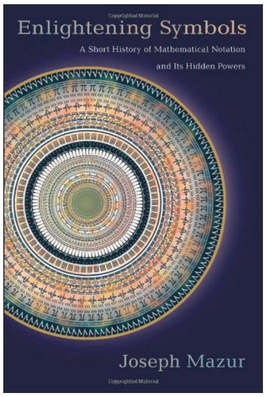
Book Review



Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers

Reviewed by Robyn Arianrhod

Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers
Joseph Mazur
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It is easy to forget, with our four centuries of post-Cartesian hindsight, that for thousands of years arithmetic and algebra were done rhetorically or geometrically, without the benefit of our familiar x's and y's, our pluses, minuses, square root signs, indices, and other symbols. These symbols make algebraic processes so transparent, so universal, and so easily generalized that mathematical thought becomes simpler and more economical.

This simplicity is possible because mathematical symbols can "evoke subliminal, sharply focused perceptions and connections" [p. xiii], as Joseph Mazur puts it in his entertaining and insightful new book. "Just as with the symbolism in music and poetry, these mathematical symbols might also transfer metaphorical thoughts capable of conveying meaning through similarity, analogy, and resemblance, and hence are as capable of such transferences as words on a page. In reading an algebraic expression, the experienced mathematical mind leaps through an immense number of connections in relatively short neurotransmitter lag times" [p. xiii].

Thinking about the history of mathematics in terms of the history of symbolic mathematical notation offers an interesting perspective on the dramatic increase in mathematical progress from the seventeenth century onwards. It also makes us wonder how ancient mathematicians made as much progress as they did without an internalized, subconscious symbolic language to aid their thinking. *Enlightening Symbols* offers food for thought on both these themes: the history of mathematics and symbolic cognition. But Joe Mazur's skill lies in discussing deep ideas in an engaging and accessible style: this is a book aimed at the lay reader, although it will also be of interest to mathematicians and (perhaps especially) mathematics educators.

Some readers might wonder what exactly is the definition of "symbol" that applies to mathematics. In his introduction Mazur gives the etymology of the word: it comes from the Greek for "token" or "token of identity" and refers to "an ancient way of proving one's identity or one's relationship to another. A stick or bone would be broken in two, and each person in the relationship would be given one piece. To verify the relationship, the pieces would have to fit together perfectly" [p. xi].

On a deeper level, the word "suggests that when the familiar is thrown together with the unfamiliar, something new is created. Or, to put it another way, when an unconscious idea fits a conscious one, a new meaning emerges" [p. xi]. A classic example of the power of mathematical symbolism is Maxwell's use of (a component form of)differential vector calculus to describe the known facts of electromagnetism in terms of a *field* (his contemporaries had been using "action-at-a-distance" integral calculus); the bonus was that differentiating Maxwell's differential equations then produces mathematical wave equations, and mathematical wave equations suggest physical waves. Hertz was the first to experimentally produce wireless electromagnetic waves, nearly twenty-five years after Maxwell's theoretical prediction of their existence.

But Mazur points out that there is a difference between powerful, evocative symbolism and simple notation, the latter being simply a form of shorthand for words used in the rhetorical formulation of mathematical problems. For instance,

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technically speaking the sign "+" comes into the category of notation, because + is shorthand for the letter "t" in the Latin word *et*, which means "and". Nevertheless, the meaning of "+" continued to evolve after Johannes Widmann introduced it in his 1489 book on mercantile calculation, where, according to Mazur, it did not refer to abstract operation of addition, but simply "meant 'excess', as in '+2 is two more than what was expected"' [p. 162]. Consequently, many fifteenth- and sixteenth-century mathematicians favored the notation "p" and "m" for "plus" and "minus", and there were many other signs used for these operations until "+" and "−" became universal in the eighteenth century.

The same was true for all our modern mathematical signs: abbreviations became ever more abbreviated until they became true symbols. So, while "a purist approach would be to distinguish symbolic representation from simple notation," Mazur favors the view that "numerals and all non-literal notation are different, but still considered symbols, for they represent things that they do not resemble" [pp. xi–xii]. With this definition in place, Part I of *Enlightening Symbols* is a brief history of numerals, and Part II is a history of symbolism in algebra. Part III is more speculative, a brief enquiry into the nature of symbolic cognition, including discussion of the similarities and differences between mathematical and other symbols.

Parts I and II each begin with a useful summary of key innovators; a quick glance at these lists shows just how long it took for mathematics to become the elegant and universal language it is today. The list for Part I focuses on the evolution of our Hindu-Arabic numeral system, beginning in the middle of the first millennium with the Bakhshâlî manuscript, followed by the work of Brahmagupta, whose *Brahmasphutasiddhanta* of 628 CE contains the first-known use of the concept of zero as a number. The list finishes with the thirteenth-century Europeans (including Fibonacci), who introduced the Indian numerals to Europe, so it took six centuries to accomplish the transition from India: first to the Middle East, whose Golden Age of translation and intellectual development included the work of the ninth-century Persian mathematician al-Khwarizmi, and then to Europe. It seems the Europeans knew about Indian numerals from the tenth century, but the concept of zero was so novel that it took another three centuries for the Indian numerical system to become fully accepted there: "The difficulty is in distinguishing placeholder from number. Accepting zero as a number representing the absence of quantity would have been a fantastically daring idea" [p. 64].

But Part I covers a much broader period than the rise of the Hindu-Arabic numerals: it includes the ancient number systems of the Babylonians, the Egyptians, Greeks, Hebrews, Chinese, Aztecs, and Mayans. To choose just one of the many interesting facts and points of discussion in this section, I was intrigued by Mazur's question, Why didn't the Greeks, with all their mathematical brilliance, "adopt the genius of the Babylonian system, such as its placeholders and relative ease of writing large numbers? The Babylonians had the right idea of positional notation, the clever idea of using the same digits to represent multiples of different powers of 60" [p. 21]. Alternatively, the ancient Chinese had come up with a clever decimal system that included symbols for powers of ten up to the fourth, so no placeholder symbol was needed. Mazur suggests that perhaps the Greeks used an abacus for calculations (later in Part I, he discusses the evolution of various abaci) or perhaps they were more interested in the "grand scope of mathematics itself" than in calculation. At any rate, it is astonishing to think how long it took before the Indian system—"the smartest system of all"—was developed.

To take another example from this section, Mazur discusses the widespread medieval art of finger-counting, as illustrated, for example, in Luca Pacioli's *Summa de Arithmetica* of 1494, a sample page of which is included as an illustration. But here as elsewhere in the book, Mazur does more than simply give an account of the art and its history: in this case, he also mentions intriguing research that suggests our brains may be hard-wired for counting, in the sense that both counting and finger movement are located in the left parietal lobe.

Part II begins with an anecdote about Mazur's visit to Oxford's Bodleian Library in order to peruse the oldest surviving copy of Euclid's *Elements*. Before being allowed to see this treasure, he had to take an oath to respect the library's property, and then he was asked to sign a special guestbook; to his amazement, he saw that just twelve lines above his own signature was that of Isaac Newton! As for Euclid, Mazur says, "I did not expect and could not find any symbols for addition, multiplication, or equality" [p. 86]. Euclid's work was entirely geometrical and rhetorical. The first known symbolic innovation did not occur until the third century, six hundred years after Euclid, when Diophantus (or his scribes) used abbreviations for unknowns, powers, and subtraction, although they were not the symbols we use today.

Mazur traces the long history of symbolic algebra, including the work of Diophantus, Brahmagupta and al-Khwarizmi, a host of fifteenth-, sixteenth-, and seventeenth-century Europeans (from Pacioli to Descartes to Newton and Leibniz), and finally, in the eighteenth and nineteenth centuries, Euler, William Jones (who introduced the Greek letter pi to denote the ratio of the circumference to the diameter of a circle), Dirichlet, and Hamilton. Along the way, he provides a fascinating tour of algebraic innovations, including the development of new number systems, notably complex numbers and quaternions, which, of course, are a world away from the ordinary numbers that developed from counting and measuring concrete "things." The square roots of negative numbers had been seen as "meaningless" right up to the seventeenth century, when, "with more general notation, more attention was paid to the 'meaningless' than had ever been paid before. So that attention called out the question: What is number?"[p. 148]. Key to this "more general notation" was Viète's decision, in 1591, to use vowels to represent unknowns and consonants to represent knowns and his "magnificent idea that those letters were also to be subject to algebraic reasoning and rules just as much as numbers." [p. 144]. Mazur's enthusiasm makes one thrill to the grand sweep of big ideas that too often we moderns mistake for small ones!

It wasn't until Descartes's *Geometria* of 1637 that almost all our modern algebraic notation was finally in place, so that "on page 69, for the first time, we find a perfectly readable account [of polynomial equations] that almost looks as if it is out of a twentieth-century textbook.…The symbol had finally arrived to liberate algebra from the informality of the word" [p. 156, xvii].

Fifty years later, Newton and Leibniz had systematized calculus, and Leibniz's brilliant notation is the one we use today. Newton's "pricked" letters, such as an x with a dot on top, also survive and are used to denote derivatives with respect to time. Mazur gives a brief but interesting comparison between Newton's and Leibniz's concepts of a derivative and notes the explosion of practical applications that followed in their wake.

He also notes, however, that amidst all this wondrous growth of symbolic mathematics, something was lost: mathematics became more specialized, less accessible to the public. Nowadays, he says, mathematics can seem like Lewis Carroll's nonsense rhyme *Jabberwocky* [p. 179]: "The Jabberwocky is what we get when we first encounter mathematics—or anything—we don't understand." Even applied mathematics "can be done without reference to any physically imaginable object other than a graphic symbol" [p. xviii], and this is why mathematics is more difficult to explain to the public than is, say, physics.

It's interesting to recall, in this context, that Newton famously chose to write *Principia* primarily in the language of geometry rather than that of his new symbolic calculus. And in developing the general theory of general relativity, Einstein, in contrast to Hilbert, chose to eschew a purely abstract mathematical formulation based on Lagrangian and Hamiltonian action principles; instead, he favored "psychologically natural" physical principles [1, p. 118]. Maxwell, too, was wary of banishing all physical content from our understanding of the symbolic equations of dynamics [2, p. 210], although he acknowledged that pure mathematics has given science many ideas whose discovery would not have been possible otherwise. His observation certainly applies to the bizarrely counterintuitive ideas of quantum theory!

To the uninitiated, this careful attention to balancing symbolic and psychological language is lost; even for the initiated, exploring the content of general relativity, for example, can require a lot of manipulation of graphic symbols whose physical content is buried deep within the layers of symbolic scaffolding that underlie the equations. Yet it is this very complexity that enables mathematical theories of nature to be expressed so economically.

To give readers some insight into why the rise of symbols helped mathematics become so abstract and so powerful, Mazur has included Part III, an absorbing, speculative excursion into the nature of thought, including mathematical cognition and the role of symbols in our thought processes [p. 207]. He tells us, for instance, that Jared Danker and John Anderson at Carnegie Mellon found that when subjects were asked to solve simple algebraic equations, "there was a strong interactional relationship between retrieval and representation in mathematical thinking." Research by Anthony Jansen, Kim Marriott, and Greg Yelland of Monash University found, in Mazur's words, "that experienced users of mathematics had an easier time identifying previously seen syntactically well-formed expressions than ill-formed ones. They found that the encoding of algebraic expressions is based primarily on processes that occur beyond the level of visual processing. For example, the well-formed string $7 - x$ is better recalled than ill-formed strings such as $7(x$" [p. 207].

They also found that "we 'read' algebraic expressions by their syntax, just as we do when processing sentences of natural language" [p. 208]. I found this fascinating in view of the fact that girls have long been considered better at language than at mathematics. Actually, current research [3], [4] suggests it is simply spatial ability that separates girls from boys in mathematical achievement and participation. This research also suggests that the spatial-ability stereotype itself could be contributing to the problem [4, p. 8], since there seems to be no genetic or hormonal basis for a gender difference in spatial skill [4, p. 7]. Indeed, it is easy for preconceived stereotypes to become reality: in some communities, and at different times in history, boys have been exposed to more spatially oriented toys and hobbies than girls, and girls have been expected to be inferior to boys in math and physics [see also 5, pp. 10, 19]. So I was intrigued that when describing his own thought processes on viewing the algebraic equation $x^2 - ab = 0$, Mazur commented, "I immediately know that $x = \pm\sqrt{ab}$. But I would also see a square and a rectangle that are aching to be compared" [p. 193]. I was intrigued because this is certainly an extremely simple visualization task, and yet I suspect that those of us who relate better to language than to spatial visualization would not automatically see geometrical interpretations of such equations.

Of course, not all women, or only women, are spatially challenged; in fact Mazur also discusses the earlier idea of "brain type" and quotes [p. 201] Henri Poincaré's words of a century ago (when most university students were male): "Among our students…some prefer to treat their problems 'by analysis', others 'by geometry'. The first are incapable of 'seeing in space'. The others are quickly tired of long calculations and become perplexed."

Fortunately, recent research [3] suggests that spatial skills can be learned; intriguingly, it also suggests [3, p. 369] that although these skills "strongly predict performance early in STEM [science, technology, engineering, and math] learning," it is less important for specialists, who "can rely on a great deal of *semantic* knowledge of the relevant spatial structures without having to perform classic mental spatial tasks[italics added]…." I have italicized the word "semantic" because this conclusion seems to fit nicely with the Monash and Carnegie Mellon research discussed above, and with the fact that a semantic geometrical connection, as well as a visual one, can be made between the square and rectangle implicit in $x^2 - ab = 0$. Mazur notes that because a and b "do not have specific values, the [geometric] exercise can only be one of symbolic manipulation. I would resort to the rules of algebra learned in school…." [p. 193]. He goes on to give a subjective account of the algebraic thought processes involved in solving this little problem—an account that is necessarily subjective, because "we all think somewhat differently with brains that are exquisitely different, using richly assorted thinking styles that contribute to and account for the preciousness of being human" [p. 202].

I have touched only briefly on the content of Part III, and I have added my own digressions, but my point is that *Enlightening Symbols* is not only informative, it can also serve as a springboard for further thought or investigation, depending on the interests of the reader.

Of course, the book can also be read simply for its wide-ranging survey of mathematical history and its enjoyable, sometimes quirky, asides that make it more than just an accessible chronological history of mathematical symbolism. Let me take just one example at random. In 751 CE, during the battle of Talas between the Arabs and the Chinese over Kazakhstan, the Kazakhstani Arabs learned how to make paper from two Chinese prisoners-of-war. The availability of cheap paper helped foster the great period of Arab translation, which preserved ancient Greek texts that had been lost in the West.

At whatever depth one chooses to read it, *Enlightening Symbols* has something for everyone. It is entertaining and eclectic, and Mazur's personal and easy style helps connect us with those who led the long and winding search for the best ways to quantify and analyze our world. Their success has liberated us from "the shackles of our physical impressions of space"—and of the particular and the concrete—"enabling imagination to wander far beyond the tangible world we live in, and into the marvels of generality" [p. 154].

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