

# *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns*

*Reviewed by James S. Walker*

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**Creating Symmetry:  
The Artful Mathematics of Wallpaper Patterns**

*Frank A. Farris*

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This new book by Frank A. Farris, professor of mathematics at Santa Clara University, is a comprehensive introduction to the mathematics of symmetry. Symmetry has long provided a connection between mathematics and the visual arts. This book distinguishes itself from other treatments of the subject (e.g., [9], [5], and [1]) by its detailed descriptions of exactly how one creates new artistic designs. It doesn't just analyze existing patterns but provides mathematical formulas that allow you to create your own designs, exhibiting a wide variety of different types of symmetries, including not only wallpaper patterns (patterns with two independent plane translational symmetries) but several other designs as well. It is filled with many beautiful images. The publisher also deserves commendation for printing the book on photo-quality glossy paper, with high-resolution color images, and at a modest price. Farris writes in a style that invites the reader to participate in the artistic process. In fact, I was so intrigued by his approach

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that I tried my hand at creating my own designs, some of which are shown in this review. I believe many readers will also become involved in this way.

The book provides a unified development of fundamental ideas from group theory, Fourier series, complex variables, linear algebra, and geometry. It shows how all of these ideas can be brought to bear in understanding and creating symmetric planar designs. It is most suitable for one of three audiences: (1) undergraduate mathematics majors studying it as a "capstone" experience or an Independent Study. This audience might benefit from having an actively involved instructor to guide their study, someone to help them over typographical errors and one incorrect mathematical argument that I will describe later. (2) Mathematics professors looking for ideas to supplement one or more of their courses in any of the fields mentioned above or who are looking for ideas for undergraduate research projects. (3) People who the author refers to as "brave mathematical adventurers," who are studying this material on their own.

Rather than try to summarize all of the manifold topics dealt with in the book, I will concentrate on three topics that are representative of the overall content. These topics are (1) color maps of complex functions, (2) creating rosette images, and (3) creating symmetric wallpaper patterns.

## **Color Maps of Complex Functions**

At the very end of the twentieth century, Farris [3] played a principal role in developing the use of color maps to sketch graphs of functions,  $f: \mathbb{C} \rightarrow \mathbb{C}$ .

The idea is to use a *color wheel*, a well-known tool in the visual arts. An example of a color wheel is shown at the top left of Figure 1. This color wheel is used to mark locations in the complex plane. For each value of  $w$ , the value of  $C(w)$  is a unique color (at least in principle). For example, on the top left of Figure 1, the values of  $w$  that are near  $i$  are colored greenish-yellow, while values near  $-1$  have a light blue tint. As values approach zero they turn black, and beyond a certain radius they are all colored white. By composing a function  $f(z)$  with  $C$ , we get a function  $C(f(z))$  that gives a color portrait of  $f(z)$ . For instance, on the top right of Figure 1, a color map for  $w = z^2$  is shown. Notice how the colors near  $w = 0$  cycle twice through the rainbow as we move once around  $w = 0$ . The connection to winding numbers is thereby made visually evident. Also, contour lines meet at right angles away from  $w = 0$ , just as they do for  $w = z$ . This illustrates conformality of  $w = z^2$  away from the origin.

The preceding color plots were created by me. The ones created by Farris are similar—although he marks points near zero as white and exceedingly large values as black—and are shown in his book and at the webpage [3]. There is software now that produces color plots with great ease. At the bottom of Figure 1, we show a color plot of the function  $w = 3(z+1)(z-i)^2(z-1+i)^3$ . This plot was produced with the free SageMath system [6], [7]. I needed just two commands:

```
3*(z+1)*(z-i)^2*(z-1+i)^3
complex_plot(f, (-2, 2), (-2, 2))
```

The plot that SageMath produced clearly marks the location of the zeros at  $-1$ ,  $i$ , and  $1-i$  and their multiplicities of 1, 2, and 3, respectively. All of that information is encoded in the number of times the colors of the rainbow are cycled through in the neighborhood of each zero. There are several nice examples of color plots at the website [2], including plots of branching in Riemann surfaces.

### Creating Rosette Images

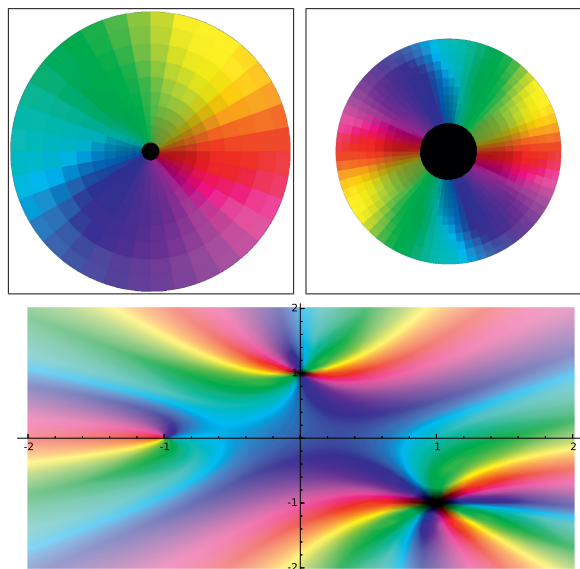
The first truly artistic images in the book occur when Farris creates rosette images. Instead of using a color wheel for the color map  $C$ , Farris decides to use “The World as my Color Wheel.” In other words, he uses for the color map  $C$  any of a series of color photographs that he has taken over the years. I will now outline his method for creating the rosette image shown on the left of Figure 2.

Farris uses the following basic fact (Theorem 4, p. 42):

*If, in the [convergent] sum*

$$f(z) = \sum a_{nm} z^n \bar{z}^m,$$

*we have  $a_{nm} = 0$  unless  $n \equiv m \pmod{p}$ ,*



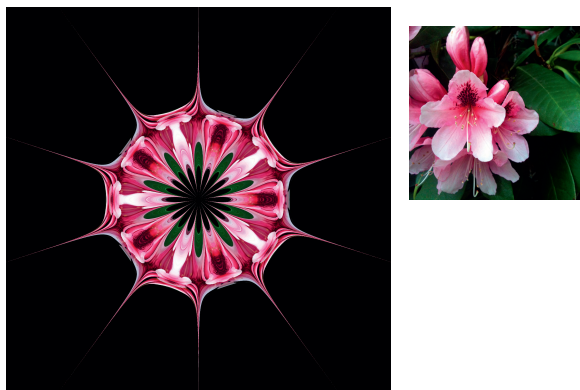
**Figure 1. Top left: Color plot for  $w = z$ , a basic color map  $C$ . Both horizontal and vertical axes span the interval  $[-2, 2]$ . Top right: Color plot for  $w = z^2$ . Bottom: SageMath color plot for  $w = 3(z+1)(z-i)^2(z-1+i)^3$ . Created by the reviewer.**

*then  $f$  is invariant under rotation through an angle of  $2\pi/p$ . In other words,  $f$  is a rosette function with  $p$ -fold symmetry.*

Here we see rotational symmetry of these functions  $f$  corresponding to number-theoretic symmetry for their coefficients. Of course,  $\mathbb{Z}_p$  is isomorphic to the rotational group about the origin of  $\mathbb{C}$  generated by  $e^{2\pi i/p}$ , as Farris points out. Farris then shows that additional symmetries can be created by invoking further symmetries on the coefficients  $a_{nm}$ . For example, for the functions  $f(z) = \sum a_{nm} z^n \bar{z}^m$  to have mirror symmetry  $\sigma_x$  about the  $x$ -axis, Farris symmetrizes using group averaging based on the symmetry group generated by  $\sigma_x$ . The group average function  $(f(z) + f(\sigma_x z))/2$  is guaranteed to have the required mirror symmetry. The power series for  $f$  then leads to the following requirement—or “recipe,” as Farris calls it—that the coefficients for  $f$  should satisfy  $a_{nm} = a_{mn}$  (since they do so for the symmetrized group average function). He has then prepared the way for stating what function  $f$  he used to create the rosette image shown in Figure 2. It is a function having the form

$$f(z) = z^5 \bar{z}^0 + z^0 \bar{z}^5 + a(z^6 \bar{z}^1 + z^1 \bar{z}^6) + b(z^4 \bar{z}^{-6} + z^{-6} \bar{z}^4)$$

for any complex constants  $a$  and  $b$ . Farris does not tell us the specific values of  $a$  and  $b$  he chose, as his aim (quite rightly) is to teach us how to create our own designs, not reproduce his. By construction, such a function  $f(z)$  will be



**Figure 2. Left: Rosette image with 5-fold rotational and mirror symmetries. Right: Photo of rhododendron used to define its color map. (From p. 45, used by permission of Princeton University Press.)**

invariant under 5-fold rotations about the origin and have mirror symmetry about the  $x$ -axis. The color mapped display,  $C(f(z))$ , then inherits all symmetries enjoyed by  $f(z)$ . Farris is careful to point out that group operations then force both  $f(z)$  and its display  $C(f(z))$  to have many more symmetries than the ones singled out in the initial design. He discusses how the symmetry group for the rosette is the dihedral group  $D_5$ , and he contrasts that group with the cyclic group  $C_5$ , isomorphic to the integer powers of  $e^{i2\pi/5}$ , which is a normal subgroup of  $D_5$ . As we can see by comparing the color map image of the rhododendron with the rosette image, none of the symmetry of the rosette comes from the color map; it all comes from the construction of symmetry enjoyed by the function  $f$ . For reasons of space, I will not show any more of the color map images  $C$ , only the designs  $C \circ f$ .

I found the rosette display that Farris has created to be just gorgeous! The method outlined above is typical of his approach throughout the book, introducing mathematical ideas from complex analysis, Fourier analysis, group theory, and geometry in the concrete setting of creating visually stunning symmetric designs.

### Creating Wallpaper Patterns

As lovely as the rosette figures are, they are just a prelude to the topic he devotes the most space to: creating color images with wallpaper symmetries. The methods he used for rosettes are now deepened and extended to handle these designs, which have infinite symmetry groups. As an example of his techniques, I shall outline his method for constructing functions possessing 4-fold rotational symmetry about the origin along with translational symmetries of period 1 along the  $x$  and  $y$  axes (square lattice symmetry), while

also including additional symmetries if desired. After that, I will show some additional examples of designs for other wallpaper groups that he discusses.

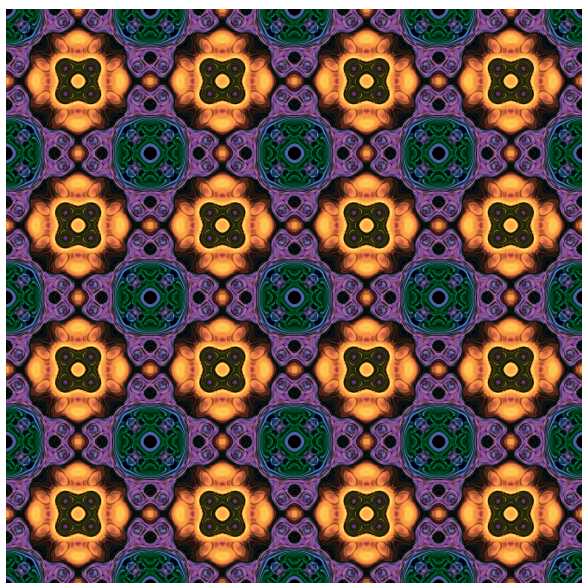
To create functions with square lattice symmetry and 4-fold rotational symmetry, Farris proceeds as follows. First, he uses as a basis the set of complex exponentials  $\{E_{n,m}(z) = e^{2\pi i(nx+my)}\}$  for all  $n$  and  $m$  in  $\mathbb{Z}$  and  $z = x + iy \in \mathbb{C}$ . Any finite (or convergent) sum  $\sum a_{n,m}E_{n,m}(z)$  is guaranteed to have the required translational symmetry. To obtain the rotational symmetry, he again employs group averaging. In this case, the group comprises the rotations generated by  $\omega_4 = e^{2\pi i/4}$ . The group average  $W_{n,m}$  of  $E_{n,m}$  is defined by  $W_{n,m}(z) = \frac{1}{4} \sum_{k=0}^3 E_{n,m}(\omega_4^k z)$ . Farris refers to all of these group averages,  $\{W_{n,m}(z)\}$ , as *wave packets*. Any finite, or convergent, linear combination  $f(z) = \sum a_{n,m}W_{n,m}(z)$  is guaranteed to have both square lattice symmetry and 4-fold rotational symmetry about the origin. By choosing various coefficients  $a_{n,m}$  and composing with various color maps  $C$ , we obtain an infinite variety of designs, all with the same symmetry group. Farris uses the notation of crystallography to refer to this symmetry group; it is  $p4$ . I like the fact that he uses crystallographic notation. Although it has weaknesses, which Farris points out, it makes his book more accessible to a wider audience.

To add additional symmetry, as with the rosette construction, we enforce symmetries on the coefficients  $a_{n,m}$  corresponding to the symmetry we desire for the functions  $f$ . For example, to obtain mirror symmetry about the line  $y = x$  we require that the coefficients satisfy  $a_{n,m} = a_{m,n}$ . In Figure 3, I show an image I constructed using this recipe. Notice that there are additional symmetries in this image other than the ones intentionally created, such as mirror symmetries about the  $x$  and  $y$  axes. Farris is careful to show that these symmetries all arise from the group properties of the symmetry group,  $p4m$ , for this design. He also provides a helpful *fundamental cell diagram* that gives a concise geometric picture of all the symmetries of the design. He does this for each of the symmetry groups he discusses.

I cannot adequately convey the joy I felt when this image popped onto my computer after about a week of preparation of the computer code [8]. All of the mathematics actually works and creates, in my opinion, a lovely image. This is praise for Farris; I was only following his detailed descriptions of what to do. I strongly encourage you to get his book and try his methods for yourself.

By way of comparison with the image I created, I show in Figure 4 a design by Farris that exhibits 4-fold symmetry as well as *glide reflection* symmetry along the line  $y = x - 1/2$ . This image was created by a different recipe involving the coefficients  $a_{n,m}$ . What's the recipe? Well, I encourage you to read





**Figure 3. Design with p4m symmetry. Created by the reviewer.**



**Figure 4. 4-fold rotational symmetry with glide reflection symmetry, type p4g. (From p. 101, used by permission of Princeton University Press.)**

the book to find out. Farris is again careful to point out that the symmetry group operations create multiple glide reflection axes in addition to  $y = x - 1/2$ . These glide reflection axes are not hard to spot, as they pass through barbell shaped figures within the image. Farris says that “undulating changes in direction and the mixture of mirror rigidity with free waviness make p4g one of my favorite pattern types.” This made sense to me, as it explains why the image appeared to vibrate when I looked for the location of the unit cell.

For another example that seems to vibrate, Figure 5 shows an image I created with 6-fold

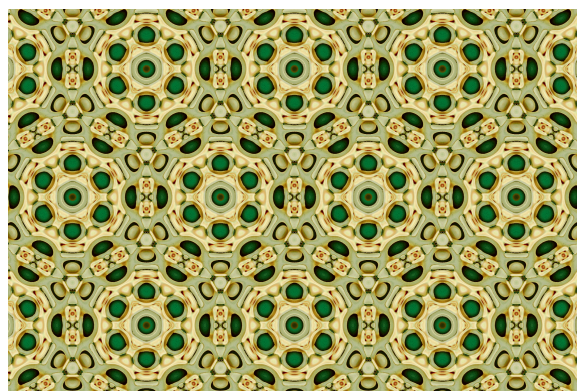
rotational symmetry and mirror symmetry, a p6m design. By similar methods to those outlined above for 4-fold symmetry, Farris creates designs with either 3-fold or 6-fold rotational symmetry. An interesting feature of the 6-fold case is that there will be centers of 3-fold rotational symmetry within unit cells. For example, in Figure 5 we can see these centers by locating points where three propeller-like arms extend outwards in a 3-fold symmetric pattern. This phenomenon is even clearer in the p6m design created by Farris, shown in Figure 6. Farris points out these 3-fold rotational centers and gives a concise explanation, via group operations again, for their existence in the 6-fold rotationally symmetric design. Another symmetry that is forced upon the design is the horizontal glide reflection axis that one can see below the row of flower-like hexagons in Figure 5 and between the rows of carrot-colored hexagon objects in Figure 6. As Farris demonstrates, there are multiple glide reflection axes in addition to the horizontal one.

I have highlighted here just a few of the many intriguing mathematical topics Farris covers in his book. For more details, go through the link to the book’s website at [4].

### **A Near Masterpiece**

While the book is a near masterpiece, it does have quite a lot of typos. These pesky errors could lead readers to question the book’s validity, which would be a shame, because the book is mathematically sound. There is unfortunately one exception to this soundness: The proof of convergence of Fourier series for continuously differentiable periodic functions is incorrect. It is correctable, but probably not by students. Fortunately, this material is in an optional section and is not used in the rest of the book. Farris told me he plans to post errata on his website [4].

The book neither includes nor provides links to computer code. Some readers might consider this a weakness. I do not. I agree with Farris



**Figure 5. A design with p6m symmetry. Created by the reviewer.**



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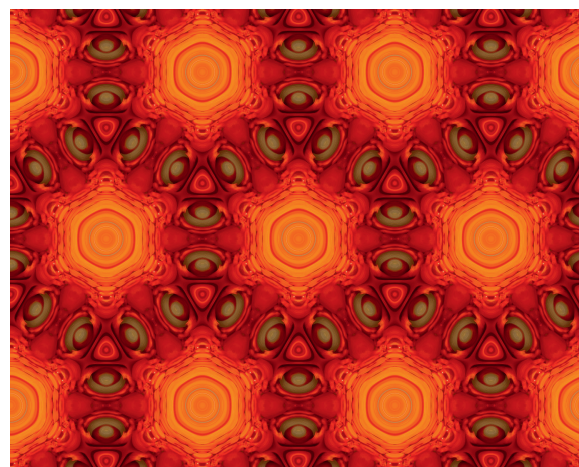


Figure 6. A design with  $p6m$  symmetry. Created by Farris. (From p. 79, used by permission of Princeton University Press.)

that there are ample resources (software such as SageMath, Maple, Mathematica, or Matlab and programming languages such as C++) and help for image graphing on the Internet. Farris explains carefully what to do with this software, i.e., the mathematics needed. In the not-too-distant future it is unlikely that any of those computer programs will exist in anything like their present form. But the mathematics described in this book surely will remain vital for centuries to come.

This book thoroughly engaged me with its application of fundamental and important mathematics to produce striking artistic designs. It is a major contribution that aids in our understanding of symmetry, art, and how mathematics unifies them. I recommend it most highly.

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