Two-Person Fair Division of Indivisible Items: An Efficient, Envy-Free Algorithm

Steven J. Brams, D. Marc Kilgour, and Christian Klamler

The problem of fairly dividing a divisible good, such as cake or land, between two people probably goes back to the dawn of civilization. The first mention we know of in Western literature of the well-known procedure, "I cut, you choose," occurs in the Hebrew Bible, wherein Abraham and Lot divide the land that lies before them, with Abraham obtaining Canaan and Lot obtaining Jordan (Genesis 13: 5-13).

Since then, a plethora of procedures have been suggested for dividing a cake among two or more players [8], [14]. Although not all the desirable properties one might hope for can be achieved with a finite number of cuts [3], this problem pales in comparison to that of fairly allocating indivisible items.

Steven J. Brams is professor of politics at New York University. His email address is `steven.brams@nyu.edu`.

D. Marc Kilgour is professor of mathematics at Wilfrid Laurier University, Canada. His email address is `mkilgour@wlu.ca`.

Christian Klamler is associate professor at the Institute of Public Economics, University of Graz, Austria . His email address is `christian.klamler@uni-graz.at`.

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In this paper we present two algorithms for the fair division of indivisible items between two players. Both assume that the players can *strictly* rank the items from best to worst, and both use *only* these rankings to make allocations. Unlike more demanding fair-division algorithms, which ask players to give more detailed information (e.g., specify their cardinal utilities for each item) or make more difficult comparisons (evaluate different bundles of items), our algorithms are easy to apply and, therefore, eminently practicable.

The first algorithm asks the two players to make simultaneous or, equivalently, independent choices in sequence, starting with their most preferred item and progressively descending to less preferred items that have not already been allocated. The second algorithm requires that the players submit their complete preference rankings in advance to a referee (or computer).

The first algorithm was proposed by Brams and Taylor [8] as a "query step" for allocating indivisible items fairly between two players, *A* and *B*. We call it BT, and it works as follows: At any point in the allocation process, if *A* and *B* name different items, BT allocates them immediately; if *A* and *B* name the same item, it goes into a "contested pile", whose items are not allocated.

The second algorithm, which we describe in the section "The BT and AL Algorithms" and call

AL, also allocates items sequentially, one to each player, based on the players' rankings. Like BT, it does not necessarily allocate all the items—some may go into a contested pile.

However, under AL the contested pile is never larger, and may be smaller, than under BT. Furthermore, if the contested piles under AL and BT contain the same number of items, each player will never strictly prefer the items it receives under BT (we henceforth use the gender-neutral "it" rather than "he" or "she" for a player).

BT and AL share the property that, when they assign an item to one player, they simultaneously assign another item to the other player. Thus, A and B are allocated equal numbers of items.

The allocations given by both BT and AL are *envy-free* (EF): A's items can be matched pairwise to B's items such that A prefers each of its items to the corresponding item of B; there is a similar pairwise matching of B's items to A's. But only AL gives EF allocations that are efficient or *Pareto-optimal* (PO): There is no other EF allocation that is at least as good for A and B and better for one or both players, based on their rankings. (If there were such an allocation, the AL allocation would be *Pareto-dominated*.)[1] Also, AL allocations are *maximal*: There is no EF allocation that allocates more items to the players.

Both BT and AL are *manipulable*: It is possible for a player to improve its allocation by ranking items insincerely (i.e., not according to its preferences). Practically speaking, however, successful manipulation of either algorithm would require that a player have essentially complete information about the preference ranking of its opponent, which is highly unlikely in most real-life situations.

In many disputes, including divorce and estate division, only an allocation in which the disputants receive about the same number of items will be perceived as fair. BT and AL work well for that purpose, especially when they allocate most, if not all, the items. While BT is a paragon of simplicity, AL is not much harder to apply, as we show in "The BT and AL Algorithms", which should facilitate its acceptance as a practicable procedure.

The paper proceeds as follows. In the next section we define envy-freeness formally, illustrate it with examples, provide a necessary and sufficient condition for an allocation to be EF, and give a condition on the players' preferences that is necessary for the existence of an EF allocation.

[1] *A BT allocation, like an AL allocation, is what we later call locally Pareto-optimal: There is no other allocation of the items that each algorithm allocates that is at least as good for A and B and better for one or both players. Because an AL allocation can allocate more or better items to one or both players, however, it may globally Pareto-dominate a BT allocation.*

In "The BT and AL Algorithms" we define and illustrate BT and AL, showing that AL generally allocates more or better items to the players than BT. Then we use AL to prove that the necessary condition of "Envy-Free Allocations" is also sufficient for the existence of an EF allocation.

In "Other Properties of EF Allocations" we prove that an AL allocation is PO and maximal, but it, like a BT allocation, may be manipulable, as we illustrate with an example.

In "The Probability of Envy-Free Allocations", we calculate the probability that an EF allocation of all the items exists when all possible rankings are equiprobable. As the number of items approaches infinity, this probability approaches 1. In the last section, "Summary and Conclusions", we summarize our results, comparing AL and BT to other fair-division algorithms, and draw several conclusions.

Envy-Free Allocations

Consider the task of dividing a set of indivisible items between two players, A and B, so that each player receives an equal number of items. For example, if the items are numbered 1 to 6, the allocation might be $\{1,3,5\}$ to A and $\{2,4,6\}$ to B. We assume that each player can strictly rank all the items from most to least preferred. Roughly speaking, an allocation is EF if each player prefers the subset of items it receives to the subset of items received by its opponent and so is not envious.

The precise definition of envy-freeness uses only the players' rankings to assess whether each player prefers its own subset of items to its opponent's subset. Denote the sets of items received by A and B by S_A and S_B, respectively. Recall that $|S_A| = |S_B|$. An allocation (S_A, S_B) is EF iff there exist an injection $f_A : S_A \to S_B$ and an injection $f_B : S_B \to S_A$ such that for each item x received by A, A prefers x to $f_A(x)$, and for each item y received by B, B prefers y to $f_B(y)$. Thus, a player pairwise prefers the items it receives in an EF allocation to the items received by its opponent.

Suppose the players' preferences for items, going from left to right, are as indicated below:

Example 1.

$$A: \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}\ 6$$
$$B: \underline{2}\ \underline{4}\ \underline{6}\ 1\ 3\ 5$$

The underscored allocation $\{1,3,5\}$ to A and $\{2,4,6\}$ to B is EF, as demonstrated by 1-1 mappings from A's items to B's, and B's items to A's, such that each player prefers each of its own items to the item of its opponent to which it is mapped. These mappings are: for A, $f_A(1) = 2, f_A(3) = 4, f_A(5) = 6$; and for B, $f_B(2) = 1$, $f_B(4) = 3$, and $f_B(6) = 5$. To simplify notation, we write these mappings as $f_A(1,3,5) = (2,4,6)$ and $f_B(2,4,6) = (1,3,5)$.

We emphasize that each player pairwise prefers its own item to the corresponding item of its opponent. For example, A receives item 3 and prefers 3 to $f_A(3) = 4$, which B receives, but A does not prefer item 3 to item 2, another item received by B. However, A prefers item 1, another item it receives, to item 2. In this example, the functions f_A and f_B are inverses, but this property is not essential and, indeed, cannot be achieved in some examples, as we will show later.

By comparison, the allocation $\{1,2,3\}$ to A and $\{4,5,6\}$ to B is not EF. This can be proven by checking exhaustively all possible injections from $\{1,2,3\}$ to $\{4,5,6\}$ and from $\{4,5,6\}$ to $\{1,2,3\}$, showing that no pair of them has the required property. But there is an easier proof, based on the following characterization:[2]

Lemma 1. *An allocation is EF iff, for each item x received by a player (say, A), the number of items received by B that A prefers to x is not greater than the number of items received by A that A prefers to x.*

Proof. To show that the given property is necessary for an allocation to be EF, suppose that A receives x in an EF allocation, and it prefers r of its own items to x, and s of B's items to x. We show that $r \geq s$. Consider the mapping f_A defined above. Suppose that, for some item y received by A, $f_A(y)$ is preferred to x. Because A prefers y to $f_A(y)$, A must also prefer y to x. It follows that each of B's s items that A prefers to x must be the image under f_A of an item received by A that A also prefers to x. There are r such items, which implies that $r \geq s$. A similar argument, beginning with an item received by B, completes the proof of necessity.

To show sufficiency, suppose that an allocation satisfies the given property. We construct a 1-1 mapping, f_A, of A's items to B's items such that A always prefers the item it receives to the corresponding item that B receives. To see that A must receive its most preferred item, x_1, assume otherwise. Then B receives at least one item that A prefers to the most preferred item it receives, whereas A receives no such items, contradicting the required property. Therefore, x_1 must have been assigned to A. Let x_k denote A's kth most preferred of the items it receives, and define $f_A(x_k)$ to be A's kth most preferred of the items B receives. Because the number of B's items that A prefers to x_k cannot exceed $k-1$, it follows that A prefers x_k to $f_A(x_k)$. The mapping f_A thus defined and the mapping f_B constructed analogously show that the allocation is EF. □

[2] *As pointed out by a referee, our characterization is related to Hall's marriage theorem* [10]. *Hall's marriage condition is stated in terms of set cardinalities, whereas ours incorporates preferences (relations on sets).*

An alternative way to state Lemma 1 is as follows: If an allocation is EF, then whenever a player receives an item x, it must also receive at least half of all the allocated items that it strictly prefers to x. Assuming that all items are allocated, if a player receives an item x that it ranked kth in its original ranking, then it must also receive at least $(k-1)/2$ items that it strictly prefers to x.

It is clear that the allocation $\{1,2,3\}$ to A and $\{4,5,6\}$ to B in Example 1 is EF for A: Because A receives its top three items, it cannot prefer any items that B receives.

But the story is different for B, as can be shown using Lemma 1. It receives item 5, which is 6th in its ranking, and prefers only two of the items it receives, 4 and 6, to item 5. The allocation cannot be EF for B, because it prefers more items in A's subset (three: 1, 2, and 3) than in its own subset (two: 4 and 6). (In general, in an EF allocation a player must receive its most preferred item, and it cannot receive its least preferred item.) Another proof can be based on the fact that B receives item 4, which it ranks 2nd, but it receives no item that it prefers to item 4 (A receives item 2, which B ranks 1st).

We can now characterize all pairs of preference rankings for which EF allocations exist. Specifically, we present Condition D below, which we will show is necessary and sufficient for the existence of an EF allocation. The proof of necessity is given below; the proof of sufficiency will be given after we describe AL, which we show in the next section always produces an EF allocation if Condition D is satisfied.

Assume there are n items that A and B rank. For an EF allocation to be possible, the number of items allocated to each player must be the same, so the total number of items allocated is even.

Before stating Condition D, we begin with a sequence of simpler conditions. We say that A's and B's rankings satisfy Condition $C(k)$ iff

Condition C*(k)*. *The set consisting of A's k most preferred items is equal to the set consisting of B's k most preferred items.*

Note that Condition $C(k)$ refers to an equality of sets: A's *ranking* of its first k items may or may not be the same as B's. What is required is that the same k items be most preferred by A and by B.

It turns out to be important whether $C(k)$ is true when k is odd. In Example 1, where $n = 6$, $C(k)$ is false for every odd k:

$k = 1$: $\{1\}$ for A is different from $\{2\}$ for B.

$k = 3$: $\{1,2,3\}$ for A is different from $\{2,4,6\}$ for B.

$k = 5$: $\{1,2,3,4,5\}$ for A is different from $\{2,4,6,1,3\}$ for B.

We can now state Condition D in terms of Condition C(k).

Condition D. *Condition C(k) fails for all odd values of k, $1 \le k \le n$.*

In other words, Condition D states that, for all odd k, at least one of A's top k items is not a top k item for B, which, as we showed above, is true in Example 1.

Note that Condition D cannot be true if n is odd, because C(k) must hold for $k = n$ (as the set of all items is the same for both players). Thus, Condition D can be true only if the number of items to be allocated is even.

An EF allocation is *complete* iff it allocates all n items. Then our first theorem gives the following characterization.

Theorem 1. *Let n be even. A pair of strict preference rankings of n items admits a complete EF allocation iff it satisfies Condition D.*

Proof (of Necessity). To show that Condition D must hold in order for an EF allocation of all n items to exist, we show that if Condition D fails, then there can be no EF allocation. Now Condition D fails iff there is some odd value of k such that C(k) holds. Assume such a value of k, and let S be the subset consisting of A's (or B's) top k items.

Suppose that an EF allocation exists. Because S contains an odd number k of items, it follows that one of A and B, say A, must receive fewer than half of the items in S. Suppose that A receives $r < k/2$ items from S. Moreover, because each player must receive the same number of items in an EF allocation, A must receive at least one item that does not lie in S; that is, it is not among A's k most preferred items.

Let y be the item most preferred by A among the items that A receives that are not in S. If y is hth ranked by A in A's original ranking, we must have $h \ge k + 1$. Moreover, A receives exactly r items that it prefers to y. According to Lemma 1, we must have $r \ge (h - 1)/2 \ge k/2$. But, as noted above, $r < k/2$. This contradiction shows that no EF allocation can exist, establishing Condition D as necessary. □

We postpone the proof of sufficiency, which depends on the performance of AL. We describe this algorithm and BT next.

The BT and AL Algorithms

In this section we formally state the rules of BT and AL. Both algorithms allocate a set of indivisible items in a series of stages. In the case of BT, the players can be thought of as simultaneously or independently choosing the most preferred unallocated item at each stage, so the players need not give a complete ranking of items at the outset. By contrast, in the case of AL the players submit their complete rankings to a referee (or a computer), which makes choices solely on the basis of the rankings.

BT Rules

1) Players A and B name their most preferred item of those that have not yet been allocated.

2) If A and B name different items, each player receives the item it names. If they name the same item, it goes into the *contested pile* (CP).

3) If all items have been allocated to the players or put in CP, stop. Otherwise, go to step 1.

AL Rules

We begin with an informal description of AL, which also works by descending the preference rankings of the players. If the players have not yet been assigned any items, then if there is an item at the top of both players' rankings, it is put into CP, and this step is repeated until each player most prefers a different unallocated item. When this happens, AL assigns each player its preferred item.

After the first assignment of items to the players is made, new assignments are made

(i) when the players prefer different items or
(ii) when they prefer the same item, provided a new assignment—of the preferred item to one player and a less preferred item to the other—does not cause envy and so is *feasible.*

When there is a commonly preferred item, the feasibility of assigning it to either player is assessed, one player at a time. Only if there is no such assignment is the commonly preferred item put in CP.

Formally, we start AL at stage 0, which may be repeated. In each stage t ($t = 0, 1, 2, \dots$), exactly t items have already been assigned to each player. AL proceeds until there are no unallocated items.

Stage 0

Compare the most preferred unallocated items of A and B. If they are identical, place the commonly preferred item in CP and repeat stage 0. If they are different, assign each player its most preferred item. Then go to stage $t = 1$.

Stage t

1) If one unallocated item remains, place it in CP and stop. If no unallocated items remain, stop. Otherwise, compare A's and B's most preferred unallocated items. If they are the same, go to step 2. If they are different, assign each player its most preferred item and go to stage $t + 1$.

2) Determine whether the unallocated item that A and B both most prefer, say i, which we call the *tied item*, can be assigned to either A or B as follows: Let $j_{A1}, j_{A2}, \ldots$ represent, in order of A's preference, the unallocated items that A finds less preferable than i. Let $j_{B1}, j_{B2}, \ldots$ represent, in order of B's preference, the unallocated items that B finds less preferable than i.

3) Consider all possible assignments of i to B and j_{A1} first, then j_{A2}, etc., to A. Such an assignment is *feasible* as long as the number of items assigned to B or unassigned, including i, that A prefers to the *compensation item* it receives, j_{A1} or j_{A2} or ..., is at most t.[3] Stage $t+1$ must be implemented for each feasible assignment of i to B. If the number of items assigned to B, or unassigned, that A prefers to j_{A1}, including i, is greater than t, then no assignment of i to B is feasible.

4) Consider all possible assignments of i to A and j_{B1} first, then j_{B2}, etc., to B. Such an assignment is feasible as long as the number of items assigned to A or unassigned, including i, that B prefers to its compensation item, j_{B1} or j_{B2} or ..., is at most t. Stage $t+1$ must be implemented for each feasible assignment of i to A. If the number of items assigned to A, or unassigned, that B prefers to j_{B1}, including i, is greater than t, then no assignment of i to A is feasible.

5) If the assignment of i to A is infeasible, and the assignment of i to B is infeasible, then put i in CP. Then repeat stage t for the remaining unallocated items.

Whereas BT gives only one EF allocation, AL may give many, because for $t > 1$ there may be multiple ways to implement AL, as we will illustrate later. Although AL is more complex than BT, it is *not* so for the players, who only need to submit their rankings of items.

The chief difference between BT and AL is in how CP is defined, as we next illustrate with two examples. In each example we assume that the players are *sincere*, ranking each item according to their true preferences. Later we assume that the players may not be sincere; in particular, they may seek to manipulate BT or AL to their advantage.

Example 2.

$$A: \ 1\ 2\ 3\ 4$$
$$B: \ 2\ 3\ 4\ 1$$

When BT is applied to Example 2, A indicates that its first choice is item 1, and B that its first choice is item 2; by BT rule 2, the players receive their preferred items because they are different. At stage 2 both A and B indicate that item 3 is their preferred item of those remaining, so it goes into CP, as does item 4 at stage 3, by BT rule 2. In summary, A receives item 1, B receives item 2, and CP $= \{3, 4\}$.

Under AL the players' top-ranked items—1 for A and 2 for B—are different, so item 1 goes to A and item 2 goes to B in stage 0. Now proceed to stage $t = 1$. Of the unallocated items, both players most prefer $i = 3$, making it the tied item. For A, one unallocated item, $j_{A1} = 4$, is less preferred than 3. We consider assigning 3 to B and $j_{A1} = 4$ to A, but then B will be assigned two items, namely, 2 and 3, that A prefers to $j_{A1} = 4$, which exceeds $t = 1$, so we cannot assign 3 to B.

For B, too, the only unallocated item less preferred than $i = 3$ is $j_{B1} = 4$. We consider assigning 3 to A and $j_{B1} = 4$ to B. This assignment is feasible, because B prefers only one item to $j_{B1} = 4$ that is allocated to A (item 1). Thus, AL produces the allocation $S_A = \{1, 3\}$, $S_B = \{2, 4\}$, in which CP $= \emptyset$. Example 2 shows that AL may sometimes produce a complete allocation when BT does not.

Example 2 also shows that, under AL, the 1-1 mappings f_A and f_B—of A's items into B's and B's into A's—need not be inverse functions. In particular, the allocation given by AL is EF for A because $f_A(1, 3) = (2, 4)$, and it is EF for B because $f_B(2, 4) = (3, 1)$.[4]

Our next example shows that AL, as well as BT, may produce only partial allocations, and these allocations may differ.

Example 3.

$$A: \ 1\ 2\ 3\ 4\ 5\ 6$$
$$B: \ 2\ 3\ 5\ 4\ 1\ 6$$

When BT is applied to Example 3, A and B initially receive their most preferred items, 1 and 2, respectively. Next, because both players name item 3, it goes into CP. Then A and B receive the items they name, 4 and 5, respectively. Finally, both players name item 6, so it goes into CP. Altogether, A receives $\{1, 4\}$, B receives $\{2, 5\}$, and CP $= \{3, 6\}$. This allocation is EF, where $f_A(1, 4) = (2, 5)$ and $f_B(2, 5) = (1, 4)$ or $(4, 1)$.

Under AL, because the players' top-ranked items are different, item 1 goes to A and item 2 goes to B in stage 0. In stage 1 both players prefer $i = 3$. For A the most preferred unallocated item less preferred than $i = 3$ is $j_{A1} = 4$. But we cannot assign $i = 3$ to B and $j_{A1} = 4$ to A, because B would

[3] *The "compensation" is in lieu of not receiving the tied item i, which A prefers.*

[4] *The mappings f_A and f_B are inverses iff $f_B(f_A(x)) = x$ for all x in A's subset. When an EF allocation exists despite a common preference (e.g., for item 3 at stage $t = 1$ in Example 2), it can be shown that the mappings f_A and f_B cannot be inverses. Thus, in Example 2, $f_A(1) = 2$, so $f_B(f_A(1)) = 3 \neq 1$.*

be assigned more than one item (namely, items 2 and 3) that A prefers to $j_{A1} = 4$.

For B the first unallocated item less preferred than $i = 3$ is $j_{B1} = 5$. We can assign $i = 3$ to A and $j_{B1} = 5$ to B, because only one item assigned to A (item 3) is preferred by B to $j_{B1} = 5$. But we cannot proceed further, because after $j_{B1} = 5$, the next unallocated item in B's preference ranking is $j_{B2} = 4$. However, assigning $i = 3$ to A and $j_{B2} = 4$ to B is infeasible, because more than one item—in fact, two items, namely, 3 and 5—that B prefers to $j_{B2} = 4$ would be unallocated or assigned to A. Therefore, there is only one way to proceed to stage 2, namely by assigning items 1 and 3 to A and items 2 and 5 to B.

In stage 2, A and B both prefer item 4 and the next most preferred item 6. As already noted, in an EF allocation neither player can be assigned item 6, the common last choice. Consequently, both 4 and 6 are put in CP. In summary, AL produces exactly one allocation in Example 3: $S_A = \{1, 3\}$, $S_B = \{2, 5\}$, and CP $= \{4, 6\}$.

Example 3 illustrates another difference between BT and AL. Neither algorithm may produce a complete allocation. Each yields a CP that contains two items, one of which is item 6. In the case of BT, the other item is 3, whereas under AL it is 4. Necessarily, the AL and BT allocations also differ, with $S_A = \{1, 4\}$ under BT and $S_A = \{1, 3\}$ under AL. Note that $S_B = \{2, 5\}$ under both BT and AL.

Consider two allocations, (S_A, S_B) and (S'_A, S'_B), where all four subsets are of equal cardinality but do not necessarily contain the same items. We say that (S_A, S_B) *Pareto-dominates* (S'_A, S'_B) iff there are injections $g_A : S_A \to S'_A$ and $g_B : S_B \to S'_B$ such that A finds x at least as preferable as $g_A(x)$ for all $x \in S_A$, B finds y at least as preferable as $g_B(y)$ for all $y \in S_B$, and for at least one of x or y this preference is strict. In words, one allocation Pareto-dominates another if it is at least as good for both players and better for at least one of them, based on pairwise comparisons.

Note that the Pareto-comparison of (S_A, S_B) and (S'_A, S'_B) depends only on the assumptions that the four subsets have equal cardinality, that S_A does not overlap S_B, and that S'_A does not overlap S'_B. In particular, the sets of items allocated, $S_A \cup S_B$ and $S'_A \cup S'_B$, need not be identical, making it possible to Pareto-compare two allocations when unallocated items remain or when the CPs are different.

In Example 3, A prefers its AL allocation, $\{1, 3\}$, to its BT allocation, $\{1, 4\}$, because while both allocations contain item 1, A prefers item 3 to item 4. Here B is indifferent between its BT and AL allocations, which are both $\{2, 5\}$.

Thus the AL allocation Pareto-dominates the BT allocation in Example 3. Note also that both players agree that CP $= \{3, 6\}$, given by BT, is preferable to CP $= \{4, 6\}$, given by AL, reflecting the fact that one player (A) prefers its AL allocation to its BT allocation, while the other player (B) is indifferent.

Examples 2 and 3 illustrate the following proposition:

Theorem 2. *The number of items allocated to the players under AL is never less, and may be more, than under BT. If the number of items allocated to the players is the same under BT and AL but some items are different, then the AL allocation Pareto-dominates the BT allocation.*

Proof. A commonly preferred item i, which we called a tied item, may be assigned to a player under AL but is never assigned under BT. Thus, one or more tied items may go into CP under BT that would not under AL, so the number of items allocated under AL may be greater and will never be less than the number allocated under BT.

When a tied item is allocated under AL, the consequence may be the creation of later tied items, which would not have occurred if the tied item had been put in CP, as it would have under BT. Thus, the total number of items in CP may be the same as under BT, but ties that occur later involve less preferred items, so an AL allocation—even if it does not reduce the cardinality of CP—Pareto-dominates the corresponding BT allocation if they differ. □

Theorem 3. *An AL allocation is a maximal EF allocation: There is no other EF allocation that allocates more items to the players.*

Proof. AL continues until all items are either assigned to one player or put in CP. Hence, any EF allocation that *contains* an AL allocation must transfer some items from CP to the players. But AL puts an item, i, in CP only if it is tied and the assignment of i to either player and any less preferred item to its opponent cannot preclude the opponent from being envious. Thus, items cannot be transferred from the CP to the AL allocation. □

This is not to say that AL finds all maximal EF allocations. In Example 3, we found two maximal EF allocations—of two items to each player: one by AL and a different one by BT—but the AL allocation Pareto-dominates the BT allocation. Indeed, Theorem 2 shows that such dominance must be the case when these two allocations are the same size but not identical.

So far we have shown that, for any pair of strict preference rankings of n items:

1. an AL allocation may give each player more items than the BT allocation;

2. an AL allocation may give each player the same number of items as the BT allocation, but the sets may not be the same, in which case the AL allocation Pareto-dominates the BT allocation;
3. the AL and BT allocations may be exactly the same.

Possibility 3 occurs in Example 1, wherein both algorithms give $\{1, 3, 5\}$ to A and $\{2, 4, 6\}$ to B. It also occurs in two extreme cases: (i) when the players rank all items exactly the same (in which case all items go into CP) and (ii) when their rankings are diametrically opposed and n is even (in which case each player will obtain its more preferred half of the items and CP will be empty).

It is apparent that BT always gives an EF allocation, because it allocates items to players only when they prefer different ones at the same time. This implies that f_A and f_B are inverses. But recall that Example 2 showed that it is possible that the mappings of an AL allocation are not inverses. Examples 2 and 3 also showed that AL may give larger or more preferred EF allocations than BT.

Earlier we proved the necessity part of Theorem 1: that Condition D—for every odd k, $1 \le k \le n$, at least one of A's top k items is not a top k item of B—is necessary for the existence of an EF allocation of all n items, i.e., a complete EF allocation. We next show that Condition D is also sufficient by adding the proof of sufficiency to Theorem 1, which we repeat below.

Theorem 1 (continued). *Let n be even. A pair of strict preference rankings of n items admits a complete EF allocation iff it satisfies Condition D.*

Proof (of Sufficiency). We show that Condition D is sufficient for the existence of a complete EF allocation by proving that AL produces a complete EF allocation unless Condition D fails. Specifically, we show that, if AL puts any item in CP, then for some odd k, the subset comprising A's k most preferred items must equal the subset comprising B's k most preferred items.

Suppose that we are applying AL to find an EF allocation. At stage 0, if A's and B's top-ranked items are the same, AL will put this item in CP. Thus, if AL puts an item in CP at stage 0, then Condition $C(k)$ must be satisfied for $k = 1$; i.e., A's and B's most preferred items are identical.

Next suppose that A's and B's top-ranked items are different and that AL has reached stage $t > 0$, so that both players have received t items without violating envy-freeness. For an item to be added to CP, it must be the case that (i) both players prefer it to all other unallocated items (i.e., it is a tied item) and (ii) the allocation of the tied item to either player will cause its opponent to be envious.

Assume the tied item is i. If it is possible to assign i to B and j_{A1}—A's most preferred unallocated item after i—to A while preserving envy-freeness, the number of items assigned to B, including i, that A prefers to j_{A1} must be at most t. If it is not possible to assign i to B and j_{A1} to A, then the number of items assigned to B that A prefers to j_{A1} must exceed t. Because only t items were assigned to each player prior to i, then the number of items assigned to B, including i, that A prefers to j_{A1} must equal exactly $t + 1$. In particular, i itself plus the items previously assigned to A or to B must be the first $2t + 1$ items in A's preference ranking.

An analogous argument can be made for B. If it is not possible to assign i to A and j_{B1} to B, then it must be the case that the subset consisting of i, the items previously assigned to A, and the items previously assigned to B must be the first $2t + 1$ items in B's preference ranking.

When the players have the same $2t + 1$ items in their preference rankings—no matter which player receives tied item i—Condition $C(k)$ holds for $k = 2t+1$, so Condition D fails. To conclude, AL puts an item into CP when Condition D fails, which means that Condition $C(k)$ must hold for some odd k. On the other hand, when Condition D holds, AL never puts an item in CP, so a complete EF allocation must exist. □

Although Condition D is both necessary and sufficient for the existence of an EF allocation, it does not say what the EF allocation(s) are.[5] For that purpose we need AL.

As noted previously, both AL and BT always allocate to each player the same number of items, although AL may allocate more items in toto (Example 2). Therefore, the number of items allocated to CP, if it is not empty, will be even or odd depending on whether the total number of items to be allocated is even or odd. In particular, if n is odd, then CP must contain at least one item.

We showed earlier (Theorem 2) that, if AL and BT give different EF allocations to the players, then AL's allocation must include more, or more preferred, items; furthermore, it gives a maximal EF allocation (Theorem 3). We next assess how well AL and BT do according to other properties.

Other Properties of EF Allocations

We begin with an example that illustrates how AL may produce more than one complete EF allocation.

Example 4.

$$A: \; 1\,2\,3\,4\,5\,6\,7\,8$$
$$B: \; 3\,4\,5\,6\,7\,8\,1\,2$$

[5] *We postpone until the next section examples showing that AL may produce multiple EF allocations.*

In stage 0, AL assigns item 1 to A and item 3 to B. In stage 1, AL assigns item 2 to A and item 4 to B. Then, in stage 2, there is a tie on item 5. The tie cannot be resolved by assigning $i = 5$ to B, because $j_{A1} = 6$, and the assignment of items 3, 4, and 5 to B would mean that of the items that A prefers to item 6, fewer than half (i.e., only items 1 and 2) are assigned to A.

But the tie can be resolved by assigning $i = 5$ to A, in which case B can receive either $j_{B1} = 6$ or $j_{B2} = 7$. Thus stage 3 can begin with A assigned $\{1, 2, 5\}$ and B assigned $\{3, 4, 6\}$, or with A assigned $\{1, 2, 5\}$ and B assigned $\{3, 4, 7\}$. In the first case, A is assigned item 7 and B item 8 in stage 3; in the second case, A is assigned item 6 and B item 8 in stage 3. The two resulting EF allocations are underscored below:

$$\text{(i) } A: \ \underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ 6\ \underline{7}\ 8 \qquad \text{(ii) } A: \ \underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ \underline{6}\ 7\ 8$$
$$B: \ \underline{3}\ \underline{4}\ 5\ \underline{6}\ 7\ \underline{8}\ 1\ 2 \qquad\qquad B: \ \underline{3}\ \underline{4}\ 5\ 6\ \underline{7}\ \underline{8}\ 1\ 2$$

In (i) a player's minimal ranking for an item it receives is 7th (item 7 for A), whereas in (ii) this minimal ranking is 6th (item 6 for A and item 8 for B). We call (ii) the *maximin allocation*: It maximizes the minimum rank of the players, which may be desirable in certain situations.

A complete allocation is called *locally Pareto-optimal* (LPO) if there is no other allocation of the same items that Pareto-dominates it; i.e., the items cannot be redistributed between the players so that each player is at least as well off, and some player is better off, where comparisons are always pairwise. For example, if there are $n = 2$ items and A prefers item 1 to item 2 and B prefers item 2 to item 1, then the allocation of 2 to A and 1 to B is not LPO, because both players would be better off if 1 were assigned to A and 2 to B. Recall from the "The BT and AL Algorithms" section that we defined the Pareto-optimality of allocations that were not constrained by the "same items" condition.

Call an allocation *sequential* if it assigns each player its most preferred item when it is that player's turn to choose according to some sequence (e.g., *ABAB* or *AABB*). Note that the players need not alternate in a sequence, though each player must have the same number of turns to choose. The resulting allocation of items, called a *sincere sequence of choices*, clearly depends on the sequence.[6]

Theorem 4 (Brams and King, 2005). *An allocation of a fixed set of items is LPO iff it is the product of a sincere sequence of choices.*

[6] *Choices may be strategic, not sincere, if the players know each other's preferences. Backward induction can then be used to determine subgame perfect Nash equilibria using algorithms discussed in* [12], [7], [9, *chs.* 2 *and* 3], [2, *ch.* 9], *and* [13].

Strictly speaking, BT and AL are not sequential algorithms, because items are assigned to the players simultaneously. But if at some stage the players' first choices are different, then the players can be considered to receive items in either order, *AB* or *BA*, because the items received by A and B would be the same. Therefore, when A and B most prefer (and receive) different items, the assignment can be considered as part of a sincere sequence of choices.

Now suppose that, at some stage, the players' top choices are the same. Under BT, this item always goes into CP and hence will not be part of the allocation to A and B. Under AL, by comparison, this item will go into CP iff the item cannot be assigned to either player so as to maintain envy-freeness. Nonetheless, the resulting allocation will be LPO under both algorithms in the sense that no reallocation of items can Pareto-dominate what each algorithm yields, as we next prove.

Theorem 5. *Both BT and AL produce LPO allocations.*

Proof. We have already noted that the BT allocation of items that do not end up in CP is a sincere sequence of choices. To show that the same is true of an AL allocation, we need only check that it is true at any point when both players prefer the same item. Suppose that the tied item, i, is assigned to B, while some compensation item, j_{A1} or j_{A2} or $\ldots$, is assigned to A. Recall that $j_{A1}, j_{A2}, \ldots$ represent, in order of A's preference, the unallocated items that A finds less preferable than i. Clearly, an allocation in which B receives i and A receives j_{A1} is the result of a sincere choice sequence, in the order BA. If B receives i and A receives, say, j_{Ah} where $h > 1$, then the allocation is the result of a sincere choice sequence, $B \ldots A$, where the missing entries are determined by the eventual allocation of the unallocated items, including $j_{A1}, j_{A2}, \ldots, j_{Ah-1}$. (This may be considered an "out-of-order" assignment in that it does not make assignments strictly according to the players' preferences.)

In an AL allocation, every item that is assigned to a player who would prefer a different item among all unallocated items receives a *deferred-compensation item*, such as j_{Ah}. When all items that precede j_{Ah} in A's order have been allocated, it will be possible to identify a sincere choice sequence containing equally many A's and B's that corresponds to an AL allocation. By Theorem 4, this allocation will be LPO. □

We have shown that complete allocations under BT and AL are both EF and LPO, but partial allocations will satisfy both properties only for the items that are allocated to the players (i.e., that do not go into CP). Moreover, as Theorem 2 shows, when a BT allocation produces the same number of items as an AL allocation but the items are different, then the AL allocation will Pareto-dominate the BT allocation.

The reason that the AL allocation in Example 3 Pareto-dominates the BT allocation is that, while each algorithm allocates four of the six items to A and B, AL assigns a preferred item (3) to A and BT does not, which puts this item in CP before assigning item 4 to A. This enables A to do better under AL than it does under BT without changing the allocation to B (but changing the contents of CP).

We note that LPO allocations need not be EF. In Example 2, for instance, the allocation of $\{1,2\}$ to A and $\{3,4\}$ to B is LPO in that any other allocation of the four items is less preferred by A. But B might envy A (for receiving the two items that bracket its two middle items), so we call such an allocation *envy-possible* (it does not ensure envy). In contrast, allocating $\{2,4\}$ to A and $\{1,3\}$ to B is *envy-ensuring* [6], because it ensures that each player envies the other.

In Example 4 both EF allocations are LPO, because they can be produced by sincere sequences. A sincere sequence that produces (i) is $ABABABAB$, whereas a sincere sequence that produces (ii) is $ABABAABB$ (there are several other sincere sequences that give each allocation).

In all examples so far in which there is a complete EF allocation (Examples 1, 2, and 4), A and B rank all the items differently (they also do so in Example 3 for the four items that do not go into CP). By contrast, if they ranked all items the same, there would be no EF allocation, because all items would go into CP.

It seems plausible, therefore, that different rankings of the items by the players might be a sufficient condition for there to be a complete EF allocation. However, this conjecture fails for

Example 5.

$$A: \ 1\ 2\ 3\ 4\ 5\ 6$$
$$B: \ 2\ 3\ 1\ 5\ 6\ 4$$

Because the top $k = 3$ items $\{1,2,3\}$ are the same for both A and B, Condition C(3) holds. Therefore, Condition D fails, so by Theorem 1 there can be no complete EF allocation.

The fact that Condition D fails in Example 5 does not tell us what *partial* EF allocation is possible. For this purpose, we need to apply AL.

In stage 0, AL assigns item 1 to A and item 2 to B. In stage 1, there is a tie on item 3. It cannot be resolved by assigning $i = 3$ to either A or B. Therefore, we must put item 3 into CP, after which AL allocates item 4 to A and item 5 to B. The remaining unallocated item, 6, must then go into CP. In summary, $A = \{1,4\}$, $B = \{2,5\}$, and $\text{CP} = \{3,6\}$. Coincidentally, BT produces the same EF allocation.

Our next example shows that AL can give exponentially many EF allocations, all of which are complete and maximin (unlike Example 4).

Example 6.

$$A: \ 1\ 2\ 3\ 4$$
$$B: \ 4\ 2\ 3\ 1$$

It is easy to see that AL produces two allocations: $A = \{1,2\}$ and $B = \{3,4\}$, and $A = \{1,3\}$ and $B = \{2,4\}$. Now add four more items for which the players' preferences copy those of Example 6.

Example 6′.

$$A: \ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$
$$B: \ 4\ 2\ 3\ 1\ 8\ 6\ 7\ 5$$

AL allocates the first four items in two ways, as before, and then allocates the second four items in two ways. Thus, there are $2 \times 2 = 4$ different EF allocations in Example 6′: $S_A = \{1,2,5,6\}$, $S_B = \{3,4,7,8\}$; $S_A = \{1,2,5,7\}$, $S_B = \{3,4,6,8\}$; $S_A = \{1,3,5,6\}$, $S_B = \{2,4,7,8\}$; and $S_A = \{1,3,5,7\}$, $S_B = \{2,4,6,8\}$.

Adding an additional four items in a similar way produces eight different EF allocations, and this doubling pattern continues. Examples of this family contain n items to be allocated; AL produces $2^{n/4}$ distinct EF allocations, all of which are complete and maximin. It follows that the number of EF allocations can grow exponentially in n, so no polynomial-time algorithm will find all EF allocations in this family.[7]

But finding just one EF allocation can be done in polynomial time by checking at every stage in which there is a tied item at most two possible

[7] *The rate of growth of the number of complete EF allocations in the family based on Example 6 is not maximal. For example, it is not difficult to show that there are six distinct complete and maximin EF allocations for the following 8-item example:*

$$A: \ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$
$$B: \ 7\ 8\ 3\ 4\ 5\ 6\ 1\ 2$$

which exceeds the four distinct EF allocations of the 8-item example in the text. Copying preferences in the manner discussed in the text yields an exponent of approximately 0.323n in this example, compared to 0.25n in the example in the text. We recently discovered that Bouveret, Endriss, and Lang [1], *using a different methodology (SCI-nets), analyze algorithms for finding EF and LPO allocations and describe their computational complexity. Some of our findings echo theirs (e.g., on "necessary envy-freeness"), but others (e.g., our Condition D and our results on maximality and manipulability) do not.*

assignments: (i) the tied item is assigned to A, with B getting its next-best item; and (ii) the tied item is assigned to B, with A getting its next-best item. If both (i) and (ii) fail, the tied item goes into CP.

We need to check only the next-best item of the player not getting the tied item because, if (i) and (ii) fail, then no lower-ranked item will give an EF allocation. Thus failure can be confirmed by testing two allocations at every stage.

To conclude, AL is an exponential-time algorithm if one wishes to generate all EF allocations. But if one EF allocation suffices, with the algorithm terminating at a stage as soon as one assignment (either to one of the players or to CP) has been found, it is polynomial time, making it applicable to the division of large numbers of items.

Up to now we have assumed that the players rank items sincerely.[8] Call an algorithm *manipulable* if a player, by submitting an insincere preference ranking, can obtain a preferred allocation.

Theorem 6. *AL and BT are manipulable.*

Proof. We begin with AL, for which there are two EF allocations in Example 7:

Example 7.

$$\text{(i) } \begin{array}{rl} A: & \underline{1}\ 2\ \underline{3}\ \underline{4}\ 5\ 6 \\ B: & \underline{2}\ \underline{6}\ 4\ \underline{5}\ 3\ 1 \end{array} \qquad \text{(ii) } \begin{array}{rl} A: & \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}\ 6 \\ B: & \underline{2}\ \underline{6}\ \underline{4}\ 5\ 3\ 1 \end{array}$$

Allocation (i) is maximin (the lowest rank of a player is 4th), whereas allocation (ii) is not (the lowest rank of a player is 5th). BT gives only a partial EF allocation—$A = \{1, 3\}$, $B = \{2, 6\}$, and $\text{CP} = \{4, 5\}$—which presumably will be unsatisfactory for the players compared to one of the two complete AL allocations.

Now assume that, instead of reporting its sincere preferences in Example 7, B reports its preferences to be B'—interchanging items 4 and 6—whereas A continues to be sincere. This yields the following unique AL allocation:

Example 7′ (manipulated by B).

$$\begin{array}{rl} A: & \underline{1}\ 2\ \underline{3}\ 4\ \underline{5}\ 6 \\ B': & \underline{2}\ \underline{4}\ \underline{6}\ 5\ 3\ 1 \end{array}$$

Thereby B obtains its top three items, whereas without manipulation B's allocation of these items was only one of two possibilities—and not the maximin one (had this property been used to choose between the two AL allocations without manipulation). BT gives exactly the same result, so B's misrepresentation helps it under BT, compared with obtaining only its top two items when it is sincere. □

[8] *The implications of insincere behavior are studied in* [15]. *Variations on the rules for making fair allocations, such as accepting or rejecting one or more items in a round, are analyzed in* [16].

We conclude that both AL and BT are manipulable if one player (B in Example 7) knows its adversary's (A's) sincere ranking and exploits its knowledge. But such manipulation seems improbable, short of A's having complete information about B's ranking of items, and A's being in the dark about the possibility of B's misrepresentation. Furthermore, the determination of an optimal misrepresentation strategy, especially when the number of items is large, is far from trivial, particularly in the case of AL because of its greater complexity. It is further complicated if there is a random selection from multiple EF allocations.

In the face of these difficulties, we think that A and B, especially when using AL, are likely to be sincere in submitting preference rankings to a referee. This presumption is reinforced by the fact that, if the players are sincere, they can ensure themselves of an EF, LPO, and maximal allocation, though it may not be complete.

The Probability of Envy-Free Allocations

There are many pairs of preference rankings for which there is no complete EF allocation. This is certainly true if both players rank all items the same, but it is also true if both players agree only on their top-ranked item, because whoever does not obtain that item will envy the other player. Similarly, no complete EF allocation is possible if the two players rank only their last-choice item the same, because whoever obtains it may envy the other.

On the other hand, if a complete EF allocation exists, it need not be unique, as we showed with several examples. To calculate the probability of a complete EF allocation, fix A's preference ranking as 1 2 3 . . . and assume all preference rankings of B are equiprobable. If $n = 2$ items and A's ranking is 1 2, then B's ranking can be 1 2 or 2 1. In the former case, there will be envy if A receives item 1 and B receives item 2, whereas in the latter there will not be envy, so the probability that an EF allocation exists is $\frac{1}{2}$.

If $n = 4$, then B can have any of $4! = 24$ preference rankings. To calculate the probability of a complete EF allocation, we note that Condition D requires that (i) the first choices of A and B be different and (ii) the first three choices of A and B be different.

Let us instead count the number of ways that Condition D can fail. For (i) to fail, B's first choice must be 1, for which there are $3! = 6$ orderings. For (ii) to fail, B's fourth choice must be 4, of which there are $3! = 6$ orderings. But Condition D fails if either (i) fails or (ii) fails, and both may fail simultaneously. However, we have double-counted the cases in which both (i) and (ii) fail, which requires that B's first choice be 1 and B's fourth

Even Number of Items n	2	4	6	8	10	12
Probability of Complete EF Allocation	0.500	0.583	0.678	0.750	0.800	0.834

choice be 4, for which there are 2 orderings. We conclude that Condition D can fail in $6+6-2=10$ ways. Thus, there are $24-10=14$ preference rankings for B for which Condition D holds.

We have shown that, when there are $n=4$ items, $14/24 \approx 0.583$ of the possible allocations admit a complete EF allocation, which can be extended to other values of n (see table above).[9] For even values of n, the probability that an EF allocation exists is on the order of $(n-2)/n$, so it tends to 1 as n approaches infinity. To see this, note that the probability that $C(k)$ holds—that a randomly chosen permutation of $\{1,\ldots,n\}$ fixes the subset $\{1,\ldots,k\}$—is $k!(n-k)!/n!$. Condition D fails iff $C(k)$ holds for at least one odd k. Therefore, the probability that $C(k)$ fails cannot exceed the sum of these probabilities over odd k from 1 to $n-1$. The terms $k=1$ and $k=n-1$ are each $1/n$, and the other terms are $0(1/n^3)$, so this sum is $2/n-0(1/n^2)$.[10]

Summary and Conclusions

Given that two players can rank a set of indivisible items from best to worst, the main algorithm we have analyzed (AL) finds an allocation giving the players the same number of items that is EF, PO, and maximal—and complete if such an allocation exists. A simpler algorithm (BT), which is also EF and LPO, may allocate fewer preferred items to the players and so may not be maximal or, if it is maximal, will be Pareto-dominated by an AL allocation if the BT allocation is different.

A possible advantage of BT, besides its simplicity, is that the players can make sequential decisions: they can decide, based on the items they have already acquired, which of the remaining items to try to obtain next. By contrast, AL requires that the players rank all items in advance, so if the players' valuations are interdependent (i.e., the acquisition of one item affects the value of others), they cannot take advantage of possible synergies among the items. This suggests the importance of packaging individual items into subsets whose elements are complementary (e.g., matching sofas instead of two individual sofas) so that the packages are as independent as possible.

Because AL and BT are manipulable, players can sometimes do better by misrepresenting their preferences. But without complete information about an opponent's preferences, BT, and especially AL (because of its greater complexity), would be difficult to exploit. Indeed, trying but failing to do so could result in an allocation that is neither EF nor PO. Thus players would seem to have good reason to be sincere in using these algorithms.

At least one, but not necessarily all, allocations produced by AL will be maximin. This seems to be an important property to ensure balanced allocations—one player does not suffer because it receives an especially low-ranked item.

There may be many complete maximin EF allocations. If they are all known, one could be selected at random. But, to avoid algorithms that require exponential time, it might be preferable to stop AL at the first EF allocation (if any) that it finds to ensure that it can be implemented in polynomial time.

If all possible preference rankings of players are equiprobable, then the probability that a complete EF allocation exists increases rapidly with the number of items and approaches 1 as this number approaches infinity. But equiprobability is not a realistic assumption in many real-life situations, wherein the players' preferences are correlated. How the degree of correlation affects the proportion of items that are allocated to the players—versus those that go into CP—remains to be investigated.

In order to allocate the items in CP for which the players have identical rankings, Brams, Kilgour, and Klamler [5] developed an algorithm called the *undercut procedure*, whereby a player proposes a "minimal bundle" of items to keep for itself. Its opponent can either accept the complementary subset or undercut the minimal bundle by one item, which becomes the division that is implemented.[11] The allocations it produces are EF. Combined with AL, however, it can be used to allocate *all* the items, including those that AL puts into CP.

Alternative two-person procedures—including *adjusted winner* [8], [9], in which players assign points to items, and a *swapping procedure* in which players can make trades after an initial allocation [4]—produce fair divisions that satisfy other desiderata.[12] However, both procedures

[9] *We thank Richard D. Potthoff for assistance with this calculation.*

[10] *We thank a referee for this proof of convergence to a limit probability of 1.*

[11] *The extent to which this procedure is vulnerable to strategic manipulation is analyzed in* [15], [16].

[12] *One desideratum is equitability, in which players perceive that they receive the same fraction of the total value. Procedures for finding equitable as well as envy-free allocations of indivisible items are analyzed in* [11]. *Unlike BT and AL, they require that players specify preferred bundles of items, which makes them more akin to the undercut procedure for allocating the items in CP.*

require that the players provide more information than a simple ranking of items and, in the case of adjusted winner, that one item, which is not identifiable in advance, be divisible.

The fact that only AL requires that the players indicate their preference rankings is clearly an advantage, but in some applications it may be desirable to elicit and use information about the intensity of the players' preferences. But when obtaining such information is difficult, AL offers a compelling alternative—for example, in allocating the marital property in a divorce or the items in an estate, especially when the players have different tastes (e.g., for memorabilia or artworks).

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