/awmmath/programs/travel-grants; telephone: 703-934-0163; or email: awm@awm-math.org; or contact Association for Women in Mathematics, 11240 Waples Mill Road, Suite 200, Fairfax, VA 22030.

November 1, 2014: Applications for November review for National Academies Research Associateship Programs. See "Mathematics Opportunities" in this issue.

National Science Board

The National Science Board is the policymaking body of the National Science Foundation. Listed below are the current members of the NSB. For further information, visit the website http://www.nsf.gov/nsb/.

Dan E. Arvizu (Chair)
Director and Chief Executive
National Renewable Energy Laboratory

Deborah L. Ball
William H. Payne Collegiate Chair
Arthur F. Thurnau Professor, Dean, School of Education
University of Michigan

Bonnie Bassler
Howard Hughes Medical Institute Investigator
Squibb Professor of Molecular Biology
Princeton University

Arthur Bienenstock
Professor Emeritus of Photon Science
Stanford University

Ray M. Bowen
President Emeritus
Texas A&M University
Vinton G. Cerf
Vice President, Google

France A. Cordova
President Emeritus
Purdue University

Ruth David
President and CEO
Analytic Services, Inc.

The contact information for the Board is: National Science Board, 4201 Wilson Boulevard, Room 1225N, Arlington, VA 22230; telephone 703-292-7000; fax 703-292-9008; email `NationalScienceBrd@nsf.gov`; World Wide Web `http://www.nsf.gov/nsb/`.

Book List

The Book List highlights recent books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. Suggestions for books to include on the list may be sent to `notices-booklist@ams.org`.

*Added to "Book List" since the list's last appearance.

**Algorithms Unlocked,* by Thomas H. Cormen. MIT Press, March 2013. ISBN-13:978-02625-188-02.

An Accidental Statistician: The Life and Memories of George E. P. Box, by George E. P. Box. Wiley, April 2013. ISBN-13: 978-1-118-40088-3.

Assessing the Reliability of Complex Models: Mathematical and Statistical Foundations of Verification, Validation, and Uncertainty Quantification, by the National Research Council. National Academies Press, 2012. ISBN-13: 978-0-309-25634-6.

**A Calculus of Ideas: A Mathematical Study of Human Thought,* by Ulf Grenander. World Scientific, September 2012. ISBN-13: 978-98143-831-89. (Reviewed in this issue.)

Charles S. Peirce on the Logic of Number, by Paul Shields. Docent Press, October 2012. ISBN-13: 978-0-9837004-7-0.

Classic Problems of Probability, by Prakash Gorroochurn. Wiley, May 2012. ISBN-13: 978-1-1180-6325-5. (Reviewed November 2013.)

Conflict in History, Measuring Symmetry, Thermodynamic Modeling and Other Work, by Dennis Glenn Collins. Author House, November 2011. ISBN-13: 978-1-4670-7641-8.

The Continuity Debate: Dedekind, Cantor, du Bois-Reymond, and Peirce on Continuity and Infinitesimals, by Benjamin Lee Buckley. Docent Press, December 2012. ISBN-13: 978-0-9837004-8-7.

The Crest of the Peacock: Non-European Roots of Mathematics, by George Gheverghese Joseph. Third edition. Princeton University Press, October 2010. ISBN-13: 978-0-691-13526-7. (Reviewed December 2013.)

Decoding the Heavens: A 2,000-Year-Old Computer—and the Century-Long Search to Discover Its Secrets, by Jo Marchant. Da Capo Press, February 2009. ISBN-13: 978-03068-174-27. (Reviewed June/July 2013).

Do I Count?: Stories from Mathematics, by Günter Ziegler (translation of *Darf ich Zahlen?: Geschichte aus der Mathematik,* Piper Verlag, 2010). CRC Press/A K Peters, July 2013. ISBN-13: 978-1466564916

Figures of Thought: A Literary Appreciation of Maxwell's Treatise on Electricity and Magnetism, by Thomas K. Simpson. Green Lion Press, February 2006. ISBN-13: 978-18880-093-16. (Reviewed October 2013.)

The Fractalist: Memoir of a Scientific Maverick, by Benoît Mandelbrot. Pantheon, October 2012. ISBN-13: 978-03073-773-57.

Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century, by the National Research Council. National Academies Press, 2012. ISBN-13: 978-0-309-25473-1.

Girls Get Curves: Geometry Takes Shape, by Danica McKellar. Plume, July 2013. ISBN-13: 978-04522-987-43.

The Golden Ticket: P, NP, and the Search for the Impossible, by Lance

Fortnow. Princeton University Press, March 2013. ISBN-13: 978-06911-564-91.

**Good Math: A Geek's Guide to the Beauty of Numbers, Logic, and Computation,* by Mark C. Chu-Carroll. Pragmatic Bookshelf, July 2013. ISBN-13: 978-19377-853-38.

Google's PageRank and Beyond: The Science of Search Engine Rankings, by Amy Langville and Carl Meyer. Princeton University Press, February 2012. ISBN-13: 978-06911-526-60.

Gösta Mittag-Leffler: A Man of Conviction, by Arild Stubhaug (translated by Tiina Nunnally). Springer, November 2010. ISBN-13: 978-36421-167-11. (Reviewed September 2013.)

Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry, by Glen Van Brummelen. Princeton University Press, December 2012. ISBN-13: 978-06911-489-22.

How to Study As a Mathematics Major, by Lara Alcock. Oxford University Press, March 2013. ISBN-13: 978-0199661312.

I Died for Beauty: Dorothy Wrinch and the Cultures of Science, by Marjorie Senechal. Oxford University Press, December 2012. ISBN-13:978-01997-325-93.

Ibn al-Haytham's Theory of Conics, Geometrical Constructions and Practical Geometry, by Roshdi Rashed. Routledge, February 2013. ISBN-13: 978-0-415-58215-5.

**If A, Then B: How the World Discovered Logic,* by Michael Shenefelt and Heidi White. Columbia University Press, June 2013. ISBN-13:978-02311-610-53.

Imagined Civilizations: China, the West, and Their First Encounter, by Roger Hart. Johns Hopkins University Press, July 2013. ISBN-13:978-14214-060-60.

Invisible in the Storm: The Role of Mathematics in Understanding Weather, by Ian Roulstone and John Norbury. Princeton University Press, February 2013. ISBN-13: 978-06911-527-21. (Reviewed September 2013.)

Levels of Infinity: Selected Writings on Mathematics and Philosophy, by Hermann Weyl. Edited by Peter Pesic. Dover Publications, February 2013. ISBN-13: 978-0486489032.

The Logician and the Engineer: How George Boole and Claude Shannon Created the Information Age, by Paul J. Nahin, Princeton University Press, October 2012. ISBN-13: 978-06911-510-07. (Reviewed October 2013.)

Manifold Mirrors: The Crossing Paths of the Arts and Mathematics, by Felipe Cucker. Cambridge University Press, June 2013. ISBN-13: 978-05217-287-68.

The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics, by Clifford A. Pickover. Sterling. February 7, 2012. ISBN-13: 978-14027-882-91.

Math is Murder, by Robert C. Brigham. iUniverse, March 28, 2012. ISBN-13 978-14697-972-81.

Math on Trial: How Numbers Get Used and Abused in the Courtroom, by Leila Schneps and Coralie Colmez. Basic Books, March 2013. ISBN-13: 978-04650-329-21. (Reviewed August 2013.)

A Mathematician Comes of Age, by Steven G. Krantz. Mathematical Association of America, December 2011. ISBN-13: 978-08838-557-82.

A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form, by Paul Lockhart. Bellevue Literary Press, April 2009. ISBN-13: 978-1-934137-17-8. (Reviewed April 2013.)

Mathematics in Nineteenth-Century America: The Bowditch Generation, by Todd Timmons. Docent Press, July 2013. ISBN-13: 978-0-9887449-3-6.

Mathematics in Victorian Britain, by Raymond Flood, Adrian Rice, and Robin Wilson. Oxford University Press, October 2011. ISBN-13: 978-019-960139-4.

Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice, by Alexandre V. Borovik. AMS, January 2010. ISBN-13: 978-0-8218-4761-9.

Maverick Genius: The Pioneering Odyssey of Freeman Dyson, by Phillip F. Schewe. Thomas Dunne Books, February 2013. ISBN-13:978-03126-423-58.

Meaning in Mathematics, edited by John Polkinghorne. Oxford University Press, July 2011. ISBN-13: 978-01996-050-57. (Reviewed May 2013.)

My Brief History, by Stephen Hawking. Bantam Dell, September 2013. ISBN-13: 978-03455-352-83.

**Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity,* by Loren Graham and Jean-Michel Kantor. Belknap Press of Harvard University Press, March 2009. ISBN-13: 978-06740-329-34. (Reviewed in this issue.)

The New York Times Book of Mathematics: More Than 100 Years of Writing by the Numbers, edited by Gina Kolata. Sterling, June 2013. ISBN-13: 978-14027-932-26.

The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century, by Yvette Kosmann-Schwarzbach. Springer, December 2010. ISBN-13: 978-03878-786-76. (Reviewed August 2013.)

**The Outer Limits of Reason: What Science, Mathematics, and Logic Cannot Tell Us,* by Noson S. Yanofsky. MIT Press, August 2013. ISBN-13: 978-02620-193-54.

Paradoxes in Probability Theory, by William Eckhardt. Springer, September 2012. ISBN-13: 978-94007-513-92. (Reviewed March 2013.)

Peirce's Logic of Continuity: A Conceptual and Mathematical Approach, by Fernando Zalamea. Docent Press, December 2012. ISBN-13: 978-0-9837004-9-4.

Perfect Mechanics: Instrument Makers at the Royal Society of London in the Eighteenth Century, by Richard Sorrenson. Docent Press, September 2013. ISBN-13: 978-0-9887449-2-9.

**Probably Approximately Correct: Nature's Algorithms for Learning and Prospering in a Complex World,* by Leslie Valiant. Basic Books, June 2013. ISBN-13: 978-04650-327-16.

Relations between Logic and Mathematics in the Work of Benjamin and Charles S. Peirce, by Allison Walsh. Docent Press, October 2012. ISBN-13: 978-0-9837004-6-3.

The Search for Certainty: A Journey through the History of Mathematics, 1800–2000, edited by Frank J. Swetz. Dover Publications, September 2012. ISBN-13: 978-04864-744-27.

Seduced by Logic: Emilie Du Châtelet, Mary Somerville and the Newtonian Revolution, by Robyn Arianrhod. Oxford University Press, September

2012. ISBN-13: 978-01999-316-13. (Reviewed June/July 2013).

Selected Papers: Volume II: On Algebraic Geometry, including Correspondence with Grothendieck, by David Mumford. Edited by Amnon Neeman, Ching-Li Chai, and Takahiro Shiota. Springer, July 2010. ISBN-13: 978-03877-249-11. (Reviewed February 2013.)

The Signal and the Noise: Why So Many Predictions Fail—But Some Don't, by Nate Silver. Penguin Press, September 2012. ISBN-13:978-15942-041-11.

Sources in the Development of Mathematics: Series and Products from the Fifteenth to the Twenty-first Century, by Ranjan Roy. Cambridge University Press, June 2011. ISBN-13: 978-05211-147-07. (Reviewed November 2013.)

Strange Attractors (comic book), by Charles Soule, Greg Scott, and Robert Saywitz. Archaia Entertainment, May 2013. ISBN-13: 978-19363-936-26.

Symmetry: A Very Short Introduction, by Ian Stewart. Oxford University Press, July 2013. ISBN-13: 978-01996-519-86.

Thinking in Numbers: On Life, Love, Meaning, and Math, by Daniel Tammet. Little, Brown and Company, July 2013. ISBN-13: 978-03161-873-74.

Thinking Statistically, by Uri Bram. CreateSpace Independent Publishing Platform, January 2012. ISBN-13: 978-14699-123-32.

Transcending Tradition: Jewish Mathematicians in German Speaking Academic Culture, edited by Birgit Bergmann, Moritz Epple, and Ruti Ungar. Springer, January 2012. ISBN-13: 978-36422-246-38. (Reviewed February 2013.)

Turbulent Times in Mathematics: The Life of J. C. Fields and the History of the Fields Medal, by Elaine McKinnon Riehm and Frances Hoffman. AMS, November 2011. ISBN-13: 978-0-8218-6914-7.

Turing's Cathedral: The Origins of the Digital Universe, by George Dyson. Pantheon/Vintage, December 2012. ISBN-13: 978-14000-759-97.

Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice, by Alexandre V. Borovik. AMS, January 2010. ISBN-13: 978-0-8218-4761-9.

Visions of Infinity: The Great Mathematical Problems, by Ian Stewart. Basic Books, March 2013. ISBN-13: 978-04650-224-03.

A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing, edited by Benjamin Wardhaugh. Princeton University Press, April 2012. ISBN-13: 978-06911-477-58. (Reviewed March 2013.)

Classified Advertisements

Positions available, items for sale, services available, and more

CONNECTICUT

YALE UNIVERSITY
J. Willard Gibbs
Assistant Professorships
in Mathematics 2014-2015

The Gibbs Assistant Professorships are intended primarily for men and women who received the Ph.D. degree and show definite promise in research in pure or applied mathematics. Appointments are for three years. The salary will be at least $74,500. Each recipient of a Gibbs Assistant Professorship will be given a moving allowance based on the distance to be moved.

The teaching load for Gibbs Assistant Professors will be kept light, so as to allow ample time for research. This will consist of three one-semester courses per year. Part of the duties may consist of a one-semester course at the graduate level in the general area of the instructor.s research. Yale is an Affirmative Action/Equal Opportunity Employer. Qualified women and members of minority groups are encouraged to apply. Submit applications and supporting material through `MathJobs.org` by January 1, 2014. Submit inquiries to math.positions@yale.edu. Offers expected to be made in early February 2014.

000007

GEORGIA

GEORGIA STATE UNIVERSITY
Mathematics and Statistics

The Department of Mathematics and Statistics, of Georgia State University invites applications for three positions: Tenure-track Assistant Professor, a Lecturer and a Visiting Lecturer, respectively. All positions require a Ph.D. in Mathematics, Statistics or related field. Submit applications to `http://www.mathjobs.org`. An offer of employment will be conditional upon background verification. Georgia State University is one of four Research Universities of the University System of Georgia and an EEO/AA institution.

000003

KANSAS

UNIVERSITY OF KANSAS
Department of Mathematics

The Department of Mathematics at the University of Kansas invites applications for a tenure-track faculty position in algebra and/or combinatorics; for two visiting assistant professor positions, one in algebraic geometry and another in partial differential equations; and for the Black-Babcock Visiting Assistant Professor position in probability. These positions are expected to begin as early as August 18, 2014. For complete announcements and to apply online go to `https://employment.ku.edu`, click `Search Faculty Jobs` and search by Math Department. In addition, at least four recommendation letters (teaching ability must be addressed in at least one letter) should be submitted electronically to Mathjobs.org. Review of applications will begin November 1, 2013 (for tenure-track position) and December 1, 2013 (for visiting positions) and will continue as long as needed to identify a qualified applicant. Equal Opportunity Employer M/F/D/V.

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000005

MASSACHUSETTS

TUFTS UNIVERSITY
Department of Mathematics
Geometric Group Theory and Topology position announcement

Applications are invited for a term-limited Norbert Wiener Assistant Professorship to begin September 1, 2014. The initial contract will be for one year, renewable for an additional two years. A Ph.D. in Mathematics, evidence of strong teaching, and promise of strong research are required, with a research focus on Geometric Group Theory and/or Low-Dimensional Topology.

The successful candidate will be expected to teach four courses a year, to contribute to research within the department in the fields of Geometric Group Theory and Low-Dimensional Topology, and to participate in the weekly Geometric Group Theory and Topology seminar. Areas of research in the department include CAT(0) groups, Teichmuller Theory, and Hyperbolic 3-Manifolds.

Applications should include a cover letter, curriculum vitae, research statement, and teaching statement, which should all be submitted through `https://www.mathjobs.org/jobs/jobs/5354`. If a recommender cannot submit online, we will accept signed PDF attachments sent to `kim.ruane@tufts.edu`, or paper letters mailed to GGTT Search Committee Chair, Department of Mathematics, Bromfield-Pearson Hall, Tufts University, Medford, MA 02155. Review of applications will begin on December 15, 2013, and will continue until the position is filled.

Tufts University is an Affirmative Action/Equal Opportunity employer. We are committed to increasing the diversity of our faculty. Members of underrepresented groups are strongly encouraged to apply.

000002

TUFTS UNIVERSITY
Department of Mathematics
Non-Tenure-Track Norbert Wiener Assistant Professorship
Algebraic Geometry

Applications are invited for a non-tenure-track Assistant Professorship to begin September 1, 2014, with an initial appointment for one year, renewable for up to two more years. Applicants must show promise of outstanding research in the area of algebraic geometry or closely related fields. Preference will be given to candidates whose interests overlap with those of Tufts faculty. Applicants must also show evidence of excellent teaching. The teaching load will be two courses per semester. Applications should include a cover letter, curriculum vitae, a research statement, and a teaching statement. All of these documents should be submitted electronically through `https://www.mathjobs.org/jobs/jobs/5373`. In addition, applicants should arrange for three letters of recommendation to be submitted electronically on their behalf through `http://www.mathjobs.org`. If a recommender cannot submit online, we will also accept signed PDF attachments sent to `montserrat.teixidoribigas@tufts.edu` or paper letters mailed to AG Search

Committee Chair, Department of Mathematics, 503 Boston Avenue, Tufts University, Medford, MA 02155. Review of applications will begin on January 2, 2014, and will continue until the position is filled.

Tufts University is an Affirmative Action/Equal Opportunity Employer. We are committed to maintaining or increasing the diversity of our faculty. Members of underrepresented groups are strongly encouraged to apply.

000001

MINNESOTA

UNIVERSITY OF MINNESOTA
School of Mathematics

The School of Mathematics of the University of Minnesota is seeking outstanding candidates for 2-3 tenured or tenure-track faculty positions starting fall semester 2014. Candidates should have a Ph.D. or equivalent degree in mathematics or a closely related field and excellent records in both research and teaching.

For full consideration, applications and all supporting materials must be submitted electronically through: `http://www.mathjobs.org` by December 15, 2013. Applications received after the deadline will be considered as positions remain.

Applicants must include the following: Cover letter, Curriculum vitae, at least 4 letters of recommendation, one of which should address teaching ability, and a research and teaching statement. Reference letter writers should be asked to submit their letters on line through `http://mathjobs.org`. If they are unable to do so, they may send their letters to this address:

Harry Singh
School of Mathematics
University of Minnesota
206 Church Street SE
Minneapolis, MN
55455

In addition to your `mathJobs` application, the University of Minnesota requires all applicants to apply through the University of Minnesota employment website `https://employment.umn.edu/applicants/jsp/shared/search/Search_css.jsp`. The requisition numbers are 187390 for the tenure position and 187381 for the tenure-track position.

Any offer of employment is contingent upon the successful completion of a background check. Our presumption is that prospective employees are eligible to work here. Criminal convictions do not automatically disqualify finalists from employment.

The University of Minnesota is an Equal Opportunity Employer/Educator.

000006

NEW MEXICO

UNIVERSITY OF NEW MEXICO
Pure Math 2013 Ad

The Department of Mathematics and Statistics at the University of New Mexico invites applications in pure math at the level of Assistant Professor. This is a probationary appointments leading to a tenure decision. The successful candidate will be expected to maintain a strong research program, and demonstrate teaching excellence at both the undergraduate and graduate levels.

A Ph.D. in mathematics or a closely related field is required. Preferred qualifications include: Post-doctoral experience, demonstrated or potential publication record in recognized premier journals, commitment to establishing a strong research program with an emerging national/international reputation, commitment to excellence in teaching at the undergraduate through graduate level and desire to participate in an environment comprising significant cultural diversity, and potential service to the Mathematics Program and Department. Preference will also be given to candidates working in the areas of Algebra or Geometry.

New Mexico is a majority minority state, and the University of New Mexico has a majority minority undergraduate population. Both the university and the Department of Mathematics and Statistics in particular have a strong commitment to cultural diversity. So candidates should have the ability to work effectively with a culturally and cognitively diverse community. Women and minorities are especially encouraged to apply.

To apply visit the UNMJobs website: `https://unmjobs.unm.edu/applicants/Central?quickFind=74453`.

Please reference Posting Number 0822167. A completed application requires an application letter, curriculum vita, teaching and research statements. Applicants must also arrange for three letters of recommendation to be sent by surface mail to:

Department of Mathematics and Statistics
Pure Math Search Committee
1 University of New Mexico
MSC01 1115
Albuquerque, NM 87131

For best consideration applications, including three letters of recommendation, should be received by December 1, however the position will remain open until filled. UNM's confidentiality policy (.Recruitment and Hiring. Policy #3210), which includes information about public disclosure of documents submitted by applicants, is located at `http://policy.unm.edu/university-policies/3000/3210.html`. The University of New Mexico is an EOE/AA employer and educator.

000004

NEW YORK

BINGHAMTON UNIVERSITY
Department of Mathematical Sciences

The Department of Mathematical Sciences at Binghamton University invites applications for a tenured position in any field of mathematics, up to full professor, to lead a new initiative creating a premier calculus teaching program within the department. Applicants should have an established reputation both as a strong mathematical researcher and as a leader in college-level mathematics education.

We will also consider candidates without mathematical research credentials, but with a distinguished record of leadership in calculus education, for a non-faculty position as Director of the Calculus Program.

Applicants must apply electronically at `http://www.mathjobs.org`. Applicants should submit a cover

About the cover

Plain geometry

The colored diagrams in this issue's article on secondary school curricula in geometry by Guershon Harel recalled to us the edition of the first six books of Euclid's *Elements* produced in the mid-nineteenth century by the eccentric Irishman Oliver Byrne. The cover shows Proposition 1 of Book 1, the opening of the main text.

Geometry has been traditionally a basic tool in teaching logical reasoning, but in recent years it has been neglected or even abandoned in school curricula throughout the world. The reasons for this are not entirely clear, but then it seems to us that the reasons why it is especially suitable for the task—the association of visualization with logic—do not seem to be all that clear, either.

Byrne's *Euclid*, in which text and diagrams are mixed together, and in which color-coding replaces labels, offers an interesting alternative to traditional methods. He is successful in showing that some care in the use of diagrams accompanying Euclid's proofs can help greatly in following them. By and large his book, although by no means perfect, also demonstrates one important if rarely recognized principle—an ideal mathematical figure should tell a story independently of text.

A full PDF file of Byrne's book is now available on the Internet, at

`https://archive.org/details/firstsixbooksofe00eucl`

The book from which its images were taken was supplied by the Thomas Fisher Rare Book Library of the University of Toronto. We thank John Shoesmith and Paul Armstrong of the library for supplying us with a high-resolution image.

—Bill Casselman
Graphics Editor
(`notices-covers@ams.org`)

Mathematics Calendar

Please submit conference information for the Mathematics Calendar through the Mathematics Calendar submission form at http://www.ams.org/cgi-bin/mathcal-submit.pl. The most comprehensive and up-to-date Mathematics Calendar information is available on the AMS website at http://www.ams.org/mathcal/.

January 2014

* 6–July 31 **Research Programme on Central Configurations, Periodic Orbits and Beyond in Celestial Mechanics**, Centre de Recerca Matematica, Bellaterra, Barcelona, Spain.
Description: The study of the dynamics of n point masses interacting according to Newtonian gravity is usually called the n-body problem. It can be considered as old as the history of the science and has influenced most of the areas in mathematics. However, most of the problems in Celestial Mechanics are beyond the present limits of the knowledge and many natural questions are difficult or impossible to solve when the number of bodies n is larger than 2. In order to make progress against such complexity one must look for specific objects. From a geometrical point of view a key point consists in trying to understand the structure of the phase space looking for the equilibrium points, periodic orbits, invariant tori,... The stable and unstable manifolds associated to these objects form a kind of network of connections, which together with the previous invariant objects constitute a big part of the main skeleton of the system.
Information: http://www.crm.cat/2014/RPCentralConfigurations.

* 7–10 **Variational Methods in Elliptic Equations and Systems**, University of Lisbon, Lisbon, Portugal.
Description: Variational Methods have proven to be a powerful way to solve many problems in the field of differential equations. This scientific meeting will focus mainly on their use in the study of elliptic equations and systems, and will bring together several experts on the field. Participants are encouraged to submit an abstract. The event, which is dedicated to the memory of Miguel Ramos, will take place at the University of Lisbon, Portugal.
Information: http://ptmat.fc.ul.pt/~vmee/.

* 20–24 **Holomorphic and Symbolic Dynamics**, Université Paul Sabatier, Toulouse, France.
Focus: The focus is on holomorphic and symbolic dynamics, and their relation to group theory. The purpose is to encourage contact and collaboration between these related fields; in particular, viewing holomorphic dynamical systems as topological objects; encoding their attractors via symbolic dynamics; describing the parameter spaces of such dynamical systems topologically and combinatorially; and using monodromy representations to tie these concepts together and with group theory.
Information: http://sites.google.com/site/hsdynamics2014/.

* 27–31 **Central Configurations, Periodic Orbits and Beyond in Celestial Mechanics (DANCE Winter School)**, Centre de Recerca Matematica, Bellaterra, Barcelona, Spain.
Description: The advanced course on Central Configurations, Periodic Orbits and Beyond in Celestial Mechanics is a joint activity of the DANCE Spanish network with the CRM in the framework of the research programme Central Configurations, Periodic Orbits and Beyond. It is the 11th winter school in Dynamical Systems of the DANCE network. This series of winter schools aims at training their participants both theoretically and in applications in the field of nonlinear science; with the aim that theory and applications enforce each other. This will be done in an atmosphere of informal discussion, interchange of ideas and critical discussion of results. Attention

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.
An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.
In general, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to notices@ams.org or mathcal@ams.org.
In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.
The complete listing of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.
The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: http://www.ams.org/.

will be paid to the numerical and computational issues. These winter schools should help the basic training of young researchers, whilst opening new fields for senior ones. As a byproduct, the courses are planned to receive official recognition in some doctorate programs.
Information: http://www.crm.cat/2014/ACDance.

February 2014

* 26–27 **7th Seminar on Linear Algebra and its Applications**, Ferdowsi University, Mashhad, Iran.
Description: The seminar will provide a forum for mathematicians worldwide and graduate students to present their latest results about all aspects of linear algebra and a means to discuss their recent researches with each other. The language of the presentation may be English or Farsi. For participating in the conference, the non-Iranian mathematicians may contact the chairman Professor M. S. Moslehian.
Information: http://profsite.um.ac.ir/~math/slaa7.htm.

March 2014

* 12–21 **School "Around Vortices: from Continuum to Quantum Mechanics"**, IMPA, Rio de Janeiro, Brazil.
Description: A school on the topic of vortices in fluids, from classical Newtonian fluids, to complex fluids, to superfluids, and in superconductors. Opening event for the Thematic Program on Incompressible Fluid Dynamics at IMPA.
Information: http://www.impa.br/opencms/pt/eventos/store/evento_1402.

April 2014

* 25–27 **The Riviere-Fabes Symposium on Analysis and PDE**, University of Minnesota, Minneapolis, Minnesota.
Description: A three-day symposium in Analysis and PDE, with two hours of talks by each of the following four speakers: Sun-Yung Alice Chang (Princeton), Alexandru Ionescu (Princeton), Frank Merle (Universite de Cergy-Pontoise and IHES), Maciej Zworski (Berkeley).
Information: http://www.math.umn.edu/conferences/riv_fabes/.

May 2014

* 9–17 **Master-class: Around Thurston-Grothendieck-Teichmüller Theories**, University of Strasbourg, Strasbourg, France.
Description: The master-class is oriented to graduate and Ph.D. students and post-docs. Confirmed researchers are also welcome. The aim of this master class is to give an introduction to hyperbolic geometry, Riemann surfaces and Teichmüller spaces as these appear in the works of Thurston, Grothendieck and Teichmüller, and to show the connections between the various theories. There will also be a course on the dynamical/probabilistic aspects of hyperbolic geometry. Seven 5-hour courses will be given by: Norbert A'Campo (Basel), Louis Funar (Grenoble), Jacques Franchi (Strasbourg), Hugo Parlier (Grenoble), Gabriela Schmithüsen (Karlsruhe), Muhammed Uludag (Istanbul) and Alexandre Zvonkine (Bordeaux). There will be some additional research lectures. Limited funding for local expenses is available. Graduate and Ph.D. students are welcome. For questions and registration contact the oranizer, A. Papadopoulos, papadop@math.unistra.fr.
Information: http://www-irma.u-strasbg.fr/article1389.html.

* 19–23 **Polynomials over Finite Fields: Functional and Algebraic Properties**, Centre de Recerca Matematica, Bellaterra, Barcelona, Spain.
Description: The theory of polynomials over finite fields is fundamental for the study of finite fields, which in turn plays a central role in many areas of pure and applied mathematics. This is a classical area of mathematics with a rich history, going back to Gauss and Galois. The exciting and challenging problems concerning univariate and multivariate polynomials over finite fields are of intricate algebraic and number theoretic flavour and their study requires deep mathematical and computational tools. The determination and construction of special types of (irreducible, primitive, permutation) univariate and multivariate polynomials, for example, as well as understanding many of their functional and algebraic properties (composition, decomposition, iteration, factorization, size of value sets) are long standing problems in the theory of finite fields. These areas have attracted further attention in recent years due to their applications in cryptography, coding theory, combinatorics, design theory, quasi-Monte Carlo methods, communications.
Information: http://www.crm.cat/2014/WKFiniteFields.

* 20–23 **XXI International Seminar NONLINEAR PHENOMENA IN COMPLEX SYSTEMS (Chaos, Fractals, Phase Transitions, Self-organization)**, Joint Institute for Power and Nuclear Research: "Sosny", Minsk, Belarus, Russia.
Objective: There will be discussed some modern advances, approaches and tools for studying nonlinear problems in different fields of science (mathematics, physics, chemistry, biology, economics and others).
Subjects: Above-mentioned and other frontier topics in modern nonlinear study in mathematical foundations and methods (dynamical systems, analytical and numerical methods, number theory and cryptography); Information processing (quantum computation, neural networks, artificial intelligence, parallel computing, GRID); High energy physics (QCD, standard model, quark-gluon plasma, collective phenomena, nonperturbative effects, confinement); Nuclear and reactor physics (development and adaptation of codes for deterministic/probabilistic safety analysis); Foundation of electronics and optics (classical and quantum optics); Social and biological systems (nonlinear dynamics in economics, social and biological systems).
Deadline: For applications: April 1, 2014.
Information: http://npcs.j-npcs.org/.

* 20–24 **7th Conference on Function Spaces**, Southern Illinois University, Edwardsville, Illinois.
Description: The conference will provide a forum for mathematicians interested in Function Algebras, Banach Algebras, Spaces & Algebras of Analytic Functions, LP Spaces, Geometry of Banach Spaces, Isometries of Function Spaces, and related problems.
Support: We applied for an NSF for a grant to help the participants with the local and travel cost and the registration fee; priority will be given to young mathematicians (including graduate students) without any other source of support.
Information: http://www.siue.edu/MATH/conference2014/.

* 29–31 **Computational Management Science 2014 - CMS 2014**, Faculty of Sciences, University of Lisbon, Lisbon, Portugal.
Description: We are pleased to announce the 11th International Conference on Computational Management Science (CMS) to be held in Lisbon, Portugal, in May of 2014. With its unique mixture of cultural and social activities, Lisbon will undoubtedly provide the perfect discussion forum for academia and industry to exchange knowledge, ideas and results. The topics for discussion span over a wide range of areas relevant to the theory and practice of computational methods, models and empirical analysis for decision-making, with special emphasis this year on Energy and Finance. We hope you accept this invitation to embark with us on a journey of discovery in the computational side of management science in the city that spawned many other journeys of discovery—Lisbon!
Information: http://cms2014.fc.ul.pt.

June 2014

* 2–5 **WSCG 2014 - 22nd International Conference on Computer Graphics, Visualization and Computer Vision 2014**, Primavera Hotel and Congress Centrum, Plzen (close to Prague), Czech Republic.

Description: Computer graphics and visualization, computer vision, image processing and pattern recognition, fundamental algorithms, GPU graphics, graphical human computer interfaces, geometric modeling, computer aided geometric design, computational geometry, rendering and virtual reality, animation and multimedia, medical imaging, graphical interaction, object oriented graphics, parallel and distributed graphics, CAD and GIS systems, geometrical algebra and related topics.
Information: `http://www.wscg.eu`.

* 2–6 **Conference on Ulam's type stability**, Rytro, Poland.
Description: The conference is organized by the Department of Mathematics of the Pedagogical University in Cracow, and is devoted to various investigations motivated by the notion of Hyers-Ulam stability and related issues. The participants are invited to give talks on stability of difference, differential, functional, and integral equations; stability of inequalities and other mathematical objects; hyperstability and superstability; various (direct, fixed point, invariant mean, etc.) methods for proving Ulam's type stability results; generalized (in the sense of Aoki and Rassias, Bourgin and Găvruţa; stability; stability on restricted domains and in various (metric, Banach, non-Archimedean, fuzzy, quasi-Banach, etc.) spaces; relations between Ulam's stability and fixed point results. Moreover, some lectures on other related topics will be provided. This time, a special session on dynamical systems is planned.
Information: `http://cuts.up.krakow.pl`.

* 2–6 **Hamiltonian Systems and Celestial Mechanics (HAMSYS 2014)**, Centre de Recerca Matematica, Bellaterra, Barcelona, Spain.
Description: In 1991 started the series of the HAMSYS Symposia. These symposia brought together top researches from several countries, working mainly in Hamiltonian Systems and Celestial Mechanics, as well as many graduate students who had the opportunity to learn from and connect with the experts in the field. The VII-th HAMSYS Symposium, denoted HAMSYS-2014, will take place at the CRM (Centre de Recerca Matemàtica, at the Universitat Autònoma de Barcelona). The emphasis of the talks will be on Hamiltonian dynamics and its relationship to several aspects of mechanics, geometric mechanics, and dynamical systems in general. The HAMSYS-2014 will be dedicated to the 70th birthday of Professor Clark Robinson.
Information: `http://www.crm.cat/2014/CHamsys`.

* 5–7 **Number Theory at Illinois: A Conference in Honor of the Batemans**, University of Illinois, Urbana, Illinois.
Description: A Number Theory Conference in memory of Paul and Felice Bateman will be held at the University of Illinois. The Batemans were long-time members of the faculty and Paul was department head for 14 years. Paul was a member of the American Mathematical Society for 71 years and among his other services, was a Trustee of the AMS. This meeting continues a long tradition of number theory conferences at Illinois.
Invited talks: There will be twenty invited talks as well as opportunities for contributed talks. These will cover a broad spectrum of number theory, representing Paul's many interests. A banquet will be held on June 6. There will be a refereed proceedings volume of conference talks. The conference will be preceded by the Midwest Number Theory Conference for Graduate Students, June 3–4, 2014 (which is being announced separately).
Information: `http://www.math.illinois.edu/nt2014`.

* 9–14 **Representations, Dynamics, Combinatorics: In the Limit and Beyond. A conference in honor of Anatoly Vershik's 80th birthday**, Saint-Petersburg, Russia.
Description: The conference will focus on the areas of mathematics where Anatoly Vershik has made important contributions, including representation theory, dynamical systems, ergodic theory, asymptotic combinatorics, operator algebras, and measure theory. It will feature invited talks by leading experts in these fields.
Information: `http://www.pdmi.ras.ru/EIMI/2014/RDC/`.

* 17–20 **First International Congress on Actuarial Science and Quantitative Finance**, Universidad Nacional de Colombia, Bogota, Colombia.
Description: The event would consist of plenary sessions of invited speakers, oral sessions of contributed talks, poster sessions and short courses in topics of interest in actuarial science and quantitative finance, given by some of the invited speakers.
Invited speakers: Hansjoerg Albrecher, Université de Lausanne; Richard Davis, Columbia University; Monique Jeanblanc, Université d'Évry Val-d'Essonne; Steve Haberman, City University London; David Ingram, Willis Re; Stéphane Loisel, Universite Claude Bernard Lyon 1; Fabio Mercurio, Blomberg; Ajay Subramanian, Georgia State University; ; Carlos Vázquez Cendón, Universidad de la Coruña; Shaun Wang, Georgia State University.
Organizer: Universidad Nacional de Colombia.
Information: `http://www.matematicas.unal.edu.co/icasqf/`.

* 30–July 5 **25th International Conference in Operator Theory**, West University of Timisoara, Timisoara, Romania.
Description: The conference is devoted to operator theory, operator algebras and their applications (differential operators, complex functions, mathematical physics, matrix analysis, system theory, etc.).
Information: `http://operatortheory25.wordpress.com`.

July 2014

* 2–4 **The 2014 International Conference of Applied and Engineering Mathematics**, Imperial College London, London, United Kingdom.
Description: The conference ICAEM'14 is held under the World Congress on Engineering 2014. The WCE 2014 is organized by the International Association of Engineers (IAENG), and serves as good platforms for the engineering community members to meet with each other and to exchange ideas. The last IAENG conferences attracted more than one thousand participants from over 30 countries. All submitted papers will be under peer review and accepted papers will be published in the conference proceeding (ISBN: 978-988-19252-7-5). The abstracts will be indexed and available at major academic databases. The accepted papers will also be considered for publication in the special issues of the journal Engineering Letters, in IAENG journals and in edited books.
Information: `http://www.iaeng.org/WCE2014/ICAEM2014.html`.

August 2014

* 3–9 **XIX EBT - 19th Brazilian Topology Meeting**, State University of São Paulo (UNESP), São José do Rio Preto, Brazil.
Description: The 19th Brazilian Topology Meeting will take place at the State University of São Paulo (UNESP), São José do Rio Preto, in the state of São Paulo, Brazil, from Sunday August 3, 2014 until August 9, 2014.
Scientific Committee: Daciberg Lima Gonçalves (Coordinator), Marek Golasiński, John Guaschi, Claude Hayat, Eduardo Hoefel and Pedro Pergher.
Organizing committee: Ermiínia de Loudes Campello Fanti (Coordinator), Alice Kimie Miwa Libardi, Darlan Rabelo Girao, Denise de Mattos, Evelin Meneguesso Barbaresco, Flávia Souza Machado da Silva, João Carlos Ferreira Costa, Leonardo Navarro de Carvalho, Ligia Laís Fêmina, Luciana de Fátima Maetins, Lucilia Daruiz Borsari, Luiz Roberto Hartmann Junior, Maria Gorete Carreira Andrade, Michelle Ferreira Zanchetta Morgao and Thiago de Melo. Further details will be posted on the website as they become available.
Contact: Ermiínia de Loudes Campello Fanti, email: `ebt2014@ibilce.unesp.br`.
Information: `http://www.mat.ibilce.unesp.br/EBT2014/`.

* 17–22 **Recent Developments in Adaptive Methods for PDEs, Collaborative Workshop and Short Course**, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.
Description: The aim of this workshop is to provide an introduction to the state of the art in theory and practical applications of adaptivity in PDEs. The program will begin with a two-day short course, "Adaptive Methods for the Numerical Solution of PDEs", given by Dr. Weizhang Huang (University of Kansas). The middle component of the program will focus on presentations by researchers whose work may benefit from the use of adaptive techniques for PDEs arising as mathematical models in practical applications. The final segment of the program will feature a workshop format in which breakout teams, consisting of graduate students and postdoctoral fellows led by experts on theoretical or computational aspects of adaptive methods for PDEs, will investigate the process of introducing adaptive techniques into the numerical simulations that arise in the applications identified earlier. The segment will also include several talks by researchers working in adaptivity for PDEs.
Information: `http://www.math.mun.ca/anasc`.

* 22–29 **Seventh International Conference on Differential and Functional Differential Equations**, Peoples' Friendship University of Russia, Moscow, Russia.
Description: The Conference is a satellite of the International Congress of Mathematicians 2014, August 13–21 Coex, Seoul, Korea. The scientific program will consist of invited 45-minute lectures, 30-minute lectures, and 20-minute communications. The conference will be devoted to classical topics of the theory of differential equations and different kinds of nonlocal interactions: ordinary differential equations, dynamical systems, partial differential equations, semigroups of operators, nonlocal spatio-temporal systems, functional differential equations, applications.
Information: `http://dfde2014.mi.ras.ru`.

* 25–29 **First Brazilian Workshop in Geometry of Banach Spaces BWB 2014**, Maresias Beach Hotel, Maresias (Sao Sebastio), Brazil.
Description: This workshop, organized by the University of São Paulo (USP), will focus on the theory of geometry of Banach spaces, with emphasis on the following directions: linear theory of infinite dimensional spaces and its relations to Ramsey theory, homological theory and set theory; nonlinear theory; and operator theory.
Plenary speakers: S. A. Argyros (Nat. Tech. Univ. Athens), J. M. F. Castillo (Univ. Extremadura), P. Dodos (Univ. Athens), G. Godefroy (Paris 6), R. Haydon (Univ. Oxford), W. B. Johnson (Texas A&M), P. Koszmider (Polish Acad.), G. Pisier (Paris 6 & Texas A&M), C. Rosendal (Univ. Illinois Chicago), G. Schechtman (Weizmann Inst.), Th. Schlumprecht (Texas A&M), S. Todorcevic (Paris 7 & Univ. Toronto).
Scientific Committee: J. M. F. Castillo (Univ. Extremadura), V. Ferenczi (Univ. Sao Paulo), R. Haydon (Univ. Oxford), W. B. Johnson (Texas A&M), G. Pisier (Paris 6 & Texas A&M), Th. Schlumprecht (Texas A&M), S. Todorcevic (Paris 7 & Univ. Toronto).
Information: `http://www.ime.usp.br/~banach/bwb2014/index.html`.

* 25–29 **Integrability and Cluster Algebras: Geometry and Combinatorics**, Brown University (ICERM), Providence, Rhode Island.
Description: This workshop focuses on certain kinds of discrete dynamical systems that are integrable and have interpretations in terms of cluster algebras. Some such systems, like the pentagram map and the octahedral recurrence, are motivated by concrete algebraic constructions (taking determinants) or geometric constructions based on specific configurations of points and lines in the projective plane. The systems of interest in this workshop have connections to Poisson and symplectic geometry, classical integrable PDE such as the KdV and Boussinesq equations and also to cluster algebras. The aim of the workshop is to explore geometric, algebraic, and computational facets of these systems, with a view towards uncovering new phenomena and unifying the work to date.
Information: `http://icerm.brown.edu/tw14-4-ica`.

September 2014

* 29–October 3 **International Conference on Numerical and Mathematical Modeling of Flow and Transport in Porous Media**, Centre for Advanced Academic Studies (CAAS), 20000 Dubrovnik, Croatia.
Description: The aim of the conference is to bring together researchers, scientists, engineers, and students to exchange and share their experiences, new ideas, and research results about modeling, analysis and simulation of flow and transport in porous media and application to problems including subsurface hydrology, petroleum exploration, contaminant remediation, carbon sequestration and nuclear waste storage.
Information: `http://www.caas.unizg.hr`; `http://nm2porousmedia.math.pmf.unizg.hr/index.html`.

October 2014

* 5–11 **International Conference on Algebraic Methods in Dynamical Systems (Conference in honour of the 60th birthday of Juan J. Morales-Ruiz)**, Universidad del Norte, Barranquilla, Colombia.
Main topics: Linear Differential Galois Theory, Non-linear Differential Galois Theory, Difference Galois Theory, Integrability of Dynamical Systems, Integrability of Partial Differential Equations, Integrability in Quantum Mechanics, Painlevé Transcendents.
Scientific committee: Jean-Pierre Ramis, President (Université Paul Sabatier, France), José Manuel Aroca (Universidad de Valladolid, Spain), Andrzej Maciejewski (University of Zielona. Gora, Poland), Hiroshi Umemura (Nagoya University, Japan), and Alexander Veselov (Loughborough University, United Kingdom).
Organizing committee: Primitivo Acosta-Humánez (Universidad del Norte, Colombia), David Blázquez-Sanz (Universidad Nacional de Colombia, secional Medellín), Camilo Sanabria (Universidad de los Andes, Colombia), and Sergi Simón (University of Portsmouth, United Kingdom).
Information: `http://www.scm.org.co/eventos/AMDS2014/`; `http://www.scm.org.co/eventos/AMDS2014/`; email: `amds.bcn@gmail.com`; +575 3 509 509 ext. 4844.

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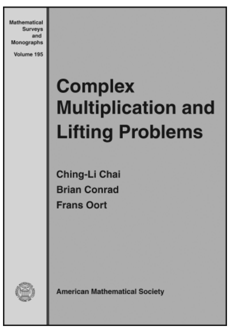
Algebra and Algebraic Geometry



Complex Multiplication and Lifting Problems

Ching-Li Chai, *University of Pennsylvania, Philadelphia, PA*, **Brian Conrad**, *Stanford University, CA*, and **Frans Oort**, *University of Utrecht, The Netherlands*

Abelian varieties with complex multiplication lie at the origins of class field theory, and they play a central role in the contemporary theory of Shimura varieties. They are special in characteristic 0 and ubiquitous over finite fields. This book explores the relationship between such abelian varieties over finite fields and over arithmetically interesting fields of characteristic 0 via the study of several natural *CM lifting problems* which had previously been solved only in special cases. In addition to giving complete solutions to such questions, the authors provide numerous examples to illustrate the general theory and present a detailed treatment of many fundamental results and concepts in the arithmetic of abelian varieties, such as the Main Theorem of Complex Multiplication and its generalizations, the finer aspects of Tate's work on abelian varieties over finite fields, and deformation theory.

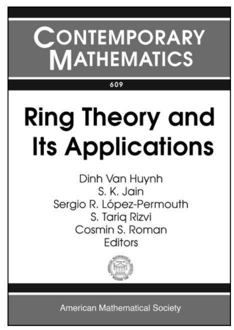
This book provides an ideal illustration of how modern techniques in arithmetic geometry (such as descent theory, crystalline methods, and group schemes) can be fruitfully combined with class field theory to answer concrete questions about abelian varieties. It will be a useful reference for researchers and advanced graduate students at the interface of number theory and algebraic geometry.

This item will also be of interest to those working in number theory.

Contents: Introduction; Algebraic theory of complex multiplication; CM lifting over a discrete valuation ring; CM lifting of p-divisible groups; CM lifting of abelian varieties up to isogeny; Some arithmetic results for abelian varieties; CM lifting via p-adic Hodge theory; Notes on quotes; Glossary of notations; Bibliography; Index.

Mathematical Surveys and Monographs, Volume 195

January 2014, 387 pages, Hardcover, ISBN: 978-1-4704-1014-8, LC 2013036892, 2010 *Mathematics Subject Classification:* 11G15, 14K02, 14L05, 14K15, 14D15, **AMS members US$80**, List US$100, Order code SURV/195



Ring Theory and Its Applications

Dinh Van Huynh, **S. K. Jain**, and **Sergio R. López-Permouth**, *Ohio University, Athens, OH*, and **S. Tariq Rizvi** and **Cosmin S. Roman**, *Ohio State University, Lima, OH*, Editors

This volume contains the proceedings of the Ring Theory Session in honor of T. Y. Lam's 70th birthday, at the 31st Ohio State Denison Mathematics Conference, held from May 25–27, 2012, at Ohio State University. Columbus, Ohio.

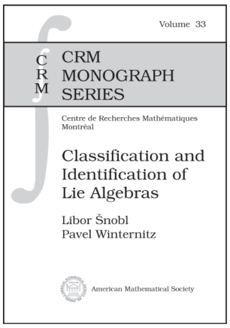
Included are expository articles and research papers covering topics such as cyclically presented modules, Eggert's conjecture, the Mittag-Leffler conjecture, clean rings, McCoy rings, QF rings, projective and injective modules, Baer modules, and Leavitt path algebras.

Graduate students and researchers in many areas of algebra will find this volume valuable as the papers point out many directions for future work; in particular, several articles contain explicit lists of open questions.

Contents: **G. M. Bergman**, Thoughts on Eggert's conjecture; **P. Bhattacharjee**, **M. L. Knox**, and **W. W. McGovern**, p-Extensions; **H. Chen**, **A. Harmancı**, and **A. C. Özcan**, Strongly J-clean rings with involutions; **J. Chen**, **W. Li**, and **L. Shen**, QF rings characterized by injectivities: A survey; **H. Q. Dinh**, Repeated-root cyclic and negacyclic codes of length $6p^8$; **A. Facchini**, **D. Smertnig**, and **N. K. Tung**, Cyclically presented modules, projective covers and factorizations; **J. Gaddis**, Isomorphisms of some quantum spaces; **P. A. Guil Asensio** and **A. K. Srivastava**, Additive unit representations in endomorphism rings and an extension of a result of Dickson and Fuller; **B. Ungor**, **S. Halicioglu**, and **A. Harmancı**, On a class of $\oplus$-supplemented modules; **D. Herbera**, Definable classes and Mitttag-Leffler conditions; **K. Joshi**, **P. Kanwar**, and **J. B. Srivastava**, A note on clean group algebras; **D. Keskin Tütüncü**, **P. F. Smith**, and **S. E. Toksoy**, On dual Baer modules; **T. Y. Lam** and **P. P. Nielsen**, Jacobson's lemma for Drazin inverses; **G. Lee**, **S. T. Rizvi**, and **C. Roman**, Transfer of certain properties from modules to their endomorphism rings; **T.-K. Lee** and **Y. Zhou**, From Boolean rings to clean rings; **A. Leroy** and **J. Matczuk**, On right strongly McCoy rings; **B. L. Osofsky**, Compatible ring structures on injective hulls of finitely embedded rings; **K. L. Price** and **S. Szydlik**, Good matrix gradings from directed graphs; **K. M. Rangaswamy**, Leavitt path algebras which are Zorn rings; **M. L. Reyes**, Sheaves that fail to represent matrix rings; **S. Singh** and **A. K. Srivastava**, Rings of invariant module type and automorphism-invariant modules.

Contemporary Mathematics, Volume 609

March 2014, approximately 314 pages, Softcover, ISBN: 978-0-8218-8797-4, LC 2013032319, 2010 *Mathematics Subject Classification:* 16-XX, 13A35, 13C10, 13E10, 14A22, 18B25, 18F20, 20G07, **AMS members US$90.40**, List US$113, Order code CONM/609



Classification and Identification of Lie Algebras

Libor Šnobl, *Czech Technical University, Prague, Czech Republic*, and **Pavel Winternitz**, *Centre de Recherches Mathématiques, Montréal, QC, Canada, and Université de Montréal, QC, Canada*

The purpose of this book is to serve as a tool for researchers and practitioners who apply Lie algebras and Lie groups to solve problems arising in science and engineering. The authors address the problem of expressing a Lie algebra obtained in some arbitrary basis in a more suitable basis in which all essential features of the Lie algebra are directly visible. This includes algorithms accomplishing decomposition into a direct sum, identification of the radical and the Levi decomposition, and the computation of the nilradical and of the Casimir invariants. Examples are given for each algorithm.

For low-dimensional Lie algebras this makes it possible to identify the given Lie algebra completely. The authors provide a representative list of all Lie algebras of dimension less or equal to 6 together with their important properties, including their Casimir invariants. The list is ordered in a way to make identification easy, using only basis independent properties of the Lie algebras. They also describe certain classes of nilpotent and solvable Lie algebras of arbitrary finite dimensions for which complete or partial classification exists and discuss in detail their construction and properties.

The book is based on material that was previously dispersed in journal articles, many of them written by one or both of the authors together with their collaborators. The reader of this book should be familiar with Lie algebra theory at an introductory level.

This item will also be of interest to those working in mathematical physics.

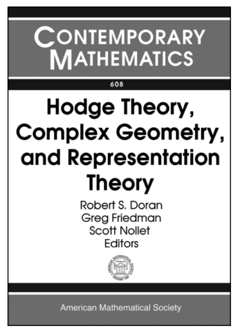
Titles in this series are co-published with the Centre de Recherches Mathématiques.

Contents: *Part 1. General theory:* Introduction and motivation; Basic concepts; Invariants of the coadjoint representation of a Lie algebra; *Part 2. Recognition of a Lie algebra given by its structure constants:* Identification of Lie algebras through the use of invariants; Decomposition into a direct sum; Levi decomposition. Identification of the radical and Levi factor; The nilradical of a Lie algebra; *Part 3. Nilpotent, solvable and Levi decomposable Lie algebras:* Nilpotent Lie algebras; Solvable Lie algebras and their nilradicals; Solvable Lie algebras with abelian nilradicals; Solvable Lie algebras with Heisenberg nilradical; Solvable Lie algebras with Borel nilradicals; Solvable Lie algebras with filiform and quasifiliform nilradicals; Levi decomposable algebras; *Part 4. Low-dimensional Lie algebras:* Structure of the lists of low-dimensional Lie algebras; Lie algebras up to dimension 3; Four-dimensional Lie algebras; Five-dimensional Lie algebras; Six-dimensional Lie algebras; Bibliography; Index.

CRM Monograph Series, Volume 33

February 2014, 306 pages, Hardcover, ISBN: 978-0-8218-4355-0, 2010 *Mathematics Subject Classification:* 17Bxx, 17B05, 81Rxx, 81R05, 70Hxx, 37J15; 17B20, 17B30, 17B40, 70Sxx, 37Jxx, **AMS members US$99.20**, List US$124, Order code CRMM/33

Geometry and Topology



Hodge Theory, Complex Geometry, and Representation Theory

Robert S. Doran, **Greg Friedman**, and **Scott Nollet**, *Texas Christian University, Ft. Worth, TX*, Editors

This volume contains the proceedings of an NSF/Conference Board of the Mathematical Sciences (CBMS) regional conference on Hodge theory, complex geometry, and representation theory, held on June 18, 2012, at the Texas Christian University in Fort Worth, TX. Phillip Griffiths, of the Institute for Advanced Study, gave 10 lectures describing now-classical work concerning how the structure of Shimura varieties as quotients of Mumford-Tate domains by arithmetic groups had been used to understand the relationship between Galois representations and automorphic forms. He then discussed recent breakthroughs of Carayol that provide the possibility of extending these results beyond the classical case. His lectures will appear as an independent volume in the CBMS series published by the AMS.

This volume, which is dedicated to Phillip Griffiths, contains carefully written expository and research articles. Expository papers include discussions of Noether-Lefschetz theory, algebraicity of Hodge loci, and the representation theory of $SL_2(\mathbb{R})$. Research articles concern the Hodge conjecture, Harish-Chandra modules, mirror symmetry, Hodge representations of Q-algebraic groups, and compactifications, distributions, and quotients of period domains. It is expected that the book will be of interest primarily to research mathematicians, physicists, and upper-level graduate students.

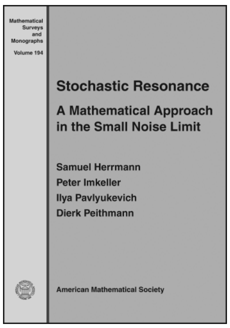
This item will also be of interest to those working in algebra and algebraic geometry.

Contents: **D. Arapura**, **X. Chen**, and **S.-J. Kang**, The smooth center of the cohomology of a singular variety; **J. Brevik** and **S. Nollet**, Developments in Noether-Lefschetz theory; **J. A. Carlson** and **D. Toledo**, Compact quotients of non-classical domains are not Kähler; **E. Cattani** and **A. Kaplan**, Algebraicity of Hodge loci for variations of Hodge structure; **M. Green** and **P. Griffiths**, On the differential equations satisfied by certain Harish-Chandra modules; **T. Hayama**, Kato-Usui partial compactifications over the toroidal compactifications of Siegel spaces; **A. Kaplan** and **M. Subils**, On the equivalence problem for bracket-generating distributions; **M. Kerr**, Notes on the representation theory of $SL_2(\mathbb{R})$; **M. Kerr**, Cup products in automorphic cohomology: The case of Sp_4; **J. D. Lewis**, Hodge type conjectures and the Bloch-Kato theorem; **C. Robles**, Principal Hodge representations; **S. Usui**, A study of mirror symmetry through log mixed Hodge theory.

Contemporary Mathematics, Volume 608

March 2014, approximately 320 pages, Softcover, ISBN: 978-0-8218-9415-6, LC 2013031105, 2010 *Mathematics Subject Classification:* 14C25, 14C30, 14D07, 14M17, 20G05, 22E45, 22E46, 22E47, 32G20, 32M10, **AMS members US$90.40**, List US$113, Order code CONM/608

Probability and Statistics



Stochastic Resonance

A Mathematical Approach in the Small Noise Limit

Samuel Herrmann, *Université de Bourgogne, Dijon, France*, **Peter Imkeller**, *Humboldt-Universität zu Berlin, Germany*, **Ilya Pavlyukevich**, *Friedrich-Schiller-Universität Jena, Germany*, and **Dierk Peithmann**, *Essen, Germany*

Stochastic resonance is a phenomenon arising in a wide spectrum of areas in the sciences ranging from physics through neuroscience to chemistry and biology.

This book presents a mathematical approach to stochastic resonance which is based on a large deviations principle (LDP) for randomly perturbed dynamical systems with a weak inhomogeneity given by an exogenous periodicity of small frequency. Resonance, the optimal tuning between period length and noise amplitude, is explained by optimizing the LDP's rate function.

The authors show that not all physical measures of tuning quality are robust with respect to dimension reduction. They propose measures of tuning quality based on exponential transition rates explained by large deviations techniques and show that these measures are robust.

The book sheds some light on the shortcomings and strengths of different concepts used in the theory and applications of stochastic resonance without attempting to give a comprehensive overview of the many facets of stochastic resonance in the various areas of sciences. It is intended for researchers and graduate students in mathematics and the sciences interested in stochastic dynamics who wish to understand the conceptual background of stochastic resonance.

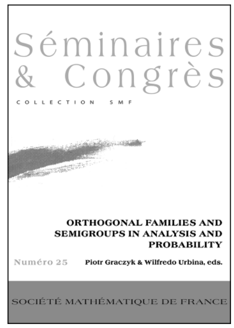
Contents: Heuristics of noise induced transitions; Transitions for time homogeneous dynamical systems with small noise; Semiclassical theory of stochastic resonance in dimension 1; Large deviations and transitions between meta-stable states of dynamical systems with small noise and weak inhomogeneity; Supplementary tools; Laplace's method; Bibliography; Index.

Mathematical Surveys and Monographs, Volume 194

January 2014, 189 pages, Hardcover, ISBN: 978-1-4704-1049-0, LC 2013034700, 2010 *Mathematics Subject Classification:* 60H10, 60J60; 34D45, 37H10, 60F10, 60J70, 60K35, 86A10, **AMS members US$61.60**, List US$77, Order code SURV/194

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Orthogonal Families and Semigroups in Analysis and Probability

Piotr Graczyk, *Université d'Angers, France*, and **Wilfredo Urbina**, *Roosevelt University, Chicago, IL*, Editors

The CIMPA-UNESCO workshop "Orthogonal Families and Semigroups in Analysis and Probability" was held in 2006 in Mirida, Venezuela and was organized with the collaboration of three Venezuelian universities (UCV, USB, and ULA). The objective of the workshop was to present the modern theory of operator semigroups, related to polynomial orthogonal expansions.

This theory comprises a vast body of knowledge and has interconnections with several other areas, including harmonic analysis, probability, random matrices, stochastic calculus, and control theory. The chapters in this volume originate from the lectures at this workshop and stress the interplay of all these domains.

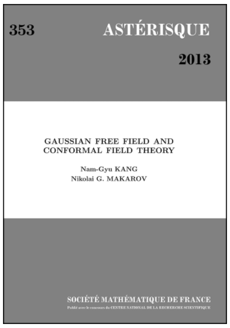
This item will also be of interest to those working in analysis.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: **D. Bárcenas** and **H. Leiva**, Semigroups and control theory; **J. Faraut**, Random matrices and orthogonal polynomials; **P. Feinsilver**, Lie algebras, representations, and analytic semigroups through dual vector fields; **P. Graczyk** and **T. Jakubowski**, Analysis of Ornstein-Uhlenbeck and Laguerre stochastic processes; **S. Thangavelu**, Hermite and Laguerre semigroups: Some recent developments; **W. O. Urbina**, Semigroups of operators for classical orthogonal polynomials and functional inequalities.

Séminaires et Congrès, Number 25

May 2013, 383 pages, Softcover, ISBN: 978-2-85629-362-1, 2010 *Mathematics Subject Classification:* 17B66, 17B99, 33C45, 33C80, 37C10, 42A38, 42A99, 42B08, 42B15, 42C25, 42C10, 46L53, 47D03, 60B10, 60E05, 60G15, 60G40, 60J45, 60H99, **AMS members US$89.60**, List US$112, Order code SECO/25



Gaussian Free Field and Conformal Field Theory

Nam-Gyu Kang, *Seoul National University, Republic of Korea*, and **Nikolai G. Makarov**, *California Institute of Technology, Pasadena, CA*

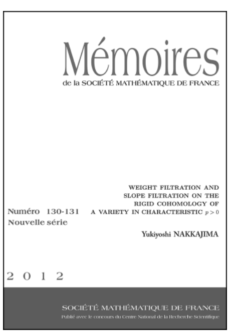
In these mostly expository lectures, the authors give an elementary introduction to conformal field theory in the context of probability theory and complex analysis. The authors consider statistical fields and define Ward functionals in terms of their Lie derivatives. Based on this approach, the authors explain some equations of conformal field theory and outline their relation to SLE theory.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Introduction; Lecture 1. Fock space fields; Appendix 2. Fock space fields as (very) generalized random functions; Lecture 3. Operator product expansion; Lecture 4. Conformal geometry of Fock space fields; Lecture 5. Stress tensor and Ward's identities; Appendix 6. Ward's identities for finite Boltzmann-Gibbs ensembles; Lecture 7. Virasoro field and representation theory; Appendix 8. Existence of the Virasoro field; Appendix 9. Operator algebra formalism; Lecture 10. Modications of the Gaussian free field; Appendix 11. Current primary fields and KZ equations; Lecture 12. Multivalued conformal Fock space fields; Appendix 13. CFT and SLE numerology; Lecture 14. Connection to SLE theory; Lecture 15. Vertex observables; Bibliography; Index.

Astérisque, Number 353

June 2013, 136 pages, Softcover, ISBN: 978-2-85629-369-0, 2010 *Mathematics Subject Classification:* 60J67, 81T40; 30C35, **AMS members US$41.60**, List US$52, Order code AST/353



Weight Filtration and Slope Filtration on the Rigid Cohomology of a Variety in Characteristic $p > 0$

Yukiyoshi Nakkajima, *Tokyo Deni University, Japan*

The author constructs a theory of weights on the rigid cohomology of a separated scheme of finite type over a perfect field of characteristic $p > 0$ by using the log crystalline cohomology of a split proper hypercovering of the scheme. The author also calculates the slope filtration on the rigid cohomology by using the cohomology of the log de Rham-Witt complex of the hypercovering.

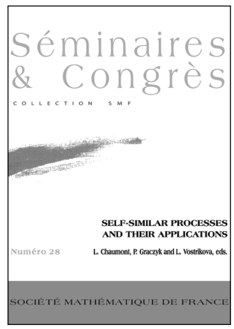
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: Introduction; Part I. Weight filtration on the log crystalline cohomology of a simplicial family of open smooth varieties in characteristic $p > 0$; Part II. Weight filtration and slope filtration on the rigid cohomology of a separated scheme of finite type over a perfect field of characteristic $p > 0$; Part III. Weight filtrations and slope filtrations on rigid cohomologies with closed support and with compact support; Bibliography.

Mémoires de la Société Mathématique de France, Number 130/131

December 2012, 250 pages, Softcover, ISBN: 978-2-85629-376-8, 2010 *Mathematics Subject Classification:* 14F30, **AMS members US$72**, List US$90, Order code SMFMEM/130/131

Analysis



Self-Similar Processes and Their Applications

Loïc Chaumont, **Piotr Graczyk**, and **Lioudmila Vostrikova**, *Université d'Angers, France*, Editors

This volume contains articles related to the conference Self-Similar Processes and Their Applications, which took place in Angers from July 20–24, 2009. Self-similarity is the property which certain stochastic processes have of preserving their distribution under a time-scale change. This property appears in all areas of probability theory and offers a number of fields of application.

The aim of this conference is to bring together the main representatives of different aspects of self-similarity currently being studied in order to promote exchanges on their recent research and enable them to share their knowledge with young researchers.

- Self-similar Markov processes
- Matrix valued self-similar processes
- Self-similarity, trees, branching and fragmentation
- Fractional and multifractional processes
- Stochastic Löwner evolution
- Self-similarity in finance

The organization of the conference was achieved in cooperation with probabilists and statisticians from the research federation Mathématiques des Pays de la Loire. The ANR Géometrie différentielle stochastique et Auto-similarité, based at the University Toulouse III, and the Franco-Mexican project ECOS-Nord, Étude des processus markoviens auto-similaires also contributed to the organization.

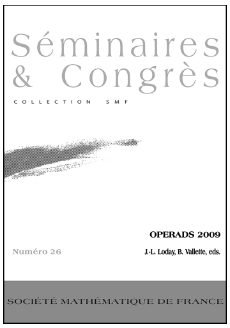
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: **K. Falconer**, Localisable, multifractional and multistable processes; **A. Echelard**, **J. L. Véhel**, and **C. Tricot**, A unified framework for 2-microlocal and large deviation spectra; **M. Maejima** and **Y. Ueda**, Quasi-selfsimilar additive processes; **P.-O. Amblard**, **J.-F. Coeurjolly**, **F. Lavancier**, and **A. Philippe**, Basic properties of the multivariate fractional Brownian motion; **J. B. Levy** and **M. S. Taqqu**, On the codifference of linear fractional stable motion; **M. Yor**, On weak and strong Brownian filtrations: definitions and examples; A. Program; B. List of participants.

Séminaires et Congrès, Number 28

July 2013, 121 pages, Softcover, ISBN: 978-2-85629-365-2, 2010 *Mathematics Subject Classification:* 26A16, 28A80, 42C40, 60E07, 60G10, 60G18, 60G22, 60G51, 60J65, **AMS members US$31.20**, List US$39, Order code SECO/28

Discrete Mathematics and Combinatorics



Operads 2009

Jean-Louis Loday, *Université de Strasbourg, CNRS, France*, and **Bruno Vallette**, *Université de Nice-Sophia Antipolis, France*, Editors

An operad is a mathematical device used to encode universally a wide variety of algebraic structures. The name operad appeared first in the 1970s in algebraic topology to recognize *n*-fold loop spaces. Operads enjoyed a renaissance in the nineties, mainly under the impulse of quantum field theories. This universal notion is now used in many domains of mathematics such as differential geometry (deformation theory), algebraic geometry (moduli spaces of curves, Gromov-Witten invariants), noncommutative geometry (cyclic homology), algebraic combinatorics (Hopf algebras), theoretical physics (field theories, renormalization), computer science (rewriting systems) and universal algebra.

The purpose of this volume is to present contributions about the notion of operads in these fields, where they play an important role. This volume is a result of a school and a conference, “Operads 2009”, both of which took place at the CIRM (Luminy, France) in April 2009.

This item will also be of interest to those working in algebra and algebraic geometry.

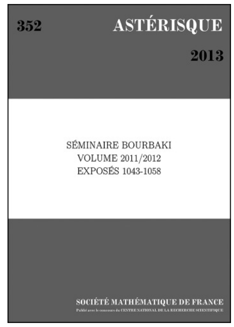
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: M. Batanin, **C. Berger**, and **M. Markl**, Operads of natural operations I: Lattice paths, braces and Hochschild cochains; **F. Chapoton**, Categorification of the dendriform operad; **V. Dotsenko**, Freeness theorems for operads via Gröbner bases; **V. Dotsenko** and **M. V. Johansson**, Implementing Gröbner bases for operads; **B. Fresse**, Batanin’s category of pruned trees is Koszul; **Y. Guiraud** and **P. Malbos**, Identities among relations for higher-dimensional rewriting systems; **Y. Lafont**, Diagram rewriting and operads; **Y. I. Manin**, Renormalization and computation I: Motivation and background; **T. Schedler**, Connes-Kreimer quantizations and PBW theorems for pre-Lie algebras; **D. P. Sinha**, The (non-equivariant) homology of the little disks operad *A*.

Séminaires et Congrès, Number 26

July 2013, 279 pages, Softcover, ISBN: 978-2-85629-363-8, 2010 *Mathematics Subject Classification:* 05C05, 06A11, 08B20, 16S15, 16G20, 17B35, 17D99, 18D05, 18C10, 18G15, 18D20, 18D50, 18G55, 55P48, 55R80, 55U99, 57T30, 68Q05, 68Q12, 68N18, 68Q25, 68W30, 68Q42, **AMS members US$51.20**, List US$64, Order code SECO/26

General Interest



Séminaire Bourbaki

Volume 2011/2012
Exposés 1043–1058

A note to readers: Many of the papers in this volume are in French.

This 64th volume of the Bourbaki Seminar contains the texts of the sixteen survey lectures presented during 2011/2012: one about functional analysis, one about complexity of algorithms, two on partial differential equations, four on algebraic geometry, one about differential geometry, one about ergodic theory, three on number theory, and three other lectures on mathematical physics.

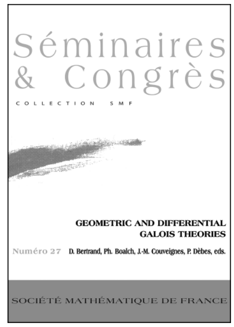
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Contents: *NOVEMBRE 2011:* **M. Brion**, Restriction de représentations et projections d’orbites coadjointes; **C. Mouhot**, Stabilité orbitale pour le système de Vlasov-Poisson gravitationnel; **P. Pansu**, Difficulté d’approximation; **P. Raphaël**, Concentration compacité à la Kenig-Merle; *JANVIER 2012:* **K. Ball**, The Ribe programme; **P. Deligne**, Multizêtas, d’après Francis Brown; **B. Poonen**, Average rank of elliptic curves; **C. Sabbah**, Théorie de Hodge et correspondance de Hitchin-Kobayashi sauvages; *MARS 2012:* **M. Dafermos**, The formation of black holes in general relativity; **C. Garban**, Quantum gravity and the KPZ formula; **D. Lannes**, Space time resonances; **J. Wolf**, Arithmetic and polynomial progressions in the primes; *JUIN 2012:* **N. Bergeron**, La conjecture des sous-groupes de surfaces; **A. Ducros**, Les espaces de Berkovich sont modérés; **J.-M. Fontaine**, Perfectoïdes, presque pureté et monodromiepoids; **F. Ledrappier**, Mesures stationnaires sur les espaces homogènes.

Astérisque, Number 352

July 2013, 556 pages, Softcover, ISBN: 978-2-85629-371-3, 2010 *Mathematics Subject Classification:* 03C64, 03C65, 03C99, 05C10, 05C12, 05C80, 05C85, 11B25, 11B30, 11E76, 11G05, 11G25, 11G99, 11N13, 14F20, 14G22, 14J60, 14L24, 14M15, 17B08, 20G05, 22E40, 22E46, 30F99, 32C38, 35B34, 35B60, 35E20, 35L67, 35Q35, 35Q60, 35Q83, 35Qxx, 37D40, 37N20, 37Nxx, 46B85, 47N10, 53C07, 60B99, 60C05, 60F17, 68Q17, 68R10, 68W25, 82B05, 82B20, 82B27, 82Cxx, 83C05, 83C57, 83C75, 85Axx, 90C22, 91B14, **Individual member US$112.80**, List US$141, Order code AST/352

Number Theory



Geometric and Differential Galois Theories

Daniel Bertrand, *Université Pierre et Marie Curie, Paris, France*, **Philip Boalch**, *École Normale Supérieure et CNRS, Paris, France*, **Jean-Marc Couveignes**, *Université Bordeaux I, Talence, France*, and **Pierre Dèbes**, *Université de Lille I, Villeneuve d'Ascq, France*, Editors

On March 29–April 2, 2010, a meeting was organized at the Luminy CIRM (France) on geometric and differential Galois theories, to recognize the close ties these theories have woven in recent years. This volume contains the proceedings of this meeting. Although it may be viewed as a continuation of the meeting held six years earlier on arithmetic and differential Galois groups (see *Groups de Galois Arithmétiques et Differentiels*, Séminaires et Congrès, volume 13), several new and promising themes have appeared.

The articles gathered here cover the following topics: moduli spaces of connexions, differential equations and coverings in finite characteristic, liftings, monodromy groups in their various guises (tempered fundamental group, motivic groups, generalized difference Galois groups), and arithmetic applications.

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Contents: L. Bary-Soroker and **F. Arno**, Open problems in the theory of ample fields; **A. Buium**, Galois groups arising from arithmetic étale equations; **A. Cadoret**, Motivated cycles under specialization; **A. Cadoret** and **A. Tamagawa**, Note on torsion conjecture; **F. Heiderich**, Introduction to the Galois theory of Artinian simple module algebras; **E. Lepage**, Tempered fundamental group; **F. Loray**, **M.-H. Saito**, and **C. Simpson**, Foliations on the moduli space of connections; **B. Matzat**, Monodromy of Frobenius modules; **A. Maurischat**, On the finite inverse problem in iterative étale Galois theory; **A. Obus**, Toward Abhyankar's inertia conjecture for $PSL_2(\ell)$; **M. Van Der Put**, Families of linear étale equations and the Painlevé equations; **M. Wibmer**, On the Galois theory of strongly normal étale and difference extensions; Annexe A. Programme; Annexe B. Liste des participants.

Séminaires et Congrès, Number 27

July 2013, 247 pages, Softcover, ISBN: 978-2-85629-364-5, 2010 *Mathematics Subject Classification:* 11F32, 11S20, 12E30, 12F10, 12F12, 12F15, 12H05, 12H10, 12H20, 12H25, 13B05, 14C25, 14D20, 14D22, 14F20, 14F42, 14G20, 14G22, 14H30, 14K15, 16T10, 20D06, 20G40, 32G20, 32G34, 34M15, 34M55, **AMS members US$48**, List US$60, Order code SECO/27

General Information Regarding Meetings & Conferences of the AMS

Speakers and Organizers: The Council has decreed that no paper, whether invited or contributed, may be listed in the program of a meeting of the Society unless an abstract of the paper has been received in Providence prior to the deadline.

Special Sessions: The number of Special Sessions at an Annual Meeting is limited. Special Sessions at annual meetings are held under the supervision of the Program Committee for National Meetings and, for sectional meetings, under the supervision of each Section Program Committee. They are administered by the associate secretary in charge of that meeting with staff assistance from the Meetings and Conferences Department in Providence. (See the list of associate secretaries on page ??? of this issue.)

Each person selected to give an Invited Address is also invited to generate a Special Session, either by personally organizing one or by having it organized by others. Proposals to organize a Special Session are sometimes solicited either by a program committee or by the associate secretary. Other proposals should be submitted to the associate secretary in charge of that meeting (who is an ex officio member of the program committee) at the address listed on page ???. These proposals must be in the hands of the associate secretary at least seven months (for sectional meetings) or nine months (for national meetings) prior to the meeting at which the Special Session is to be held in order that the committee may consider all the proposals for Special Sessions simultaneously. Special Sessions must be announced in the *Notices* in a timely fashion so that any Society member who so wishes may submit an abstract for consideration for presentation in the Special Session.

Talks in Special Sessions are usually limited to twenty minutes; however, organizers who wish to allocate more time to individual speakers may do so within certain limits. A great many of the papers presented in Special Sessions at meetings of the Society are invited papers, but any member of the Society who wishes to do so may submit an abstract for consideration for presentation in a Special Session, provided it is submitted to the AMS prior to the special early deadline for consideration. Contributors should know that there is a limit to the size of a single Special Session, so sometimes all places are filled by invitation. An author *may* speak by invitation in more than one Special Session at the same meeting. Papers submitted for consideration for inclusion in Special Sessions but not accepted will receive consideration for a contributed paper session, unless specific instructions to the contrary are given.

The Society reserves the right of first refusal for the publication of proceedings of any Special Session. If published by the AMS, these proceedings appear in the book series *Contemporary Mathematics*. For more detailed information on organizing a Special Session, see `www.ams.org/meetings/specialsessionmanual.html`.

Contributed Papers: The Society also accepts abstracts for ten-minute contributed papers. These abstracts will be grouped by related *Mathematical Reviews* subject classifications into sessions to the extent possible. The title and author of each paper accepted and the time of presentation will be listed in the program of the meeting. Although an individual may present only one ten-minute contributed paper at a meeting, any combination of joint authorship may be accepted, provided no individual speaks more than once.

Other Sessions: In accordance with policy established by the AMS Committee on Meetings and Conferences, mathematicians interested in organizing a session (for either an annual or a sectional meeting) on employment opportunities inside or outside academia for young mathematicians should contact the associate secretary for the meeting with a proposal by the stated deadline. Also, potential organizers for poster sessions on a topic of choice should contact the associate secretary before the deadline.

Abstracts: Abstracts for all papers must be received by the meeting coordinator in Providence by the stated deadline. Unfortunately, late papers cannot be accommodated.

Submission Procedures: Visit the Meetings and Conferences homepage on the Web at `http://www.ams.org/meetings` and select "Submit an abstract".

Site Selection for Sectional Meetings

Sectional meeting sites are recommended by the associate secretary for the section and approved by the Secretariat. Recommendations are usually made eighteen to twenty-four months in advance. Host departments supply local information, ten to fifteen rooms with overhead projectors and a laptop projector for contributed paper sessions and Special Sessions, an auditorium with twin overhead projectors and a laptop projector for Invited Addresses, space for registration activities and an AMS book exhibit, and registration clerks. The Society partially reimburses for the rental of facilities and equipment and for staffing the registration desk. Most host departments volunteer; to do so, or for more information, contact the associate secretary for the section.

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `http://www.ams.org/meetings/`. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

Baltimore, Maryland

Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 15–18, 2014
Wednesday – Saturday

Meeting #1096

Joint Mathematics Meetings, including the 120th Annual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia M. Benkart
Announcement issue of *Notices*: October 2013
Program first available on AMS website: November 1, 2013
Program issue of electronic *Notices*: January 2013
Issue of *Abstracts*: Volume 35, Issue 1

Deadlines

For organizers: Expired
For abstracts: Expired

Knoxville, Tennessee

University of Tennessee, Knoxville

March 21–23, 2014
Friday – Sunday

Meeting #1097

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: January 2014
Program first available on AMS website: February 6, 2014
Program issue of electronic *Notices*: March 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: January 28, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Maria Chudnovsky, Columbia University, *Coloring graphs with forbidden induced subgraphs* (Erdős Memorial Lecture).

Ilse Ipsen, North Carolina State University, *Introduction to randomized matrix algorithms*.

Daniel Krashen, University of Georgia, *Algebraic structures and the arithmetic of fields*.

Suresh Venapally, Emory University, *Quadratic forms and Galois cohomology*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the *abstract submission form found at* `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Methods in Graph Theory and Combinatorics (Code: SS 7A), **Felix Lazebnik**, University of Delaware, **Andrew Woldar**, Villanova University, and **Bangteng Xu**, Eastern Kentucky University.

Arithmetic of Algebraic Curves (Code: SS 9A), **Lubjana Beshaj**, Oakland University, **Caleb Shor**, Western New England University, and **Andreas Malmendier**, Colby College.

Commutative Ring Theory (in honor of the retirement of David E. Dobbs) (Code: SS 1A), **David Anderson**, University of Tennessee, Knoxville, and **Jay Shapiro**, George Mason University.

Completely Integrable Systems and Dispersive Nonlinear Equations (Code: SS 12A), **Robert Buckingham**, University of Cincinnati, and **Peter Perry**, University of Kentucky.

Complex Analysis, Probability, and Metric Geometry (Code: SS 11A), **Matthew Badger**, Stony Brook University, **Jim Gill**, St. Louis University, and **Joan Lind**, University of Tennessee, Knoxville.

Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (Code: SS 18A), **Xiaobing Feng** and **Ohannes Karakashian**, University of Tennessee, Knoxville, and **Yulong Xing**, University of Tennessee, Knoxville, and Oak Ridge National Laboratory.

Diversity of Modeling and Optimal Control: A Celebration of Suzanne Lenhart's 60th Birthday (Code: SS 3A), **Wandi Ding**, Middle Tennessee State University, and **Renee Fister**, Murrray State University.

Fractal Geometry and Ergodic Theory (Code: SS 2A), **Mrinal Kanti Roychowdhury**, University of Texas Pan American.

Galois Cohomology and the Brauer Group (Code: SS 26A), **Ben Antieau**, University of Washington, **Daniel Krashen**, University of Georgia, and **Suresh Venapally**, Emory University.

Geometric Topology (Code: SS 21A), **Craig Guilbault**, University of Wisconsin-Milwaukee, and **Steve Ferry**, Rutgers University.

Geometric Topology and Number Theory (Code: SS 22A), **Eriko Hironaka** and **Kathleen Petersen**, Florida State University.

Geometric and Algebraic Combinatorics (Code: SS 16A), **Benjamin Braun** and **Carl Lee**, University of Kentucky.

Geometric and Combinatorial Methods in Lie Theory (Code: SS 15A), **Amber Russell** and **William Graham**, University of Georgia.

Graph Theory (Code: SS 8A), **Chris Stephens**, **Dong Ye**, and **Xiaoya Zha**, Middle Tennessee State University.

Harmonic Analysis and Nonlinear Partial Differential Equations (Code: SS 5A), **J. Denzler**, **M. Frazier**, **Tuoc Phan**, and **T. Todorova**, University of Tennessee, Knoxville.

Invariant Subspaces of Function Spaces (Code: SS 6A), **Catherine Beneteau**, University of South Florida, **Alberto A. Condori**, Florida Gulf Coast University, **Constanze Liaw**, Baylor University, and **Bill Ross**, University of Richmond.

Mathematical Modeling of the Within- and Between-Host Dynamics of Infectious Diseases (Code: SS 25A), **Megan Powell**, University of St. Francis, and **Judy Day** and **Vitaly Ganusov**, University of Tennessee, Knoxville.

Mathematical Physics and Spectral Theory (Code: SS 10A), **Roger Nichols**, The University of Tennessee at Chattanooga, and **Günter Stolz**, University of Alabama at Birmingham.

Metric Geometry and Topology (Code: SS 23A), **Catherine Searle**, Oregon State University, **Jay Wilkins**, University of Connecticut, and **Conrad Plaut**, University of Tennessee, Knoxville.

Nonlinear Partial Differential Equations in the Applied Sciences (Code: SS 19A), **Lorena Bociu**, North Carolina State University, **Ciprian Gal**, Florida International University, and **Daniel Toundykov**, University of Nebraska-Lincoln.

Randomized Numerical Linear Algebra (Code: SS 4A), **Ilse Ipsen**, North Carolina State University.

Recent Development on Hyperbolic Conservation Laws (Code: SS 20A), **Geng Chen**, **Ronghua Pan**, and **Weizhe Zhang**, Georgia Tech.

Scientific Computing, Numerical Analysis, and Mathematical Modeling (Code: SS 17A), **Vasilios Alexiades**, **Xiaobing Feng**, and **Steven Wise**, University of Tennessee, Knoxville.

Singularities and Physics (Code: SS 13A), **Mboyo Esole**, Harvard University, and **Paolo Aluffi**, Florida State University.

Stochastic Processes and Related Topics (Code: SS 14A), **Jan Rosinski** and **Jie Xiong**, University of Tennessee, Knoxville.

von Neumann Algebras and Free Probability (Code: SS 24A), **Remus Nicoara**, University of Tennessee, Knoxville, and **Arnaud Brothier**, Vanderbilt University.

Session for Contributed Talks

There also will be sessions for 10-minute contributed talks. Please see the abstracts submission form at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`. **The deadline for all submissions is January 28, 2014**.

Accommodations

Participants should make their own arrangements directly with the properties listed below. All hotels listed are within a 15- to 20-minute drive from McGhee Tyson Airport. Special rates for the meeting have been negotiated and are available at the properties shown below for Friday and Saturday nights, March 21 and 22, 2014; rates may be extended to one day before or after these dates if rooms are available; please ask when you make your reservation. The AMS is not responsible for the availability of rates or for the quality of the accommodations. When making reservations **participants should state that they are with the American Mathematical Society (AMS) meeting on the UT-Knoxville campus**. Hotels have varying cancellation or early checkout penalties; be sure to ask for details when making your reservation. The room rates listed do not include applicable taxes; the current tax rate on hotel rooms is 17.25%. All properties are smoke free.

Hilton Garden Inn, 1706 West Cumberland Ave., Knoxville, TN 37916; 865-437-5500 (phone) or 865-437-5501 (fax); www.stayhgi.com. US$119/single or double, US$129/triple, or US$139/quad (king or two queens, or two-queen suite). Downtown's newest hotel includes a full service restaurant and bar, business center, fitness center, complimentary Internet access throughout the hotel, including sleeping rooms; in-room coffee makers, refrigerators, and microwaves; complimentary electric shuttle service within the local area; US$10 for indoor parking (valet only). Located .5 mile from the meeting site on campus. **Deadline for reservations is February 28, 2014**.

Holiday Inn World's Fair Park, 525 Henley St., Knoxville, TN 37902; 565-522-2800; www.holidayinnwfp.com. US$97/single or double; restaurant on premises; fitness facility, complimentary Internet access throughout the hotel, including sleeping rooms; free on-site parking for AMS participants; indoor swimming pool; complimentary local area shuttle service. Located .5 mile from the meeting site on campus. **Deadline for reservations is February 27, 2014.**

Four Points by Sheraton, 1109 White Ave., Knoxville, TN 37916; 865-971-4663; reservations available through https://www.starwoodmeeting.com/StarGroupsWeb/booking/reservation?id=1310240435&key=81A57. US$99/single or double (king or double queens). MK's restaurant on premises, open 6:30 a.m.–10:00 p.m.; complimentary Internet access throughout hotel, including sleeping rooms, which also include coffee makers; fitness center and business office on site. The current rate for on-site parking is US$8 (subject to change). Located .15 mile from the meeting site on campus. **Deadline for reservations is February 26, 2014**; the stated rate is subject to availability.

Hilton Knoxville, 501 West Church Ave., Knoxville, TN 37902, 865-251-2578 or 800-445-8667 (phone), 865-525-6532 (fax); http://www3.hilton.com/en/hotels/tennessee/hilton-knoxville-KNXKHHF/index.html. US$99/single or double (two queens); complimentary Internet throughout the hotel, including sleeping rooms; fitness room; Market Cafe features a breakfast buffet and lunch items, with room service available; full service Starbucks and The Orange Martini lounge on premises. AMS guests may take advantage of the discounted self-parking rate of US$7. Located .78 mile from the meeting site on campus. Please cite MATH when making your reservation. **Deadline for reservations is February 5, 2014.**

Dining on Campus

Information about dining options on campus and in the surrounding area will be published on the meeting website at a later date.

Local Information and Maps

The University of Tennessee Department of Mathematics website is found at http://www.math.utk.edu/; the street address of the department is 1403 Circle Dr., Knoxville, TN 37996.

A good campus map with searchable directions is found at http://www.utk.edu/maps/campus/.

Other Activities

AMS Book Sale: Stop by the on-site AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Parking

Free parking for participants will be in the 11th Street Parking Garage, near the intersection of 11th St. and Cumberland Ave., next to the University Police building. The meeting venues are a very short walk from the garage. Parking is on levels 3–9; please be aware that the Level 2-Staff area is for the exclusive use of university police. Signage in the garage clearly marks this area.

Registration and Meeting Information

Knoxville, Tennessee is located in the Eastern Time Zone.

The meeting will take place on the main campus of the University of Tennessee, Knoxville. Sessions and Invited Addresses will be held in Ayers Hall, Dabney-Beuhler Hall, Dougherty Engineering, and the Min Kao Building. Advance registration will become available on January 20, 2014, through the links for this meeting on the AMS website at http://www.ams.org/meetings/sectional/sectional.html. Fees are US$54 for AMS members, US$76 for nonmembers; and US$5 for students, unemployed mathematicians, and emeritus members, payable by credit card.

The on-site registration desk will be located in the Min Kao Building and will be open Friday, 1:00 p.m.–5:00 p.m., and Saturday, 7:30 a.m.–4:00 p.m. Fees are payable on-site by cash, check, or credit card.

Travel Information

The nearest large airport is McGhee Tyson Airport (**TYS**), 2055 Alcoa Hwy, Alcoa, TN 37701; 865-342-3000. The airport is about twelve miles south of campus. Taxi fare to campus would be about US$25.

Driving Directions to UT Knoxville Parking Garage

From the Airport: Follow US129 (Alcoa Highway) north to Knoxville and take the exit to I-40 east; change lanes left to enter I-40. Take the next exit (388) toward downtown Knoxville. Keep right at the fork to 11th St. The garage is one block after the second traffic light.

From the South or West: Follow I-75N/I-40W to Exit 388. Keep right at the fork to 11th St. The garage is one block after the second traffic light.

From the East: Follow I-40W to Exit 389 and stay left toward the James White Parkway. Take the exit for Summit Hill Dr. and turn right. Folllow Summit Hill Dr. about 3/4 mile to 11th St. The garage is one block after the traffic light.

From the North: Take I-75 south to Knoxville. After Exit 108 follow the signs to I-275. After Exit 1, follow the exit signs to I-40 East and downtown. Keep right at the first fork for the exit (toward downtown). Keep right at the second fork to 11th St. The garage is one block after the second traffic light.

Car Rental

Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at `www.hertz.com`, click on the box "I have a discount", and type in our convention number (CV) **04N30004**. You can also call Hertz directly at 800-654-2240 (U.S. and Canada) or 405-749-4434 (other countries). At the time this announcement was prepared, rates started at US$29.47 per day on the weekend. At the time of your reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

Weather

Mid-March weather is usually pleasant, with daytime temperatures ranging in the mid-50s F. and dropping into the 30s F. in the evenings, so dressing in layers is advised. Passing showers are a possibility.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `http://sites.nationalacademies.org/pga/biso/visas/` and `http://travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `dls@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence

- property ownership

- bank accounts

- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Baltimore, Maryland

University of Maryland, Baltimore County

March 29–30, 2014

Saturday – Sunday

Meeting #1098

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: January 2014
Program first available on AMS website: February 26, 2014
Program issue of electronic *Notices*: March 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: January 28, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Maria Gordina, University of Connecticut, *Stochastic analysis and geometric functional inequalities*.

L. Mahadevan, Harvard University, *Shape: Mathematics, physics and biology*.

Nimish Shah, Ohio State University, *Dynamics of subgroup actions on homogeneous spaces and its interaction with number theory*.

Dani Wise, McGill University, *Cube complexes*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Data Assimilation Applied to Controlled Systems (Code: SS 15A), **Damon McDougall**, University of Texas at Austin, and **Richard Moore**, New Jersey Institute of Technology.

Difference Equations and Applications (Code: SS 8A), **Michael Radin**, Rochester Institute of Technology.

Discrete Geometry in Crystallography (Code: SS 14A), **Egon Schulte**, Northeastern University, and **Marjorie Senechal**, Smith College.

Harmonic Analysis and Its Applications (Code: SS 10A), **Susanna Dann**, University of Missouri, **Azita Mayeli**, Queensborough College, City University of New York, and **Gestur Olafsson**, Louisana State University.

Interaction between Complex and Geometric Analysis (Code: SS 13A), **Peng Wu**, Cornell University, and **Yuan Yuan**, Syracuse University.

Invariants in Low-Dimensional Topology (Code: SS 1A), **Jennifer Hom**, Columbia University, and **Tye Lidman**, University of Texas at Austin.

Knots and Applications (Code: SS 3A), **Louis Kauffman**, University of Illinois at Chicago, **Samuel Lomonaco**, University of Maryland, Baltimore County, and **Jozef Przytycki**, George Washington University.

Low-dimensional Topology and Group Theory (Code: SS 16A), **David Futer**, Temple University, and **Daniel Wise**, McGill University.

Mathematical Biology (Code: SS 6A), **Jonathan Bell** and **Brad Peercy**, University of Maryland Baltimore County.

Mathematical Finance (Code: SS 2A), **Agostino Capponi**, John Hopkins University.

Mechanics and Control (Code: SS 9A), **Jinglai Shen**, University of Maryland Baltimore County, and **Dmitry Zenkov**, North Carolina State University.

Novel Developments in Tomography and Applications (Code: SS 4A), **Alexander Katsevich**, **Alexandru Tamasan**, and **Alexander Tovbis**, University of Central Florida.

Open Problems in Stochastic Analysis and Related Fields (Code: SS 7A), **Masha Gordina**, University of Connecticut, and **Tai Melcher**, University of Virginia.

Optimization and Related Topics (Code: SS 11A), **M. Seetharama Gowda**, **Osman Guler**, **Florian Potra**, and **Jinlai Shen**, University of Maryland at Baltimore County.

Substitution and Tiling Dynamical Systems (Code: SS 17A), **Natalie Priebe Frank**, Vassar College, and **E. Arthur Robinson Jr.**, George Washington University.

Theory and Applications of Differential Equations on Graphs (Code: SS 5A), **Jonathan Bell**, University of Maryland, Baltimore County, and **Sergei Avdonin**, University of Alaska Fairbanks.

Session for Contributed Talks

There also will be a session for 10-minute contributed talks. Please see the abstracts submission form at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`. **The deadline for all submissions is January 28, 2014.**

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include hotel tax. At the time of publication current Baltimore sales tax per room per night was 13% (a combined rate of 6% Sales tax and 7% occupancy tax). Participants must state that they are with the **American Mathematical Society (AMS) Meeting at University of Maryland, Baltimore County** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

The Doubletree by Hilton Baltimore-BWI Airport, 890 Elkridge Landing Rd., Linthicum Heights, MD, 21090 (410) 859-8400; `http://doubletree3.hilton.com/en/hotels/maryland/doubletree-by-hilton-baltimore-bwi-airport-BWIBADT/index.html`. Rates are US$99 per night for single/double occupancy. Amenities include complimentary wireless Internet in guest rooms, complimentary parking, complimentary airport shuttle service, business center, fitness center, restaurant and lounge on property. This hotel is pet-friendly. This property is located approximately 6 miles from the campus by car. Check in time is 3:00 p.m.; check out time is 12:00 p.m. **The deadline for reservations at this rate is February 26, 2014.**

Fairfield Inn and Suites Baltimore BWI Airport, 1020 Andover Road, Linthicum Heights, MD, 21090, (410) 691-1001; `http://www.marriott.com/hotels/travel/bwiaf-fairfield-inn-and-suites-baltimore-bwi-airport/`. Rates are US$79 per night for single/double occupancy. Amenities include complimentary hearty continental breakfast, complimentary wireless Internet access throughout property, business center, indoor pool, 24-hour fitness center, complimentary newspaper, and complimentary parking on site. This property is located approximately 6 miles from the campus by car. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations at this rate is March 7, 2014.**

Hampton Inn BWI/Baltimore, 829 Elkridge Road, Linthicum, MD, 21090, (410) 850-0600; `http://www.hamptoninnbwiairport.com/`. Rates are US$89 per night for single/double/quad occupancy. Amenities include complimentary wireless Internet, complimentary hot breakfast buffet, complimentary airport shuttle service and limited shuttle service to local restaurants, complimentary coffee service, fitness center, and complimentary parking on-site. This property is pet friendly. This property is located approximately 6 miles from the campus by car. Check in time is 3:00 p.m. and check out time is 12:00 noon. **This rate will be offered by the property until March 29, 2014, based upon availablity of rooms.**

The BWI Airport Marriott, 1743 West Nursery Road, Linthicum, MD, 21090, (410) 859-8300; `http://www.marriott.com/hotels/travel/bwiap-bwi-airport-marriott/`. Rates are US$109 per night for single/double occupancy in a room with a king or two double beds. Amenities include complimentary wireless Internet in lobby and public areas, guest room wireless Internet access for US$12.95/day, complimentary on-site parking, complimentary airport shuttle, two restaurants on site serving breakfast, lunch, dinner, and room service, Starbucks, fitness center, indoor pool, and business center. This property is located approximately 6 miles from the campus by car. Check in time is 3:00 p.m. and check out time is 12:00 noon. **The deadline for reservations at this rate is February 28, 2014.**

Sheraton Baltimore Washington Airport-BWI, 1100 Old Elkridge Landing Road, Linthicum, MD, 21090, (443) 577-2100; `http://www.starwoodhotels.com/sheraton/property/overview/index.html?propertyID=1495`. Rates are US$99 per night for a king room, single/double occupancy. Amenities include complimentary wireless Internet in lobby, guest room wireless or wired Internet access for US$9.95/day, complimentary on-site parking, complimentary airport shuttle and shuttle to the surrounding area, restaurants on site serving breakfast, lunch, dinner, and room service, fitness center, indoor pool, and business center. This property is located approximately 6 miles from the campus by car. Check in time is 3:00 p.m. and check out time is 12:00 noon.**The deadline for reservations at this rate is February 27, 2014.**

Food Services

On Campus: At the time of publication dining options on campus include venues located in *The Commons*, a 5-10 minute walk from the location of the meeting. Options in The Commons include *True Grits*, a dining hall, open Saturday and Sunday 10:30 a.m. - 2:00 p.m. and 4:30 p.m. - 7:00 p.m., and *Salsarita's Fresh Cantina*, serving healthy Tex-Mex options, open Saturday 11:00 a.m.- 1:00 a.m. and Sunday 11:00 a.m. - 11:00 p.m. There is also a 24-hour convenience store, *Outakes*, located in The Commons, offering convenience foods and grab-and-go items. There is a Subway restaurant, serving breakfast, sandwiches and salads, located in the Research Park adjacent to campus, approximately a 5-minute walk from the meeting. Hours of operation for this location are Saturday, 8:00 a.m. - 7:00 p.m. and Sunday 9:00 a.m. - 7:00 p.m.

Off Campus: The nearby towns of Catonsville (Frederick Road) and Arbutus (East Drive) have retail districts with a variety of small shops and services. *Arundel Mills Mall*, 15 minutes away, offers shopping, entertainment and restaurants.

The Catonsville Main Street Area, to the west of campus, has many dining options including:

Dimitri's International Grille, 2205 Frederick Rd, Catonsville, (410) 747-1927, open Saturday 11:00 am.-midnight and Sunday 11:00 a.m.-10:00 p.m., `http://www.dimitris.us/catonsville_baltimore/mediterranean_greek_restaurant_baltimore.html`; serving Mediterranean and Greek cusine.

G.L. Shacks Grill of Catonsville, 583 Frederick Road, (410) 788-0879, open Saturday and Sunday 11:00 a.m.-2:00 a.m., `http://www.glshacksgrill.com/catonsville/index.php`; featuring fine food in a casual atmosphere with an adjoining sports bar.

Indian Delight Restaurant, 622 Frederick Road, Cantonsville (410) 744-4422, open Saturday and Sunday for lunch between 12:00 p.m.- 3:00 p.m. and for dinner 5:00 p.m. - 9:30 p.m. `http://tajrestaurant.com/`; serving Indian cuisine with Halal Kosher options.

Matthew's 1600 Restaurant and Bar, 1600 Fredrick Road, Catonsville, (410) 788-2500, Saturday open 11:30 a.m. for lunch and dinner and Sunday open 11:00 a.m. - 2:00 p.m. for brunch; casual dining wiht a varied menu.

Other dining options can be found in nearby Linthicum and Ellicott City. Some restaurants located here include:

Bare Bones Grill and Brewery, 9150-22 Baltimore Natl. Pike, Ellicott City, (410) 461-0770, open Saturday 11:30 a.m.-1:30 a.m. and Sunday 11:30 a.m.-10 p.m.; pub fare and craft beer.

G & M Restaurant, 804 N Hammonds Ferry Road, Linthicum Heights (877) 554-3723, open for lunch and dinner Saturday 10:00 a.m. - 11:00 p.m. and Sunday 10:00 a.m.-10:00 p.m.; serving award winning crab cakes, seafood, chicken, and steaks.

Honey Pig Restaurant, 10045 Baltimore National Pike, Ellicott City, (410) 696-2426, open Saturday and Sunday; `http://www.eathoneypig.com/`; serving Gooldagee Korean BBQ.

Maiwand Kabob, 839 Elkridge Landing Rd., Linthicum Heights, (410) 850-0273; open Saturday and Sunday; casual dining serving Afghan cuisine.

Olive Grove, 705 North Hammonds Ferry Road, Linthicum Heights, (410) 636-1385; serving Italian cuisine.

Ruby Tuesday, 950 International Drive, Linthicum Heights, (410) 694-0031, open Saturday 11:00 a.m. - midnight and Sunday 11:00 a.m.- 10:00 p.m.; `http://www.rubytuesday.com/`; serving American cuisine.

Snyders Willow Grove Restaurant, 841 North Hammonds Ferry Road, Linthicum Heights (410) 789-1149, open Saturday and Sunday; `http://www.snydersrestaurant.com/`; American cuise with a Maryland flare.

Registration and Meeting Information

Registration and the AMS book exhibit will be located on the first floor of the Information Technology/Engineering (ITE) Building. Special Sessions will be held in Sherman Hall (Academic IV Building) and Sondheim Hall. All Invited Addresses will be held in the first floor lecture hall of the the Information Technology/Engineering (ITE) Building. Please refer to the general campus map at `http://about.umbc.edu/visitors-guide/campus-map/` for specific locations. The registration desk will be open on Saturday, March 29, 7:30 a.m.– 4:00 p.m. and Sunday, March 30, 8:00 a.m. – 12:00 noon. Fees are US$54 for AMS members, US$76 for nonmembers; and US$5 for students, unemployed mathematicians, and emeritus members. Fees are payable on site via cash, check, or credit card. Instructions on how to register in advance will be available starting on January 20, 2014 at `http://www.ams.org/meetings/sectional/2220_other.html`.

Other Activities

Book Sales: Stop by the on-site AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Local Information and Maps

This meeting will take place on the Main Campus of the University of Maryland, Baltimore County. A campus map can be viewed at `http://about.umbc.edu/visitors-guide/campus-map/`. Information about the University of Maryland, Baltimore County Department of Mathematics may be found at http://www.math.umbc.edu/. Please watch the website available at www.ams.org/meetings/sectional/sectional.html for additional information on this meeting. Please visit the University of Maryland, Baltimore County website at http://umbc.edu/for additional information on the campus.

Parking

Visitor parking is available at metered garages located at Administration Drive, The Commons Drive, and Walker Avenue. Meters are enforced Monday through Friday for hours posted, meters are not monitored on weekends. On weekends all street parking is available without restricitons. Street parking most convenient to the meeting can be located along Hilltop Circle (campus loop road), in front of the ITE building. Handicapped parking is available in Lot 9, the Administrative Drive Garage and the Commons Garage.

Additional information on visitor parking can be found at http://about.umbc.edu/files/2012/08/2012Map_Parking_Lettersize.pdf

Travel

The University of Maryland, Baltimore County is located in Baltimore just 15 minutes from Baltimore's Inner Harbor, 45 minutes from Washington, D.C., and four miles from BWI Airport.

By Air:

Baltimore-Washington International Airport is only a five-minute drive away from campus. Upon arrival at the airport there are many ground transportation options available.

The Light Rail offers service from the nearby BWI Rail Station L to downtown Baltimore, Timonium, and Hunt Valley from BWI Marshall Airport. The BWI Marshall Light Rail Station is located immediately outside the lower level of the terminal building, adjacent to Concourse E and fares are US$1.60 each way. For details on schedules please visit http://mta.maryland.gov/.

A taxi stand is located just outside of the baggage claim area of the Lower Level of the BWI Marshall terminal. Please note that this service is available from BWI Marshall only. For cab service to BWI Marshall, please consult your local cab company. BWI Marshall taxicabs are prohibited from charging flat rates. For more information, call 410-859-1100 or visit www.bwiairporttaxi.com.

Rental cars are available through a number of rental agencies at the airport and information phones are available at all baggage claim areas for each of the on-airport car rental agencies. BWI Marshall Airport has a rental car facility providing one-stop rental car shopping for our customers. The facility is located at Stoney Run Road and New Ridge Road. Free shuttle service carries customers to and from the airport approximately every 10 minutes. Passengers arriving on flights should take the free shuttle from the lower level terminal for a ten-minute ride to the new facility. When returning a vehicle, look for highway directional signs to the facility. The Car Rental Facility is located at: 7432 New Ridge Rd. Hanover, MD 21076.

Other options for ground transportation from the airport include hotel courtesy shuttles, commercial shuttle and sedan services. Many hotels offer courtesy shuttles, inquire when you book your room. Hotel and off-airport parking shuttles will pick-up passengers only at Zones 1 and 3. For information on commercial shuttles and sedan services please visit http://www.bwiairport.com/en/travel/ground-transportation.

By Train:

AMTRAK and MARC commuter trains serve the nearby BWI Rail Station. The Baltimore region is served by Amtrak and Acela, reservations can be made at www.amtrak.com. Amtrak trains provide service to the BWI Marshall Rail Station. MARC Commuter trains provide transportation between Baltimore and Washington D.C., as well as several Maryland Counties. MARC trains do not operate on weekends. For more information visit www.mtamaryland.com.

Local Transportation

Local Bus, Light Rail, and MARC train services: The Maryland Transit Authority (MTA) offers a variety of local transportation options to get around Baltimore. For information on utilizing public transit in Baltimore, please download the Visitors Ride Guide from the MTA website at http://mta.maryland.gov/content/visitors or use the trip planner feature to plan your travel http://mta.maryland.gov/.

The MARC train is a commuter rail system whose service areas include Harford County, Maryland; Baltimore City; Washington D.C.; Brunswick, Maryland; Frederick, Maryland; and Martinsburg, West Virginia. Please note that the MARC train only opperates Monday - Friday.

The Light Rail travels between Hunt Valley and BWI Airport Monday–Friday between 5:00 a.m.–midnight, Saturdays between 6:00 a.m.–midnight, and Sundays between 11:00 a.m.–7:00 p.m. Please visit the MTA website for details about stations and fares.

Single ride fares for local buses, metro subway, and light rail are US$1.60. An express bus costs US$2.00 and the MARC Train fare varies based upon destination. Pay one-way bus fare (exact change only) upon boarding the bus. For Light Rail and Metro Subway, purchase one-way or round trip tickets from ticket vending machines located in the rail stations, cash and credit cards are accepted.

Taxi Service: Licensed, metered taxis are available throughout the Baltimore metropolitan area. With a fleet of over 600 taxicabs in Baltimore city, Yellow Cab of Baltimore can be booked online at http://www.yellowcabofbaltimore.com/ or by phone at (410) 685-1212.

Weather

The average high temperature for March is approximately 58 degrees Fahrenheit and the average low is approximately 38 degrees Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `http://sites.nationalacademies.org/pga/biso/visas/` and `http://travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `mac@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Albuquerque, New Mexico

University of New Mexico

April 5–6, 2014

Saturday – Sunday

Meeting #1099

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: January 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 11, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Anton Gorodetski, University of California, Irvine, *Hyperbolic dynamics and spectral properties of one-dimensional quasicrystals*.

Fan Chung Graham, University of California, San Diego, *Some problems and results in spectral graph theory*.

Adrian Ioana, University of California, San Diego, *To be announced*.

Karen Smith, University of Michigan, Ann Arbor, *To be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Analysis and Topology in Special Geometries (Code: SS 14A), **Charles Boyer**, **Daniele Grandini**, and **Dimiter Vassilev**, University of New Mexico.

Arithmetic and Differential Algebraic Geometry (Code: SS 17A), **Alexandru Buium**, University of New Mexico, **Taylor Dupuy**, University of California, Los Angeles, and **Lance Edward Miller**, University of Arkansas.

Commutative Algebra (Code: SS 7A), **Daniel J. Hernandez**, University of Utah, **Karen E. Smith**, University of Michigan, and **Emily E. Witt**, University of Minnesota.

Descriptive Set Theory and its Applications (Code: SS 6A), **Alexander Kechris**, California Institute of Technology, and **Christian Rosendal**, University of Illinois, Chicago.

Flat Dynamics (Code: SS 8A), **Jayadev Athreya**, University of Illinois, Urbana-Champaign, **Robert Niemeyer**, University of New Mexico, Albuquerque, **Richard E. Schwartz**, Brown University, and **Sergei Tabachnikov**, The Pennsylvania State University.

Harmonic Analysis and Dispersive Equations (Code: SS 11A), **Matthew Blair**, University of New Mexico, and **Jason Metcalfe**, University of North Carolina.

Harmonic Analysis and Its Applications (Code: SS 19A), **Jens Gerlach Christensen**, Colgate University, and **Joseph Lakey** and **Nicholas Michalowski**, New Mexico State University.

Harmonic Analysis and Operator Theory (in memory of Cora Sadosky) (Code: SS 18A), **Laura De Carli**, Florida International University, **Alex Stokolos**, Georgia Southern University, and **Wilfredo Urbina**, Roosevelt University.

Hyperbolic Dynamics, Dynamically Defined Fractals, and Applications (Code: SS 22A), **Anton Gorodetski**, University of California Irvine.

Interactions in Commutative Algebra (Code: SS 4A), **Louiza Fouli** and **Bruce Olberding**, New Mexico State University, and **Janet Vassilev**, University of New Mexico.

Mathematical Finance (Code: SS 21A), **Indranil SenGupta**, North Dakota State University,.

Modeling Complex Social Processes Within and Across Levels of Analysis (Code: SS 15A), **Simon DeDeo**, Indiana University, and **Richard Niemeyer**, University of Colorado, Denver.

Partial Differential Equations in Materials Science (Code: SS 10A), **Lia Bronsard**, McMaster University, and **Tiziana Giorgi**, New Mexico State University.

Physical Knots, honoring the retirement of Jonathan K. Simon (Code: SS 13A), **Greg Buck**, St. Anselm College, and **Eric Rawdon**, University of St. Thomas.

Progress in Noncommutative Analysis (Code: SS 2A), **Anna Skripka**, University of New Mexico, and **Tao Mei**, Wayne State University.

Spectral Theory (Code: SS 12A), **Milivoje Lukic**, Rice University, and **Maxim Zinchenko**, University of New Mexico.

Stochastic Processes in Noncommutative Probability (Code: SS 20A), **Michael Anshelevich**, Texas A&M University, and **Todd Kemp**, University of California San Diego.

Stochastics and PDEs (Code: SS 5A), **Juraj Földes**, Institute for Mathematics and Its Applications, **Nathan Glatt-Holtz**, Institute for Mathematics and Its Applications and Virginia Tech, and **Geordie Richards**, Institute for Mathematics and Its Applications and University of Rochester.

The Common Core and University Mathematics Instruction (Code: SS 16A), **Justin Boyle**, **Michael Nakamaye**, and **Kristin Umland**, University of New Mexico.

The Inverse Problem and Other Mathematical Methods Applied in Physics and Related Sciences (Code: SS 1A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico and Enfitek, Inc.

Topics in Spectral Geometry and Global Analysis (Code: SS 3A), **Ivan Avramidi**, New Mexico Institute of Mining and Technology, and **Klaus Kirsten**, Baylor University.

Weighted Norm Inequalities and Related Topics (Code: SS 9A), **Oleksandra Beznosova**, Baylor University, **David Cruz-Uribe**, Trinity College, and **Cristina Pereyra**, University of New Mexico.

Session for Contributed Talks

There also will be a session for 10-minute contributed talks. Please see the abstracts submission form at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`. The deadline for all submissions is February 11, 2014.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include hotel tax. Participants must state that they are with the **American Mathematical Society (AMS) Meeting at University of New Mexico** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

Plaza Inn, 900 Medical Arts Avenue, N.E., Albuquerque, NM 87102, 800-237-1307; `http://www.plazainnabq.com`. Rates are US$56 per night for single/double occupancy. Please note that hotel tax is 13%. Amenities include complimentary hot breakfast, free wireless Internet in rooms, indoor heated swimming pool and hot tub, complimentary guest parking, fitness center, guest laundry, and shuttle service. The hotel is located 1.3 miles from campus. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations at this rate is April 1, 2014.**

Hilton Garden Inn Albuquerque Uptown, 6510 America's Parkway NE, Albuquerque, NM 87110, 505-944-0300; `http://hiltongardeninn3.hilton.com/en/hotels/new-mexico/hilton-garden-inn-albuquerque-uptown-ABQUPGI/index.html`. Rates are US$89 per night for singles/doubles. Please note applicable taxes are 7% state tax, 6% lodger's tax, and US$3.78 per room night hospitality surcharge. Amenities include complimentary wireless access, 24-hour business center, complimentary fitness center and a full-service restaurant on the property, and complimentary guest parking. This property is located within 6 miles of campus. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations at this rate is March 14, 2014.**

Crowne Plaza Albuquerque, 1901 University Blvd. NE, Albuquerque, NM 87102, 505-884-2500; `http://www.ihg.com/crowneplaza/hotels/us/en/albuquerque/abqcp/hoteldetail`. Rates are US$79 per night for single/double occupancy. Please note that hotel tax is 13%. Amenities include complimentary Internet, fitness facility, indoor and outdoor pool, and business center. This property is located within 2 miles of campus. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations at this rate is March 5, 2014.**

Hyatt Place Airport, 1400 Sunport Place SE, Albuquerque, New Mexico, 87106, 505-242-9300; `http://albuquerqueairport.place.hyatt.com/en/hotel/home.html`. Rates are US$95 per night for single and double occupancy. Please note that hotel tax is 13%. Amenities include complimentary hot breakfast, free wireless high-speed Internet access, and complimentary business center. This property is located approximately 3 miles from the campus. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations at this rate is March 21, 2014.**

Food Services

On Campus–Food can be purchased at the UNM Student Union:

Satellite Coffee–Offering freshly roasted coffee, lattes, or one of many teas, sweet treats or grilled sandwiches made to order. Open Saturday 10:30 a.m.–5:00 p.m.

Sonic–Sonic offers a variety of menu items including sandwiches, footlong quarter pound Coney, onion rings, and Tater Tots. Open Saturday 10:30 a.m.–5:00 p.m.

Chick-fil-A–Variety of chicken options including nugget entrees and combos, chicken strip meals, grilled chicken sandwiches and grilled chicken salad, as well as a variety of sides including waffle fries, a fruit cup, or brownies. Open Saturday 10:30 a.m.–5:00 p.m.

Outakes (On-Campus Convenience Store)–is a multi-faceted solution to your convenience needs, providing an upscale convenience store or a scaled down dining center.

Off Campus–There is a wide variety of restaurants and fast food within walking distance to the main campus.

Artichoke Cafe, 424 Central Ave SE. (1.1 miles). Fine dining, beer and wine.

66 Diner, 1405 Central Ave NE. (0.4 miles) offers nostalgia diner food.

Yasmine's Cafe, 1600 Central Ave SE. (0.3 miles). Middle Eastern. Closed Sunday.

Coaches Sports Grill, 2929 Monte Vista Blvd NE. (0.4 miles). Burgers, sandwiches, Buffalo wings, beer.

El Bandido Hideout, 2128 Central Ave SE. (0.4 miles). Mexican food, beer.

Grandma's Mexican Diner, 1606 Central Ave (0.3 miles). Mexican and American food.

Pita Pit, 2106 Central Ave SE. (0.4 miles). Sandwiches on flatbread. Lunch only.

Times Square Deli, 2132 Central Ave SE. (0.4 miles). New York style sandwiches, plus baklava. Closed Sunday.

Pericos de Albuquerque, 109 Yale Blvd SE. (0.4 miles). Family owned taco shop.

Olympia Cafe, 2210 Central Ave SE. (0.5 miles). Greek food. Closed Sunday.

Brickyard Pizza, 2216 Central Ave SE. (0.5 miles). Pizza and beer.

Winning Coffee, 111 Harvard Dr SE. (0.5 miles). Coffee, light breakfast, and lunch.

Rocka Taco, 115 Harvard SE. (0.5 miles). Seafood and other tacos, beer.

El Patio de Albuquerque, 142 Harvard Dr SE. (0.6 miles). New Mexican food, beer.

Frontier Restaurant, 2400 Central Ave SE. (0.6 miles). New Mexican food in a landmark student hangout. Open late.

Saggios Restaurant, 107 Cornell Dr SE. (0.6 miles). Pizza and beer. Closed Sunday.

Bailey's on the Beach, 2929 Monte Vista Blvd NE. (1.0 miles). Seafood, burgers, beer, wine. Breakfast too.

Registration and Meeting Information

Registration and the book exhibit will be located in the first floor lobby of the Science and Math Learning Center (SMLC) which is on the far west side of the main campus. Please see building 14 on the campus map. Special Sessions will be held in Mitchell Hall which is located in building 23 on the campus map. The Invited Adresses will be held in the Science and Math Learning Center (SMLC) Auditorium which is located in building 14 on the campus map. You can find the campus map here: `http://iss.unm.edu/PCD/SM/doc/VisitorMapCentral_Numeric.pdf`.

The registration desk will be open on Saturday, April 5, 7:30 a.m.-4:00 p.m. and Sunday, April 6, 8:00 a.m.-12:00 p.m. Fees are US$54 for AMS members, US$76 for non-members; and US$5 for students, unemployed mathematicians, and emeritus members. Fees are payable on-site via cash, check, or credit card; advance registration will be available on the AMS website.

Other Activities

Book Sales: Stop by the onsite AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Local Activities: The "13th New Mexico Analysis Seminar" will take place April 3-4, 2014 and "An Afternoon in Honor of Cora Sadosky" is scheduled for April 4, 2014. Further information for both events can be found here: `http://www.math.unm.edu/conferences/13thAnalysis/index.html`

Local Information and Maps

This meeting will take place on the Central Campus of the University of New Mexico. A campus map can be found at `http://iss.unm.edu/PCD/SM/doc/VisitorMapCentral_Numeric.pdf`. Information about the The Department of Mathematics and Statistics may be found at `http://www.math.unm.edu/`.Please visit the university's website at `http://www.unm.edu/` for additional information on the campus.

Parking

Questions about parking on the University of New Mexico Campus can be addressed to the UNM Parking and Transportation Office (505-277-1938).

Core Campus Parking: Parking restrictions are enforced Monday through Saturday between the hours of 7:00 a.m.–4:00 p.m. Anyone who parks in core campus parking without a legal pass is subject to receiving a ticket and/or being towed.

Hourly Parking: University of New Mexico operates a system of hourly paid parking across campus. The parking rate is currently US$1.75 per hour, for both surface and structure lots. There is no incremental rate for part of an hour, i.e., if a customer stays any part of an hour, the fee will be for the entire hour. Daily and weekly permits are available for purchase, see details here: `http://pats.unm.edu/permits/students/index.html`

Cornell Parking Structure: The Cornell parking structure is located on Redondo Rd. The Cornell parking structure is open during events. Pay stations in Cornell accept both credit cards and cash, otherwise there is a US$7.00 charge for parking in the structure during events. Handicap parking is available. Parking regulations are enforced Monday–Saturday.

Yale Parking Structure: The Yale parking structure is accessed from Lomas Blvd. heading east, or Yale Blvd. heading south from Lomas Blvd. The Yale parking structure is open 24 hours per day. Visitor parking is available on levels 1–3. Pay stations in Yale accept only credit cards—no cash. Handicap parking is available. Parking regulations are enforced Monday–Saturday.

Handicap Parking is available throughout the campus. Visitors with a vehicle displaying a handicap placard may park in designated handicap parking or any regular parking spot free of charge. Do not park in any spot with a reserved sign or spots designated for UNM departments or personnel. UNM does not require registration of your handicap placard, therefore a UNM permit is required.

Travel

The University of New Mexico is located in Albuquerque, NM. The Albuquerque International Sunport (code: ABQ) is located approximately 2.5 miles south of the university.

By Car: From the Albuquerque Airport (Sunport): Take Sunport Loop to Yale Blvd. Follow Yale Blvd to Central Ave. Cross over Central onto the UNM Campus. The main road that circles the campus is Redondo Drive. The Yale parking structure is to your left (see parking instructions.) The Cornell parking structure is to your right (see parking instructions.)

From Interstate 40 (I-40): Exit to Interstate 25 (I-25) traveling south toward Las Cruces. Exit at Dr. Martin Luther King Jr. Ave. and follow MLK Jr. Ave. to University Blvd. Cross over University onto the UNM Campus. The main road that circles the campus is Redondo Drive. The Yale parking structure is to your left (see parking instructions). The Cornell parking structure is to your right (see parking instructions).

From Interstate 25 (I-25): Exit at Dr. Martin Luther King Jr. Ave, and follow MLK Jr. Ave, to University Blvd. Cross over University onto the UNM Campus. The main road that circles the campus is Redondo Drive. The Yale parking structure is to your left (see parking instructions). The Cornell parking structure is to your right (see parking instructions).

By Train: The Amtrak station is located approximately 2 miles from the University of New Mexico Campus. Information for Amtrack can be found here: `http://www.amtrak.com/home`.

Car Rental: Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at `www.hertz.com`, click on the box “I have a discount”, and type in our convention number (CV): **04N30004**. You can also call Hertz directly at 800-654-2240 (U.S. and Canada) or 405-749-4434 (other countries).

Local Transportation

A wide variety of ground transportation serves Albuquerque International Sunport. Contact the companies listed below for more information. You can find approximate cost from airport to local hotels here: `http://sunport-shuttle.com/hotels.pdf`.

Local Shuttle Service: Sunport Shuttle, 505-883-4966 or `http://sunportshuttle.com`.

Local Bus Service: ABQ Ride, 505-243-7433 or `http://www.cabq.gov/transit`.

You can find information about Albuquerque’s public transit system here: `http://www.cabq.gov/transit/bus-routes-and-schedules`. Visit website for bus service times and routes.

Roadrunner Shuttle & Charter: Daily door-to-door shared ride shuttle and private ride charters from the Albuquerque International Sunport to Santa Fe, Los Alamos, Espanola, and surrounding areas. Information can be found here: `www.RideRoadRunneR.com` or 505-424-3367.

Taxi Service: Albuquerque Cab Company, 505-883-4888.

Weather

During the month of April the average high temperature is in the low 70s and the average low temperature is in the low 40s. Generally, the weather can be windy but rarely will you have more than 10% chance of rain. The average humidity during April is 33%. Springtime can bring out allergies for those who are susceptible. Pollen counts for Ash, Cottonwood, and Juniper can be very high during this time of the year.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `sites.nationalacademies.org/pga/biso/visas/` and `travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `mkl@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and

dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Lubbock, Texas

Texas Tech University

April 11–13, 2014

Friday – Sunday

Meeting #1100

Central Section
Associate secretary: Georgia M. Benkart
Announcement issue of *Notices*: January 2014
Program first available on AMS website: February 27, 2014
Program issue of electronic *Notices*: April 2014
Issue of *Abstracts*: Volume 35, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 10, 2014

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Nir Avni, Northwestern University, *To be announced.*

Alessio Figalli, University of Texas, *To be announced.*

Jean-Luc Thiffeault, University of Wisconsin-Madison, *To be announced.*

Rachel Ward, University of Texas at Austin, *To be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic Geometry (Code: SS 9A), **David Weinberg**, Texas Tech University.

Analysis and Applications of Dynamic Equations on Time Scales (Code: SS 2A), **Heidi Berger**, Simpson College, and **Raegan Higgins**, Texas Tech University.

Applications of Special Functions in Combinatorics and Analysis (Code: SS 12A), **Atul Dixit**, Tulane University, and **Timothy Huber**, University of Texas Pan American.

Approximation Theory in Signal Processing (Code: SS 17A), **Rachel Ward**, University of Texas at Austin, and **Rayan Saab**, University of California San Diego.

Complex Function Theory and Special Functions (Code: SS 7A), **Roger W. Barnard** and **Kent Pearce**, Texas Tech University, **Kendall Richards**, Southwestern University, and **Alex Solynin** and **Brock Williams**, Texas Tech University.

Developments from PASI 2012: Commutative Algebra and Interactions with Related Disciplines (Code: SS 26A), **Kenneth Chan**, University of Washington, and **Jack Jeffries**, University of Utah.

Differential Algebra and Galois Theory (Code: SS 23A), **Lourdes Juan** and **Arne Ledet**, Texas Tech University, **Andy R. Magid**, University of Oklahoma, and **Michael F. Singer**, North Carolina State University.

Fractal Geometry and Dynamical Systems (Code: SS 3A), **Mrinal Kanti Roychowdhury**, The University of Texas-Pan American.

Geometry and Geometric Analysis (Code: SS 25A), **Lance Drager** and **Jeffrey M. Lee**, Texas Tech University.

Homological Methods in Algebra (Code: SS 8A), **Lars W. Christensen**, Texas Tech University, **Hamid Rahmati**, Miami University, and **Janet Striuli**, Fairfield University.

Hysteresis and Multi-rate Processes (Code: SS 19A), **Ram Iyer**, Texas Tech University.

Interactions between Commutative Algebra and Algebraic Geometry (Code: SS 11A), **Brian Harbourne** and **Alexandra Seceleanu**, University of Nebraska-Lincoln.

Issues Regarding the Recruitment and Retention of Women and Minorities in Mathematics (Code: SS 5A), **James Valles Jr.**, Prairie View A&M University, and **Doug Scheib**, Saint Mary-of-the-Woods College.

Lie Groups (Code: SS 13A), **Benjamin Harris**, **Hongyu He**, and **Gestur Ólafsson**, Louisiana State University.

Linear Operators in Representation Theory and in Applications (Code: SS 20A), **Markus Schmidmeier**, Florida Atlantic University, and **Gordana Todorov**, Northeastern University.

Mathematical Models of Infectious Diseases in Plants, Animals and Humans (Code: SS 21A), **Linda Allen**, Texas Tech University, and **Vrushal Bokil**, Oregon State University.

Navier-Stokes Equations and Fluid Dynamics (Code: SS 14A), **Radu Dascaliuc**, Oregon State University, and **Luan Hoang**, Texas Tech University.

Noncommutative Algebra, Deformations, and Hochschild Cohomology (Code: SS 10A), **Anne Shepler**, University of North Texas, and **Sarah Witherspoon**, Texas A&M University.

Numerical Methods for Systems of Partial Differential Equations (Code: SS 27A), **JaEun Ku**, Oklahoma State Universtity, and **Young Ju Lee**, Texas State University.

Optimal Control Problems from Neuron Ensembles, Genomics and Mechanics (Code: SS 24A), **Bijoy K. Ghosh** and **Clyde F. Martin**, Texas Tech University.

Qualitative Theory for Non-linear Parabolic and Elliptic Equations (Code: SS 6A), **Akif Ibragimov**, Texas Tech University, and **Peter Polacik**, University of Minnesota.

Recent Advancements in Differential Geometry and Integrable PDEs, and Their Applications to Cell Biology and Mechanical Systems (Code: SS 4A), **Giorgio Bornia**, **Akif Ibragimov**, and **Magdalena Toda**, Texas Tech University.

Recent Advances in the Applications of Nonstandard Finite Difference Schemes (Code: SS 15A), **Ronald E. Mickens**, Clark Atlanta University, and **Lih-Ing W. Roeger**, Texas Tech University.

Recent Developments in Number Theory (Code: SS 18A), **Dermot McCarthy** and **Chris Monico**, Texas Tech University.

Statistics on Manifolds (Code: SS 22A), **Leif Ellingson**, Texas Tech University.

Topology and Physics (Code: SS 1A), **Razvan Gelca** and **Alastair Hamilton**, Texas Tech University.

Undergraduate Research (Code: SS 16A), **Jerry Dwyer**, **Levi Johnson**, **Jessica Spott**, and **Brock Williams**, Texas Tech University.

Session for Contributed Talks

There also will be a session for 10-minute contributed talks. Please see the abstracts submission form at `http://www.ams.org/cgi-bin/abstracts/abstract.pl`. The deadline for all submissions is February 10, 2014.

Accommodations

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Staybridge Suites, 2515 19th Street, Lubbock, TX 79410, 806-765-8900; `www.staybridge.com/lubbocktx`. Rates are US$109 for a queen suite and US$119 for a king suite. Please note that hotel tax is 13%. Ammenities include free high-speed Internet and complimentary breakfast. The hotel is located 1.5 miles from campus. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations is March 11, 2014.**

Hawthorn Suites By Wyndham, 4435 Marsha Sharp Freeway, Lubbock, TX 79407, 806-792-3600; `www.hawthornlubbock.com`. Rates are US$99 for a single, US$109 for a double, and US$149 for a suite. Please note that the hotel tax is 13%. Amenities include free airport shuttle, complimentary high-speed Internet, free breakfast, campus shuttle service, and a fully equipped kitchen or kitchenette with microwave, refrigerator, and stove. The hotel is located 3 miles from campus. Check in time is 3:00 p.m. and check out time is 11:00 a.m. **The deadline for reservations is March 19, 2014.**

La Quinta Inn and Suites Lubbock North, 5006 Auburn Street, Lubbock, TX 79416, 806-749-1600; `http://www.lq.com/lq/properties/propertyProfile.do?ident=LQ6018&propId=6018`. Rates are US$94 for a single and US$114 for a suite. Please note the hotel tax is 13%. Amenities include free airport shuttle, complimentary high-speed Internet, free breakfast, campus shuttle service as well as a heated indoor pool, hot tub and onsite fitness center. The hotel is located 4.3 miles from campus. Check in time is 3:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations is March 6, 2014**

MCM Elegante Hotel and Suites (Formerly Holiday Inn Lubbock-Towers), 801 Avenue Q, Lubbock, TX 79401, 806-763-1200; `http://mcmelegantelubbock.com`. Rates are US$89 single/double. Please note the hotel tax is 13%. Amenities include complimentary shuttle service to and from the airport, shuttle service to campus, free high-speed Internet, fitness center and indoor pool. This hotel is located 2.3 miles from campus. Check in time is 4:00 p.m. and check out time is 12:00 p.m. **The deadline for reservations is March 13, 2014.**

Overton Hotel and Conference Center, 2322 Mac Davis Lane, Lubbock, TX 79401, 888-776-7001; `http://www.overtonhotel.com`. Rates are US$100 for a double room. Please note the hotel tax is 13%. Amenities include shuttle service to and from the airport, free high-speed Internet, full-service Business Center, heated outdoor pool and jacuzzi, and Fitness Center. **The deadline for reservations is March 21, 2014.**

Food Services

On Campus–the Student Union Building is located at 15th Street & Akron Avenue. The Market at Stangel/Murdough is located at 3009 Main Street.

Sam's Place @ the Student Union is the place to get your favorite south-of-the-border food items, including nachos, burritos, burrito bowls, and quesadillas, as well as many Grab-n-Go options. Closed Saturday, open 12:00 p.m.–6:00 p.m. Sunday.

The Market @ Stangel/Murdough hosts numerous options to satisfy the appetites of all, including sandwiches on fresh baked breads, authentic Mexican foods, grilled items, Asian/Wok foods, pizza and Italian foods, made-to-order pasta entrées, and a café that serves specialty coffees. All of these options are open for dine in and carry out. Open Saturday 9:00 a.m.–7:00 p.m. and Sunday 9:00 a.m.–2:30 p.m.

Off Campus–There is a wide variety of restaurants and fast food. For further options: `https://maps.google.com/maps/ms?msid=206318757734525836513.0004dc4f232820d655b93&msa=0&source=gplus-ogsb`.

One Guy from Italy, 1101 University Avenue & 4320 50 Street, 806-747-1226. Featuring pizza & calzones and fine Italian cuisine. Open Saturday 11:00 a.m.–12:00 a.m. and Sunday 11:00 a.m.–11:00 p.m.

Texas Roadhouse, 4801 S Loop 289, 808-799-9900. Hand cut steaks, ribs, and margaritas, a classic American steak house. Open Saturday 11:00 a.m.–11:00 p.m. and Sunday 11:00 a.m.–10:00 p.m.

Abuelo's Mexican, 4401 82nd Steet, 806-794-1762. Open Saturday 11:00 a.m.–11:00 p.m. and Sunday 11:00 a.m.–10:00 p.m.

Triple J Chophouse & Brew Co, 1807 Buddy Holly Avenue, 806-771-6555. Brew pub and steak house with comfortable atmosphere. Open Saturday 11:00 a.m.–12:00 a.m. and closed Sunday.

Manna Bread & Wine, 2610 Salem Avenue, Ste 19, 806-791-5600. Modern American wine bar with Italian influenced cooking. Open Saturday 11:00 a.m.–10:00 p.m. and closed Sunday.

Jazz, A Louisiana Kitchen, 3703 19th St Steet C, 806-799-2124. Authentic French Quarter Cafe, featuring fine Cajun/Creole cuisine. Open Saturday 11:00 a.m.–2:00 a.m. and Sunday 11:00 a.m.–10:00 p.m.

Registration and Meeting Information

Registration and the book exhibit will be located in the first floor lobby and adjacent to Room 106 of the Department of Mathematics and Statistics Building. The location of the Special Sessions and Invited Addresses is to be determined, please check the website for the most up to date information. You can find the campus map here: `http://www.ttu.edu/map/`.

The registration desk will be open on Friday, April 11, 1:00 p.m.–5:00 p.m. and Saturday, April 12, 7:30 a.m.–4:00 p.m. Fees are US$54 for AMS members, US$76 for nonmembers; and US$5 for students, unemployed mathematicians, and emeritus members. Fees are payable on-site via cash, check, or credit card; advance registration will be available on the AMS website.

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AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Local Activities: The Texas Tech University Department of Mathematics & Statistics will host a reception for conference attendees featuring heavy hors d'oeuvres, cash bar, and a live performance by local recording artist Andy Wilkinson. Friday, April 11, 6:30 p.m. at the Jones AT&T Stadium, West Side Stadium Club.

Local Information and Maps

This meeting will take place on the campus of the Texas Tech University. A campus map can be found at `http://www.ttu.edu/map/`. Information about the the Department of Mathematics and Statistics may be found at `http://www.math.ttu.edu/`. Please visit the university's website at `http://www.ttu.edu/` for additional information on the campus.

Parking

Visitor Permits–Any vehicle parked on the Tech campus must have a permit. Visitor permits can be obtained at any of the six entry stations to campus. See our visitor parking map for entry station locations: `http://www.parking.ttu.edu/docs/mapdocuments/visitor-parking-map.png?sfvrsn=0`. Visitor permits are only good for one day.

Park-and-Pay–Some visitor parking on campus is controlled by park-and-pay machines. The charge for these spaces is US$1.50/hour or US$9/day. The machines accept bills, coins and credit cards (MasterCard, Visa, Discover, and American Express).

Please pay careful attention to signage. During the weekend, it is free to park in any spot not specifically marked as being 24-hour reserved.

Travel

The Texas Tech University is located in Lubbock, TX. The **Lubbock Preston Smith International Airport (LBB)** is located approximately 8 miles south of the University.

Driving to Lubbock

Centrally located between Dallas, Texas, and Albuquerque, New Mexico, Interstate 27 connects the city of Lubbock with two east-west interstate systems, I-20 and I-40. State highways 82, 84, 87, and 114 also provide easy access to cities throughout the region. The following numbers may also provide more information: Emergency Road Conditions - (806) 745-4411, Texas Department of Transportation - (806) 745-4411, Tourist Information - (800) 452-9292.

From the Airport–Head southwest on North Martin L King Blvd/Martin Luther King Jr Blvd toward East Bluefield Steet. Turn right onto US-82 W/Parkway Drive. Continue to follow US-82 W. Take the exit toward University Avenue and merge onto 4th Street. Turn left onto University Avenue followed by a right onto Broadway Street. The university will be on the left

Car Rental–Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at `www.hertz.com`, click on the box "I have a discount", and type in our convention number (CV): **04N30004**. You can also call Hertz directly at 800-654-2240 (U.S. and Canada) or 405-749-4434 (other countries).

Local Transportation

Transportation service from the airport, located approximately 15 minutes from the conference hotels, is available. Traditional cab fare to/from the airport is approximately US$25.

City Cab- 806-765-7474
Yellow Cab- 806-765-7777
Royal Cab- 806-749-5333
VIP Royal Coach Private Service- 806-795-3888
Limousines of Lubbock- 806-743-5466
Lone Star Limousine- 888-286-5466
White Knights Limousines- 806-799-3366

Weather

The month of April is characterized by rising daily high temperatures, with daily highs increasing from 71°F to 79°F over the course of the month, exceeding 89°F or dropping below 55°F only one day in ten. Daily low temperatures range from 42°F to 52°F, falling below 32°F or exceeding 61°F only one day in ten. Throughout April, the most common forms of precipitation are thunderstorms, light rain, and moderate rain.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `sites.nationalacademies.org/pga/biso/visas/` and `travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `mkl@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Tel Aviv, Israel

Bar-Ilan University, Ramat-Gan and Tel-Aviv University, Ramat-Aviv

June 16–19, 2014

Monday – Thursday

Meeting #1101

The Second Joint International Meeting between the AMS and the Israel Mathematical Union.

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: January 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/internmtgs.html`.

Special Sessions

Additive Number Theory, **Melvyn B. Nathanson**, City University of New York, and **Yonutz V. Stanchescu**, Afeka Tel Aviv Academic College of Engineering.

Algebraic Groups, Division Algebras and Galois Cohomology, **Andrei Rapinchuk**, University of Virginia, and **Louis H. Rowen** and **Uzi Vishne**, Bar Ilan University.

Applications of Algebra to Cryptography, **David Garber**, Holon Institute of Technology, and **Delaram Kahrobaei**, City University of New York Graduate Center.

Asymptotic Geometric Analysis, **Shiri Artstein** and **Boaz Klar'tag**, Tel Aviv University, and **Sasha Sodin**, Princeton University.

Combinatorial Games, **A. Fraenkel**, Weizmann University, **Richard Nowakowski**, Dalhousie University, Canada, **Thane Plambeck**, Counterwave Inc., and **Aaron Siegel**, Twitter.

Field Arithmetic, **David Harbater**, University of Pennsylvania, and **Moshe Jarden**, Tel Aviv University.

Financial Mathematics, **Jean-Pierre Fouque**, University of California, and **Eli Merzbach** and **Malka Schaps**, Bar Ilan University.

Geometric Group Theory and Low-Dimensional Topology, **Ian Agol**, University of California, Berkeley, and **Zlil Sela**, Hebrew University.

History of Mathematics, **Leo Corry**, Tel Aviv University, **Michael N. Fried**, Ben Gurion University, and **Victor Katz**, University of District of Columbia.

Mirror Symmetry and Representation Theory, **David Kazhdan**, Hebrew University, and **Roman Bezrukavnikov**, Massachusetts Institute of Technology.

Mirror Symmetry and Representation Theory, **Roman Bezrukavnikov**, Massachusetts Institute of Technology, and **David Kazhdan**, Hebrew University.

Nonlinear Analysis and Optimization, **Boris Mordukhovich**, Wayne State University, and **Simeon Reich** and **Alexander Zaslavski**, The Technion-Israel Institute of Technology.

Nonlinear Analysis and Optimization, **Boris Mordukhovich**, Wayne State University, and **Simeon Reich** and **Alexander Zaslavski**, Technion Israel Institute of Technology.

Qualitative and Analytic Theory of ODE's, **Andrei Gabrielov**, Purdue University, and **Yossef Yomdin**, Weizmann Institute of Science.

Qualitative and Analytic Theory of ODE's, **Yosef Yomdin**, Weizmann Institute.

Quasigroups, Loops and Applications, **Tuval Foguel**, Western Carolina University.

Random Matrix Theory, **Brendan Farrell**, California Institute of Technology, **Mark Rudelson**, University of Michigan, and **Ofer Zeitouni**, Weizmann Institute of Science.

Recent trends in History and Philosophy of Mathematics, **Misha Katz**, Bar Ilan University, and **David Sherry**, Northern Arizona University.

Teaching with Mathematical Habits in Mind, **Theodore Eisenberg**, Ben Gurion University, **Davida Fishman**, California State University, San Bernardino, and **Jennifer Lewis**, Wayne State University.

The Mathematics of Menahem M. Schiffer, **Peter L. Duren**, University of Michigan, and **Lawrence Zalcman**, Bar Ilan University.

Topological Graph Theory and Map Symmetries, **Jonathan Gross**, Columbia University, and **Toufik Mansour**, University of Haifa.

Abstract Submissions

Talks in Special Sessions are by invitation of the organizers. It is recommended that you contact an organizer before submitting an abstract for a Special Session if you have not been specifically invited to speak in the session. Please watch the website cited above for more information on abstract submission. The deadline for submissions for invited speakers is April 13, 2014.

Accommodations

Information on conference rates and other accommodation options, including those in Tel Aviv, will be provided shortly at the conference website. The organizers plan to arrange transportation between the hotel area in Tel Aviv and the conference venues. It is advisable check the websites of Internet booking services, since they might offer lower rates.

Local Information/ Tourism

Many signs are in English and Hebrew (and perhaps also Arabic). Most people on the street speak English at some level, so it is not difficult to get around. A map of Tel-Aviv-Yafo can be found here: `https://maps.google.com/maps/ms?msa=0&msid=215527707446717608936.0004b0635192768881d8f` and a map of Ramat-Gan can be found here: `https://maps.google.com/maps/ms?msa=0&msid=215527707446717608936.0004b0635192768881d8f`. A map of Tel Aviv University is found here: `https://maps.google.com/maps/ms?msa=0&msid=215527707446717608936.0004b0635192768881d8f` or Bar Ilan University here: `https://maps.google.com/maps/ms?msa=0&msid=215527707446717608936.0004b0635192768881d8f`.

For more information about visiting the Tel-Aviv area, please visit the Tel-Aviv-Yafo Tourism site `http://www.visit-tlv.com/`, or Wikitravel article on Tel-Aviv: `http://wikitravel.org/en/Tel Aviv`. A useful website for planning your time in Israel can be found at `http://www.goisrael.com/`.

The currency in Israel is the (new) Israel Shekel, abbreviated IS. At the time of publication of this announcement the exchange rate was US$1 is equal to 3.52 IS. ATM machines are available at main banks and at airports and shopping centers, but there also are licensed exchange offices which do not charge commissions. The exchange rates offered by the banks at the airport are reasonable, but not the best available. Major credit cards are accepted in large hotels, car rental companies, and almost all stores.

Israel's electrical current is 220 V; 50 cycles. Sockets have two round prongs at the top and a vertical prong for ground, see `electricaloutlet.org/type-c`.

Registration and Meeting Information

Online registration will be active soon at the meeting's website: `imu.org.il/Meetings/IMUAMS2014l`. The participation fee is US$100 and should be paid at the registration desk on-site at the meeting. It covers the conference materials and refreshments. Participants will be asked during registration about transfer. All locations for conference events are located in the hosting universities (Monday and Tuesday at Bar-Ilan University, Wednesday and Thursday Tel-Aviv University). Special Sessions at Bar-Ilan will be held at building 507, and the Invited Addresses will take place in building 901, where Registration will be located as well. It will be possible to register in Tel Aviv University as well, further information will be provided at a later date, see the website.

Opening Reception and Conference Dinner

An opening reception will take place on Sunday, June 15, in the Kfar Maccabia Hotel `www.kmc-hotel.co.il/`, in Ramat Gan.

A Conference Dinner will take place on Wednesday, June 18. Details of this event will be announced.

Travel

Tel-Aviv is located in central Israel, at the coast. No immunizations or unusual health precautions are necessary or required. The customary way to arrive in Israel is by air.

By Air: The main airport in Israel is Ben-Gurion Airport. From there, one can either take a taxi (about 120 IS) or the train or rental car.

By Taxi: If you take a taxi, be sure to take the taxi at the local taxi stop and ask for a fare according to the meter. By law, the taxi driver must oblige. When a taxi driver offers a flat rate, it may turn out to be more than the meter fare. Taxis within the cities are usually ordered by phone, and for intercity rides one usually does get a flat rate fixed in advance.

By Train: The train is an excellent way to travel between Tel Aviv and cities and towns to the north, such as Netanya, Hadera, Haifa, and Nahariya. The northbound train leaves from the Central Railway Station and the Azrieli station. The information office is open Sunday to Thursday 6:00 a.m.-11:00 p.m. and Friday 6:00 a.m.-3:00 p.m. Trains run roughly every hour on weekdays from between 5:00 a.m and 6:00 a.m. to between 10:00 p.m. and 11:00 p.m. depending on the destination; there are fewer trains on Friday and Jewish holiday eves and no service on Saturday or on holidays. There is also a line to Beersheva.

By Car: If a visitor's foreign car registration is valid and the driver is in possession of a valid driver's license, a car may be brought into Israel for a period of up to 1 year. No customs document or customs duty deposit is required. Due to the many hostile countries that surround Israel, and for security reasons, very few foreign vehicles are actually brought into Israel. Almost all tourists find it easier to rent a car in Israel.

Local Transportation

By Train: The rail service is best between Tel-Aviv and Haifa (and towns further north), with 3 stations in Tel-Aviv proper and one in Ramat-Aviv near Tel-Aviv University (called "University"). There also is a direct rail line to the airport, as well as Beer Sheva and other cities. The train line to Jerusalem is scenic but takes a long time, so the bus might be preferable. (Bus number 480 has a stop just outside Tel-Aviv Mercaz Station). There is no train station near Bar-Ilan University. The train station most convenient to Bar-Ilan is called Tel Aviv-Hashalom, near the Azrieli Center, but one then must take the bus number 56. Alternatively, one can get from Bar-Ilan to the Tel Aviv Mercaz Station (also known as the Arlozoro stateion) by the 68 or 70 bus lines. Schedules can be found at the site `www.rail.co.il/EN/`.

By Bus: Tel-Aviv has an extensive network of bus routes. The fare is IS 6.60. Information about buses can be found at `bus.co.il`. Bus lines servicing Bar-Ilan include 43, 45, 56, 68, 70 (for environs of TA), 164 to Rehovot and Petah Tiqva, and 506, 564, 620 to locations further north. Also the 30 line connects the Kfar Maccabea hotel to a 5-minute walk from Bar-Ilan, and other nearby lines are the 54, 60, and 61. Bus lines servicing Tel-Aviv University include 7, 24, 25, 45, 49, 74, 274, and 506, 564, 620 to locations further north. The 45 line connects Bar-Ilan and Tel-Aviv Universities. IMPORTANT NOTE: Most public transportation terminates Friday about an hour before sundown, until Saturday, about two hours after sundown. Likewise, many shops are closed Friday afternoon and all day Saturday.

By Rental Car: When renting a car, be sure to rent in advance and have a written contract from the car company, especially if it is one of the smaller companies. Cars are useful for visiting out-of-the way tourist sites such as national parks, but public transportation is fine for getting to other cities. You cannot take a rental car into or out of Israel. Cars rented in Israel are not generally insured for damages or liability if taken into the West Bank. The local automobile club, MEMSI (tel. 03/564-1122), has its main office in Tel Aviv at 20 Rehov Ha-Rakevet. MEMSI offers very detailed road maps, some of which are in English.

Weather:

Tel-Aviv has a warm Mediterranean climate, similar to southern California. Summer is quite warm, with extended sunny days, and you should not expect rain. Average temperatures in June are approximately 72 degrees and 82 degrees Fahrenheit.

Eau Claire, Wisconsin

University of Wisconsin-Eau Claire

September 20–21, 2014

Saturday – Sunday

Meeting #1102

Central Section
Associate secretary: Georgia M. Benkart
Announcement issue of *Notices*: June 2014
Program first available on AMS website: August 7, 2014
Program issue of electronic *Notices*: September 2014
Issue of *Abstracts*: Volume 35, Issue 3

Deadlines

For organizers: March 20, 2014
For abstracts: July 29, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Matthew Kahle, Ohio State University, *To be announced*.

Markus Keel, University of Minnesota, *To be announced*.

Svitlana Mayboroda, University of Minnesota, *To be announced*.

Dylan Thurston, Indiana University, *To be announced*.

Special Sessions

Commutative Ring Theory (Code: SS 3A), **Michael Axtell**, University of St. Thomas, and **Joe Stickles**, Millikin University.

Directions in Commutative Algebra: Past, Present and Future (Code: SS 1A), **Joseph P. Brennan**, University of Central Florida, and **Robert M. Fossum**, University of Illinois at Urbana-Champaign.

Von Neumann Algebras and Related Fields (Code: SS 2A), **Stephen Avsec** and **Ken Dykema**, Texas A&M University.
This announcement was composed with information taken from the website maintained by the local organizers at `imu.org.il/Meetings/IMUAMS2014/index.html`. Please check the website for the most up-to-date information.

Halifax, Canada

Dalhousie University

October 18–19, 2014

Saturday – Sunday

Meeting #1103

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: August 2014
Program first available on AMS website: September 5, 2014

Program issue of electronic *Notices*: October 2014
Issue of *Abstracts*: Volume 35, Issue 3

Deadlines

For organizers: March 18, 2014
For abstracts: August 19, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

François Bergeron, Université du Québec à Montréal, *Title to be announced*.

Sourav Chatterjee, New York University, *Title to be announced*.

William M. Goldman, University of Maryland, *Title to be announced*.

Sujatha Ramdorai, University of British Columbia, *Title to be announced*.

Special Sessions

p-adic Methods in Arithmetic. (Code: SS 1A), **Henri Darmon**, McGill University, **Adrian Iovita**, Concordia University, and **Sujatha Ramdorai**, University of British Columbia.

San Francisco, California

San Francisco State University

October 25–26, 2014

Saturday – Sunday

Meeting #1104

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2014
Program first available on AMS website: September 11, 2014
Program issue of electronic *Notices*: October 2014
Issue of *Abstracts*: Volume 35, Issue 4

Deadlines

For organizers: March 25, 2014
For abstracts: September 3, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Kai Behrend, University of British Columbia, Vancouver, Canada, *Title to be announced*.

Kiran S. Kedlaya, University of California, San Diego, *Title to be announced*.

Julia Pevtsova, University of Washington, Seattle, *Title to be announced*.

Burt Totaro, University of California, Los Angeles, *Title to be announced*.

Special Sessions

Algebraic Geometry (Code: SS 1A), **Renzo Cavalieri**, Colorado State University, **Noah Giansiracusa**, University of California, Berkeley, and **Burt Totaro**, University of California, Los Angeles.

Geometry of Submanifolds (Code: SS 3A), **Yun Myung Oh**, Andrews University, **Bogdan D. Suceava**, California State University, Fullerton, and **Mihaela B. Vajiac**, Chapman University.

Polyhedral Number Theory (Code: SS 2A), **Matthias Beck**, San Francisco State University, **Martin Henk**, Universität Magdeburg, and **Joseph Gubeladze**, San Francisco State University.

Greensboro, North Carolina

University of North Carolina, Greensboro

November 8–9, 2014

Saturday – Sunday

Meeting #1105

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: August 2014
Program first available on AMS website: September 25, 2014
Program issue of electronic *Notices*: November 2014
Issue of *Abstracts*: Volume 35, Issue 4

Deadlines

For organizers: April 8, 2014
For abstracts: September 16, 2014

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Susanne Brenner, Louisiana State Unviersity, *Title to be announced*.

Skip Garibaldi, Emory University, *Title to be announced*.

Stavros Garoufaldis, Georgia Institute of Technology, *Title to be announced*.

James Sneyd, University of Auckland, *Title to be announced* (AMS-NZMS Maclaurin Lecture).

San Antonio, Texas

Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio

January 10–13, 2015
Saturday – Tuesday

Meeting #1106
Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2014
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2015
Issue of *Abstracts*: Volume 36, Issue 1

Deadlines
For organizers: April 1, 2014
For abstracts: To be announced

Washington, District of Columbia

Georgetown University

March 7–8, 2015
Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: August 7, 2014
For abstracts: To be announced

Huntsville, Alabama

University of Alabama in Huntsville

March 27–29, 2015
Friday – Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: August 20, 2014
For abstracts: To be announced

Las Vegas, Nevada

University of Nevada, Las Vegas

April 18–19, 2015
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines
For organizers: September 18, 2014
For abstracts: To be announced

Porto, Portugal

University of Porto

June 11–14, 2015
Thursday – Sunday
First Joint International Meeting involving the American Mathematical Society (AMS), the European Mathematical Society (EMS), and the Sociedade de Portuguesa Matematica (SPM).
Associate secretary: Georgia M. Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Not applicable

Deadlines
For organizers: To be announced
For abstracts: To be announced

Chicago, Illinois

Loyola University Chicago

October 3–4, 2015
Saturday – Sunday
Central Section
Associate secretary: Georgia M. Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2015
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 10, 2015
For abstracts: To be announced

Fullerton, California

California State University, Fullerton

October 24–25, 2015

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2015
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 27, 2015
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 6–9, 2016

Wednesday – Saturday
Joint Mathematics Meetings, including the 122nd Annual Meeting of the AMS, 99th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2015
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2016
Issue of *Abstracts*: Volume 37, Issue 1

Deadlines

For organizers: April 1, 2015
For abstracts: To be announced

Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017

Wednesday – Saturday
Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2016
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: January 2017
Issue of *Abstracts*: Volume 38, Issue 1

Deadlines

For organizers: April 1, 2016
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 10–13, 2018

Wednesday – Saturday
Joint Mathematics Meetings, including the 124th Annual Meeting of the AMS, 101st Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia M. Benkart
Announcement issue of *Notices*: October 2017
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2017
For abstracts: To be announced

Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Central Section: **Georgia Benkart**, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: **Steven H. Weintraub**, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: **Brian D. Boe,** Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, e-mail: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: lapidus@math.ucr.edu; telephone: 951-827-5910.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at** www.ams.org/meetings/**.**

Meetings:

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 99 in this issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit http://www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences in Cooperation with the AMS: (see http://www.ams.org/meetings/ for the most up-to-date information on these conferences.)

February 13–17, 2014: 2014 Annual Meeting of AAAS, Chicago, Illinois.

American Mathematical Society Distribution Center
35 Monticello Place, Pawtucket, RI 02861 USA

0002-9920(201401)61:1;1-1