Simple but Effective: Van der Waerden's Tick Diagrams

In this issue is a book review by Reinhart Siegmund-Schulze of Alexander Soifer's biography of the renowned Dutch mathematician Bartel van der Waerden. Soifer's biography has aroused much controversy. Van der Waerden's behavior during World War II caused much criticism in his own country after the war and Soifer has brought this criticism to life again.

Van der Waerden is well known for his textbook on what was then called modern algebra, but also for a striking theorem of his youth—the proof of a conjecture attributed at times to either Issai Schur or P. J. H. Baudet:

> *If the set of positive integers is partitioned into two subsets, then arithmetic progressions of arbitrary length can be found in one or the other.*

Van der Waerden's proof appeared in volume 15 of the *Nieuw Archief voor Wiskunde*. It became famous when a slightly different proof appeared as the first pearl in Khinchin's **Three Pearls of Number Theory**.

N. G. de Bruijn has written a helpful note about the conjecture titled *Commentary*. It was written for a book that never appeared, but can be found on the Internet. De Bruijn remarks that although "Van der Waerden's proof is very clear from his paper, yet the notational difficulties seemed to be a bit awkward to the modern combinatorialists." An extremely brief proof of the conjecture presumably intended to get around that difficulty was presented by Ron Graham and B. L. Rothschild in 'A short proof of Van der Waerden's theorem on arithmetic progressions' (volume 42 of the *Proceedings of the AMS*). Some will be amused by the comment at the end of this paper that "… while previous proofs follow essentially the argument above, the one given above is hopefully clearer."

There are no figures in any of these accounts. But Van der Waerden eventually wrote an informal account of how the proof was discovered, which deserves to be ranked with the greatest short expositions of mathematics of all times. This first appeared in English as a chapter in the book **Studies in Pure Mathematics**, edited by L. Mirsky and published in 1971. It was reprinted, with Van der Waerden's permission, in Chapter 33 of Soifer's **The Mathematical Coloring Book**. The book by Soifer reviewed in this issue discusses the conjecture in an extremely short chapter, but the coloring book treats it and its history at great length as well as reproducing Van der Waerden's account *verbatim*.

In this informal account Van der Waerden narrates the events of two days in Göttingen in 1926. The central event was a discussion, after lunch on the second day, by Van der Waerden, Emil Artin, and Otto Schreier in Artin's office, in the course of which the entire proof was found. Artin and Schreier made important initial observations that reduced the conjecture to something manageable, but it was apparently Van der Waerden who used little diagrams on a blackboard to find the path from the reduction.

The cover of this issue of *The Notices* illustrates the figures for the very first step in his eventual proof. These and similar diagrams turned out to be the key; they make the proof transparent.

The early part of this two-day discussion led to a stronger formulation of the conjecture:

> *Suppose k, ℓ to be integers ≥ 2, and suppose the positive integers are partitioned into k subsets. There exists $n(k, \ell)$ such that any interval of $n(k, \ell)$ positive integers possesses an arithmetic progression of length ℓ in one of the subsets.*

The case $k = 2, \ell = 2$ is true by the pigeonhole principle. For $k = 2, \ell = 3$ a listing of all possibilities shows that one can take $n(2, 3) = 9$, but such a procedure is impossible for all other cases, and this argument gives no hint at all of how to proceed. The diagrams on the cover, on the other hand, do lead to a proof in this simple case that turns out to carry through very generally. Van der Waerden's use of his tick figures is one of the classic examples of schematic diagrams in mathematical proofs.

Unfortunately, how this goes is a little too complicated even to suggest here.

The numbers $n(k, \ell)$ grow astronomically with k and ℓ, very roughly the order of towers of exponentials. Soifer's **Coloring Book** discusses subsequent work on the true order of magnitude, as well as other later generalizations of Van der Waerden's result—for example, the well-known results of Szemeredi and Furstenberg.

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