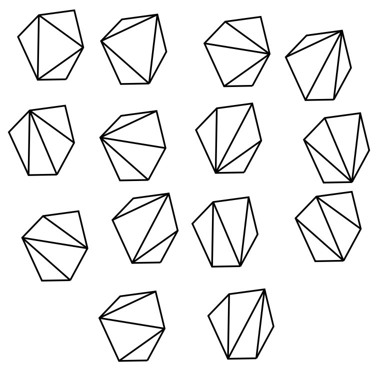
About the Cover

Remarkable combinatorics in the presence of cyclic symmetry

This month's cover illustrates in some detail one of the examples in this issue's article *What is cyclic sieving?* by Victor Reiner, Dennis Stanton, and Dennis White. If X_n is the set of triangulations of a given polygon of $n+2$ sides, then

$$|X_n| = \frac{1}{n+1} \cdot \frac{2n.2n-1.\cdots.n+1}{n.n-1.\cdots.1}.$$

On the cover, $n = 4$ and hence $|X_n| = 14$. Here is the collection of all triangulations in this case:



Inspection shows that these triangulations are not essentially distinct—some of them are combinatorially identical. There is in effect an action of the cyclic group $\mathbb{Z}/(n+2)$ on X_n, which rotates the connecting edges among the vertices. It is not a geometric symmetry since the polygon might not be regular. The cover illustrates this action.

Let R be one step of this rotation. From the formula for X_n is constructed a polynomial $X(q)$—each integer m in the formula is replaced by its q-form

$$[m]_q = \frac{q^m - 1}{q - 1},$$

a technique perhaps originating in Gauss' work on quadratic reciprocity. There is no reason to think it always leads to something significant. But here we have the remarkable fact that if ζ is a primitive n-th root of unity then

$$X(\zeta^k) = |\{x \in X_n \mid R^k(x) = x\}|.$$

For example, $X(1) = |X_n|$. You can verify this equation easily in the example illustrated on the cover.

This is the phenomenon called *cyclic sieving*. As Reiner et al. explain, it occurs in a myriad of contexts. Not all of them, by any means, are as simple even to formulate as this example, much less prove. And in fact there seems to be no uniform approach to the construction or verification of such occurrences, nor even a general criterion in which one would expect the phenomenon to arise. The short article in this issue gives a glimpse into the complexity of the subject.

The cover demonstrates that even some simple examples are associated with diagrams. There are in fact some extremely complicated and beautiful graphical associations in the subject. We looked through the literature to see what had been done, and the most interesting diagrams we came across were in the `arxiv` preprint *Invariant tensors and the cyclic sieving phenomenon* by Bruce Westbury. This in turn refers back to an earlier paper with exceptional graphics titled *Promotion and cyclic sieving via webs* by Kyle Petersen, Pavlo Pylyavskyy and Brendon Rhoades, which in turn is partly derived from Greg Kuperberg's *Spiders for rank two Lie algebras*.

—Bill Casselman
Graphics Editor
(notices-covers@ams.org)