WHAT IS... a Rauzy Fractal?

Pierre Arnoux and Edmund Harriss

There is a long tradition, going back to Hadamard and Morse, of associating symbolic infinite words with dynamical systems coming from geometry or mechanics. Conversely, one may try to give geometric representations to infinite words generated in an algebraic or combinatorial way; elementary examples, such as the Fibonacci word, produce well-known dynamical systems, but natural generalizations give rise to a diverse family of self-similar sets with fractal boundaries.

Define a sequence of words on the alphabet of two letters a, b, starting with a and at each step substituting every a by ab and every b by a. Elementary algebra shows that the lengths of the words $a, ab, aba, abaab\ldots$ are the Fibonacci numbers, the ratio of the frequencies of a and b tends to the golden number $\phi = \frac{1+\sqrt{5}}{2}$, and each word is a prefix of the next one. In this way we define the *Fibonacci word*, the only infinite word $abaababaab\ldots$ that is invariant by the substitution rule.

We can understand this word better by giving it a geometric representation as a broken line in the plane. Every a (resp. b) gives a step to the right (resp. a step up). We see (Figure 1) that this broken line not only has asymptotic slope $1/\phi$, as expected, but is also a very good discrete approximation of a line with slope $1/\phi$.

Pierre Arnoux is professor of mathematics at Université d'Aix-Marseille, France, France. His email address is `pierre@pierrearnoux.fr`.

Edmund Harriss is a clinical assistant professor of mathematics at the University of Arkansas. His email address is `eharriss@uark.edu`.

DOI: http://dx.doi.org/10.1090/noti1144

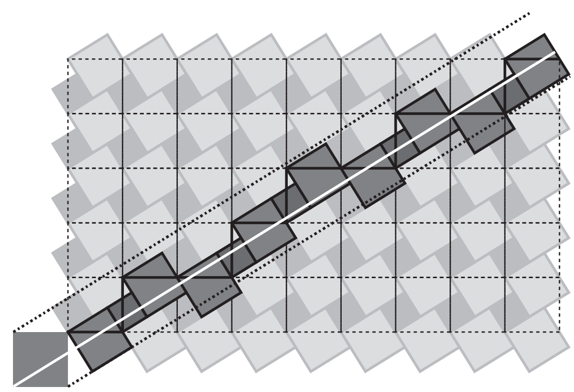

Figure 1. Geometric construction of the Fibonacci word.

This is easily proved. Denote by σ the substitution rule; the associated matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has one expanding direction associated with the eigenvalue ϕ, and one contracting direction associated with the conjugate $-1/\phi$. Therefore the points associated with the finite Fibonacci words $\sigma^n(a)$ tend to the expanding space. More generally, any initial word U of the infinite Fibonacci word can be written $U = \sigma^k(i_k)\ldots\sigma(i_1)i_0$ for a finite sequence $i_0, i_1, \ldots, i_k$, where i_n is the letter a or the empty word. Using the eigendirections for axes, the coordinates of the corresponding point are therefore given by finite sums. These are bounded by a geometric series, convergent in the contracting direction. The vertices of the broken line, therefore, lie within a bounded distance of the expanding direction.

Even better, if one draws a "corridor" by sliding the unit square with upper right corner at the origin along the expanding direction, the broken line lies within this corridor and is determined by it. This gives an arithmetic definition of the Fibonacci word: its nth letter is an a (resp. b) if $[(n+1)/\phi] - [n/\phi]$ is equal to 1 (resp. 0).

There is a physical interpretation, as a cut-and-project quasicrystal. A crystal is a periodic lattice in Euclidean space. To get a subset that is close to periodic, but not periodic, consider a cylinder in $\mathbb{R}^n$ with irrational direction and bounded base (the *window*). Projecting the points in the intersection of the integer lattice and this cylinder onto the irrational direction gives the cut-and-project quasicrystal associated with this cylinder [Sen95].

For the Fibonacci word, the projection of the vertices on the contracting direction is a bounded set whose closure is an interval; the projection on the expanding line gives a self-similar tiling that is the simplest possible example of a cut-and-project quasicrystal, with the window given by the interval in the contracting direction.

There is also a dynamical interpretation: We can *shift* the Fibonacci word by erasing its first letter; if we consider the closure of the shift-orbit of the Fibonacci word for the natural topology on infinite words, we get a very simple symbolic dynamical system. Its domain is a Cantor-like set, which projects continuously to the interval, and the projection conjugates the shift to the rotation by the golden number.

This is readily seen on the window, which is divided inyo two intervals corresponding to the two letters; the point corresponding to the next vertex of the broken line is obtained by exchanging the two intervals. This translation of intervals corresponds to the step to the right or up associated with each letter. These properties hold if we change the substitution rule, provided the associated matrix has determinant ± 1.

What happens if we increase the number of letters? Consider the three-letter substitution rule $a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$. This rule again defines an infinite word, and the Perron-Frobenius theorem shows that the associated broken line, now in three-dimensional space, has an asymptotic direction. Since all the eigenvalues except the leading eigenvalue have modulus strictly smaller than 1, the same proof shows that the broken line lies within a bounded distance of the expanding direction.

Definition. *First defined by Gérard Rauzy in 1982, the* Rauzy fractal *(also called the* central tile*) is the closure of the projection of the vertices of the broken line to the contracting space along the expanding line.* [Rau82]

To use the word "fractal" is slightly misleading, since this set is the closure of its interior and has the same dimension as the contracting plane; it is the boundary that is fractal. The Rauzy fractal has remarkable properties. Firstly, it is self-similar; more exactly, it is divided into three pieces, corresponding to the three letters, which are the solutions of a graph-directed iterated function system. Secondly, these three pieces admit both a periodic tiling and a nonperiodic self-similar tiling of the contracting plane. Finally, the broken line connects all the integer points contained in the prism given by translating the Rauzy fractal along the expanding line. These points project to a self-similar tiling of the expanding line: this gives another cut-and-project quasicrystal.

The three pieces can be assembled in two ways to partition the Rauzy fractal; this gives rise to a dynamical system, an exchange of the three pieces. This system is conjugate to a rotation of the torus and also to the symbolic dynamical system generated by the substitution rule. One can define three prisms based on the three pieces that form a fundamental domain for the integer lattice in three-dimensional space as well as a Markov partition for the automorphism of the three-torus associated with the matrix of the substitution rule.

We can define a Rauzy fractal for any substitution rule whose associated matrix has all but one eigenvalue of modulus strictly smaller than 1 (so-called *Pisot substitution rules*) and determinant ± 1. All known examples satisfy the properties above. For examples see Figure 2. This gives rise to the main unsolved problem of this area:

Conjecture (Pisot conjecture)**.** *The one-dimensional tiling generated by any Pisot substitution rule on d letters with determinant* ± 1 *is a cut-and-project quasicrystal.*

This conjecture occurs in several areas with many different formulations; the dynamical formulation states that the symbolic system (or the tiling flow) associated with such a substitution rule has pure discrete spectrum. It was proved for 2 letters by Barge and Diamond. While it is open for more letters, there is a terminating algorithm to confirm it holds for any specific substitution rule. The conjecture implies a number of consequences including producing a self-similar tiling and giving a Markov partition for the toral automorphism defined by a matrix.

Various generalizations have been developed. For nonunit determinants, the problem adds an arithmetic component, and one can define a Rauzy fractal using p-adic space. A multi-dimensional framework has been proposed recently for tiling substitution rules, the *Pisot families.* Here the expanding line is replaced by the underlying space

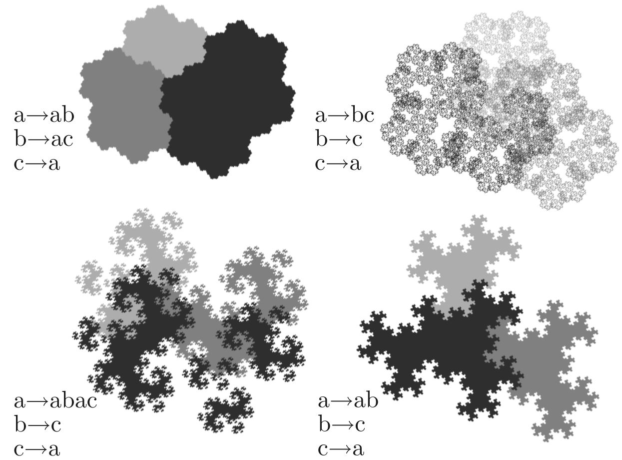

Figure 2. Examples of Rauzy fractals.

of the tiling, and the leading eigenvalue by the expansion matrix of the substitution rule; one requires that any conjugate of modulus > 1 of an eigenvalue of the expansion matrix is an eigenvalue of this matrix, with the same multiplicity. The Pisot conjecture can be generalized to this setting.

References

[BK06] M. Barge and J. Kwapisz, Geometric theory of unimodular Pisot substitutions, *Amer. J. Math.* **128** (2006), no. 5, 1219–1282.

[Pyt02] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics, Lecture Notes in Mathematics*, vol. 1794, Springer-Verlag, Berlin, 2002, edited by V. Berthé, S. Ferenczi, C. Mauduit, and A. Siegel.

[Rau82] G. Rauzy, Nombres algébriques et substitutions, *Bull. Soc. Math. France* **110** (1982), no. 2, 147–178.

[Sen95] M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press (1995).