Book Review

Mathematics of the Transcendental

Reviewed by Andrej Bauer

Mathematics of the Transcendental
Alain Badiou (translated into English by A. J. Bartlett and Alex Ling)
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When a *Notices* editor asked me to review Badiou's book [2] I objected on the grounds that I am no philosopher, which only strengthened her determination. Here then is a mathematician's review of a philosopher's mathematics book.

Alain Badiou (born 1937) is a prominent French philosopher whose work may be placed somewhere between the continental and analytic traditions, although closer to the former. He has been active outside philosophy, in literature and especially in politics as a proponent of the radical left. Out of his philosophical considerations of "the multiple" came the idea that set theory was "the pure doctrine of the multiple" and that mathematics *was* ontology.

Set theory was Badiou's first excursion into mathematics, in which he related the standard axioms of set theory to his philosophy in a precise way that left some impressed and others incredulous. In his second undertaking, category theory and topos theory appeared at first as alternatives to set theory and later complemented it to make a bigger picture. The present book is an English translation by A. J. Bartlett and Alex Ling of two sets of Badiou's unpublished French notes on category theory, toposes, and logic. The two parts of the book are titled "Topos, or Logics of "Onto-logy: An Introduction for Philosophers" (the dash in "onto-logy" serves a purpose) and "Being There: Mathematics of the Transcendental." While one can guess that the first part deals with toposes and logic, the second title gives no indication that it is about complete Heyting algebras, of which I shall say more shortly.

To a mathematician like me, reading the book feels quite unusual. The mathematics is precise and correct—I point this out because Badiou's mathematical skill had been called into question by his critics—but written in an idiosyncratic way and intertwined with philosophy. Quite intentionally, there is no clear separation between the philosophical and the mathematical parts of the book, as neither is supposed to be above, under, or beside the other. To give you a feeling for what the book is like, here is how the least element of a partial order is explained (p. 173):

> Maintaining the supposition that, with regard to what appears in the situation, the transcendental T supports evaluation of intensity, it is reasonable to assume the capacity to determine a *nil intensity*.

You will never hear a professional mathematician speak like this, which precisely is the point! Where mathematics abstracts away the nonessential and keeps a narrow focus, philosophy seeks breadth and wider context at the expense of clarity and definiteness. If a mathematician can bear this fact in mind while reading the book, she or he may catch a glimpse of philosophy. I could understand many a philosophical passage only because I already knew the mathematics it referred to.

The first part explains basic notions of category theory: category, limits, opposite categories and colimits, Cartesian closed categories, and subobject classifiers (called "central objects"). These culminate in the notion of a topos, which is of

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central interest, as its rich structure allows us to interpret higher-order intuitionistic logic. Two examples of toposes are given: a Boolean one that models classical logic and a non-Boolean one that is properly intuitionistic. There are no functors and consequently no natural transformations or adjunctions, while toposes are approached entirely from their logical side. This is probably as much as one could expect from philosophical notes on category theory, but I wonder to what degree the choice of topics is fortuitous. Could Lawvere's functorial semantics serve the philosophical considerations equally well or better? In several places it would help to have presheaves and the Yoneda lemma at hand, for instance to bolster the claim that "every determination is external (by arrows or relations)" (p. 56) and to give more substance to the example of a non-Boolean topos, which is just presheaves on an arrow. While set theory is ontology for Badiou, category theory is "the space of possible logics" (p. 57) and "a description of the possible options for thought, which does not constitute by itself such an option" (p. 161). Badiou understands that set theory and category theory play essentially different foundational roles, a point that seems to elude many investigators of foundations of mathematics.

The second part begins with an introduction to complete Heyting algebras, which we learn to be complete lattices with finite meets distributing over arbitrary joins. In mathematics, these are known as frames or locales, depending on what role they are given: as frames they serve as domains of truth values for intuitionistic predicate calculus, and as locales they embody the topological notion of a (possibly point-free) space built just from abstract open neighborhoods. Badiou's interests lie in logic, so let me call them frames, although he provides a wealth of examples by noting that the topology of a space is always a complete Heyting algebra. The fact that the book calls a frame "a transcendental" is indicative of its philosophical role: "that which, in any situation, serves as a domain for the evaluation of identities and differences in appearing" (p. 167). To put it less eloquently, the elements of a frame are used to express degrees of equality between objects and degrees of truth in general. Thus the book supplements the traditional sets with frame-valued identity relations according to which elements are equal to a certain degree, not just completely equal or completely unequal. In the same way, the existence of an element is a measured quantity and is just the degree to which the element equals itself. Complete Boolean algebras are seen to be a special case of frames, and they serve as a bridge to classical logic and classical set theory. We can never be satisfied with a single frame because different situations call for different scales of measurement. Just as in the first part of the book Badiou says nothing about functors between categories, here we learn nothing about morphisms between frames. He would need them had he felt the need to relate and systematically compare the different scales of measurement.

Badiou's notes paint a peculiar picture of category theory in which categories, toposes, and complete Heyting algebras stand isolated from each other to form a plurality of structures. Any student, whether of mathematics or philosophy, should supplement this view with introductory texts, maybe those of Awodey [1] or Lawvere and Schanuel [3], to see that functors, natural transformations, and adjunctions connect categories in rich ways. I suspect that Badiou could put the connections to good philosophical use, but cannot speculate why he has not done so.

I can hardly judge Badiou's philosophical interpretations of mathematics, although I am surprised at how tightly Badiou links his philosophy with the specific mathematical structures. Is it really necessary for Badiou's philosophy to use precisely toposes and not some other kind of categories? It looks like most of the philosophical analogies would still hold in a more general setting, maybe that of hyperdoctrines or other gadgets one finds in categorical logic. It could even be argued that Tarskian model theory could satisfy most, if not all, the philosophical needs. After all, it enjoys Gödel's completeness theorem. Similarly, why should the study of the "transcendental" be limited to frames? If Badiou adopted the treatment of quantifiers as adjoints rather than infima and suprema, then (external) completeness would not be needed anymore and a wealth of new examples would be at hand. Is generality not appreciated by philosophy? Anyhow, I shall not criticize a philosopher for not knowing everything when he expended an amazing amount of energy to build not one, but two bridges from his land to mine. I am impressed by the lucidity of Badiou's remarks on the philosophical significance of category theory, especially in relation to set theory, and I invite philosophically minded mathematicians to be so too.

References

1. Steven Awodey, *Category Theory*, Oxford Logic Guides, Oxford University Press, 2010.
2. Alain Badiou, *Mathematics of the Transcendental*, translated by A. J. Bartlett and Alex Ling, Bloomsbury Academic, 2014.
3. Francis William Lawvere and Stephen H. Schanuel, *Conceptual Mathematics: A First Introduction to Categories*, Cambridge University Press, Cambridge, New York, 1997.