The Legacy of Józef Marcinkiewicz: Four Hallmarks of Genius
In Memoriam of an Extraordinary Analyst

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This article is a tribute to one of the most prominent Polish mathematicians, Józef Marcinkiewicz, who perished eighty years ago in the Katyń massacre. He was one of nearly 22,000 Polish officers interned by the Red Army in September 1939 and executed in April–May 1940 in the Katyń forest near Smolensk and at several locations elsewhere. One of these places was Kharkov (Ukraine), where more than 3,800 Polish prisoners of war from the Starobelsk camp were executed. One of them was Marcinkiewicz; the plaque with his name (see Figure 1) is on the Memorial Wall at the Polish War Cemetery in Kharkov.[1] This industrial execution was authorized by Stalin's secret order dated 5 March 1940 and organized by Beria, who headed the People's Commissariat for Internal Affairs (the interior ministry of the Soviet Union) known as NKVD.

Turning to the personality and mathematical achievements of Marcinkiewicz, it is appropriate to cite the article [24] of his supervisor Antoni Zygmund (it is published in the *Collected Papers* [13] of Marcinkiewicz; see p. 1):

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[1]https://www.tracesofwar.com/sights/10355/Polish-War-Cemetery-Kharkiv.htm.

Considering what he did during his short life and what he might have done in normal circumstances one may view his early death as a great blow to Polish Mathematics, and probably its heaviest individual loss during the second world war.

From the Marcinkiewicz Biography [9]

On the occasion of the centenary of Marcinkiewicz's birth, a conference was held on 28 June–2 July 2010 in Poznań. In its proceedings, L. Maligranda published the detailed article [9] about Marcinkiewicz's life and mathematical results; sixteen pages of this paper are devoted to his biography, where one finds the following about his education and scientific career.

Education. Klemens Marcinkiewicz, Józef's father, was a farmer well-to-do enough to afford private lessons for him at home (the reason was Józef's poor health) before sending him to elementary school and then to gymnasium in Białystok. After graduating in 1930, Józef enrolled in the Department of Mathematics and Natural Science of the Stefan Batory University (USB) in Wilno (then in Poland, now Vilnius in Lithuania).

From the beginning of his university studies, Józef demonstrated exceptional mathematical talent that attracted the attention of his professors, in particular, of A. Zygmund. Being just a second-year student, Marcinkiewicz attended his lectures on orthogonal series, requiring some erudition, in particular, knowledge of the Lebesgue integral; this was the point where their collaboration began. The first paper of Marcinkiewicz (see [13, p. 35])

Figure 1. Plaque for Marcinkiewicz on the Memorial Wall at the Polish War Cemetery in Kharkov.

was published when he was still an undergraduate student. It provides a half-page proof of Kolmogorov's theorem (1924) guaranteeing the convergence almost everywhere for partial sums of lacunary Fourier series. Marcinkiewicz completed his MSc and PhD theses (both supervised by Zygmund) in 1933 and 1935, respectively; to obtain his PhD degree he also passed a rather stiff examination. The second dissertation was the fourth of his almost five dozen publications; it concerns interpolation by means of trigonometric polynomials and contains interesting results (see [24, p. 17] for a discussion), but a long publication history awaited this work. Part of it was published in the *Studia Mathematica* the next year after the thesis defense (these two papers in French are reproduced in [13, pp. 171–185 and 186–199]). The full, original text in Polish appeared in the *Wiadomości Matematyczne* (the *Mathematical News*) in 1939. Finally, its English translation was included in [13, pp. 45–70].

Scientific career. During the two years between defending his MSc and PhD theses, Marcinkiewicz did the one year of mandatory military service and then was Zygmund's assistant at USB. The academic year 1935–1936 Marcinkiewicz spent as an assistant at the Jan Kazimierz University in Lwów. Despite twelve hours of teaching weekly, he was an active participant in mathematical discussions at the famous Scottish Café (see [3, ch. 10], where this unique form of doing mathematics is described), and his contribution

to the *Scottish Book* compiled in this café was substantial, taking into account that his stay in Lwów lasted only nine months. One finds the history of this book in [14, ch. I], whereas problems and their solutions, where applicable, are presented in ch. II. Marcinkiewicz posed his own problem; it concerns the uniqueness of the solution for the integral equation

$$\int_0^1 y(t)f(x-t)\,dt = 0, \qquad x \in [0,1].$$

He conjectured that if $f(0) \neq 0$ and f is continuous, then this equation has only the trivial solution $y \equiv 0$ (see problem no. 124 in [14, pp. 211 and 212]). He also solved three problems; his negative answers to problems 83 and 106 posed by H. Auerbach and S. Banach, respectively, involve ingenious counterexamples. His positive solution of problem 131 (it was formulated by Zygmund in a lecture given in Lwów in the early 1930s) was published in 1938; see [13, pp. 413–417].

During the next two academic years, Marcinkiewicz was a senior assistant at USB and after completing his habilitation in June 1937 became the youngest docent at USB. The same year, he was awarded the Józef Piłsudski Scientific Prize (the highest Polish distinction for achievements in science at that time). His last academic year 1938–1939, Marcinkiewicz was on leave from USB; a scholarship from the Polish Fund for National Culture afforded him opportunity to travel. He spent October 1938–March 1939 in Paris and moved to the University College London for April–August 1939, also visiting Cambridge and Oxford.

This period was very successful for Marcinkiewicz; he published several brief notes in the *Comptes rendus de l'Académie des Sciences Paris*. One of these, namely [12], became widely cited because the celebrated theorem concerning interpolation of operators was announced in it. Now this theorem is referred to as the Marcinkiewicz or Marcinkiewicz–Zygmund interpolation theorem (see below). Moreover, an important notion was introduced in the same note: the so-called weak-L^p spaces, known as Marcinkiewicz spaces now, are essential for the general form of this theorem.

Meanwhile, Marcinkiewicz was appointed to the position of Extraordinary Professor at the University of Poznań in June 1939. On his way to Paris, he delivered a lecture there and this, probably, was related to this impending appointment. Also, this was the reason to decline an offer of professorship in the USA during his stay in Paris.

Marcinkiewicz still was in England when the general mobilization was announced in Poland in the second half of August 1939; the outbreak of war became imminent. His colleagues advised him to stay in England, but his ill-fated decision was to go back to Poland. He regarded

Figure 2. Józef Marcinkiewicz.

himself as a patriot of his homeland, which is easily explainable by the fact that he was just eight years old (very sensitive age in forming a personality) when the independence of Poland was restored.

Contribution of Marcinkiewicz to Mathematics

Marcinkiewicz was a prolific author, as demonstrated by the almost five dozen papers he wrote in just seven years (1933–1939); see *Collected Papers* [13, pp. 31–33]. He was open to collaboration; indeed, more than one third of his papers (nineteen, to be exact) were written with five coauthors, of which the lion's share belongs to his supervisor Zygmund.

Marcinkiewicz is known, primarily, as an outstanding analyst, whose best results deal with various aspects of real analysis, in particular, theory of series (trigonometric and others), inequalities, and approximation theory. He also published several papers concerning complex and functional analysis and probability theory. In the extensive paper [9] dedicated to the centenary of Marcinkiewicz's birth, one finds a detailed survey of all his results.

This survey begins with the description of five topics concerning functional analysis ([9, pp. 153–175]). No doubt, the first two of them—the Marcinkiewicz interpolation theorem and Marcinkiewicz spaces—are hallmarks

of genius. One indication of the ingenuity of the idea behind these results is that the note [11], in which they first appeared, is the most cited work of Marcinkiewicz.

Another important point about his work is that he skillfully applied methods of real analysis to questions bordering with complex analysis. A brilliant example of this mastery—one more hallmark of genius—is the Marcinkiewicz function μ introduced as an analogue of the Littlewood–Paley function g. It is worth mentioning that the short paper [10], in which μ first appeared, contains other fruitful ideas developed by many mathematicians subsequently.

One more hallmark of genius one finds in the paper [11] entitled "Sur les multiplicateurs des séries de Fourier." There are many generalizations of its results because of their important applications. This work was the last of eight papers that Marcinkiewicz published in the *Studia Mathematica*; the first three he submitted during his stay in Lwów, and they appeared in 1936.

Below, the above-mentioned results of Marcinkiewicz are outlined in their historical context together with some further developments. One can find a detailed presentation of all these results in the excellent textbook [18] based on lectures of the eminent analyst Elias Stein, who made a considerable contribution to further development of ideas proposed by Marcinkiewicz.

Marcinkiewicz Interpolation Theorem and Marcinkiewicz Spaces

There are two pillars of the interpolation theory: the classical Riesz–Thorin and Marcinkiewicz theorems. Each of these serves as the basis for two essentially different approaches to interpolation of operators known as the complex and real methods. The term "interpolation of operators" was, presumably, coined by Marcinkiewicz in 1939, because Riesz and Thorin, who published their results in 1926 and 1938, respectively, referred to their assertions as "convexity theorems."

It is worth emphasizing again that a characteristic feature of Marcinkiewicz's work was applying real methods to problems that other authors treated with the help of complex analysis. It was mentioned above that in his paper [10] published in 1938, Marcinkiewicz introduced the function μ without using complex variables but so that it is analogous to the Littlewood–Paley function g, whose definition involves these variables. In the same year, 1938, Thorin published his extension of the Riesz convexity theorem, which exemplifies the approach based on complex variables. Possibly this stimulated Marcinkiewicz to seek an analogous result with proof relying on real analysis. Anyway, Marcinkiewicz found his interpolation theorem and announced it in [12]; concurrently, a letter was sent to Zygmund that contained the proof concerning a

particular case. Ten years after World War II, Zygmund reconstructed the general proof and published it in 1956; for this reason the theorem is sometimes referred to as the Marcinkiewicz–Zygmund interpolation theorem.

An excellent introduction to the interpolation theory one finds in the book [1] based on the works of Jaak Peetre (he passed away on 1 April 2019 at age eighty-three), whose contribution to this theory cannot be overestimated. In collaboration with Jacques-Louis Lions, he introduced the "real method interpolation spaces" (see their fundamental article [8]), which can be considered as "descendants" of the Marcinkiewicz interpolation theorem.

An important fact of Peetre's biography is that his life was severely changed during World War II (another reminder about that terrible time). With his parents, Jaak escaped from Estonia in September 1944 just two days before his home town of Pärnu was destroyed in an air raid of the Red Army. He was only ten years old when his family settled in Lund (Sweden), where he spent most of his life. But let us turn to mathematics again.

The Marcinkiewicz interpolation theorem for operators in $L^p(\mathbb{R}^n)$.

We begin with this simple result because it has numerous applications, being valid for subadditive operators mapping the Lebesgue spaces $L^p(\mathbb{R}^n)$ with $p \geq 1$ into themselves (see, e.g., [18, ch. 1, sect. 4]). We recall that an operator $T : L^p \to L^p$ is subadditive if

$$|T(f_1 + f_2)(x)| = |T(f_1)(x)| + |T(f_2)(x)| \quad \text{for every } f_1, f_2.$$

Furthermore, T is of *weak type* (r, r) if the inequality

$$\alpha^r \text{mes}\{x : |T(f)(x)| > \alpha\} \leq A_r \|f\|_r^r$$

holds for all $\alpha > 0$ and all $f \in L^r$ with A_r independent of α and f. Here, mes{...} denotes the Lebesgue measure of the corresponding set, and

$$\|f\|_p = \left[\int_{\mathbb{R}^n} |f(x)|^p \, dx \right]^{1/p}$$

is the norm in $L^p(\mathbb{R}^n)$. Now, we are in a position to formulate the following.

Theorem 1. *Let $1 \leq r_1 < r_2 < \infty$, and let T be a subadditive operator acting simultaneously in $L^{r_i}(\mathbb{R}^n)$, $i = 1, 2$. If it is of weak type (r_i, r_i) for $i = 1, 2$, then for every $p \in (r_1, r_2)$ the inequality $\|T(f)\|_p \leq B \|f\|_p$ holds for all $f \in L^p(\mathbb{R}^n)$ with B depending only on A_{r_1}, A_{r_2}, r_1, r_2, and p.*

When B is independent of f in the last inequality, the operator T is of *strong type* (p, p); it is clear that T is also of weak type (p, p) in this case.

In the letter to Zygmund mentioned above, Marcinkiewicz included a proof of this theorem for the case $r_1 = 1$ and $r_2 = 2$. Presumably, it was rather simple; indeed, even when $r_2 < \infty$ is arbitrary, the proof is less than two pages long in [18, ch. 1, sect. 4].

Marcinkiewicz spaces. Another crucial step, made by Marcinkiewicz in [12], was the introduction of the *weak L^p* spaces playing the essential role in his general interpolation theorem. They are now called the *Marcinkiewicz spaces* and usually denoted $L^{p,\infty}$.

To give an idea of these spaces, let us consider a measure space (U, Σ, m) over real scalars with a nonnegative measure m (just to be specific). For a real-valued f, which is finite almost everywhere and m-measurable, we introduce its distribution function

$$m(\{x : |f(x)| > \lambda\}), \qquad \lambda \in (0, \infty)$$

and put

$$|f|_{p,\infty} = \sup_{\lambda>0} \lambda [m(\{x : |f(x)| > \lambda\})]^{1/p} \quad \text{for } p \in [1, \infty).$$

Then $L^{p,\infty} = \{f : |f|_{p,\infty} < \infty\}$, and it is clear that $L^p \subset L^{p,\infty}$ for $p \in [1, \infty)$, because $|f|_{p,\infty} \leq \|f\|_p$ in view of Chebyshev's inequality. The Marcinkiewicz space for $p = \infty$ is L^∞ by definition.

It occurs that $|f|_{p,\infty}$ is not a norm for $p \in [1, \infty)$, but a quasi-norm because

$$|f + g|_{p,\infty} \leq 2(|f|_{p,\infty} + |g|_{p,\infty})$$

(see, e.g., [1, p. 7]). However, it is possible to endow $L^{p,\infty}$, $p \in (1, \infty)$, with a norm $\|\cdot\|_{p,\infty}$, converting it into a Banach space. Moreover, the inequality

$$|f|_{p,\infty} \leq \|f\|_{p,\infty} \leq p(p-1)^{-1}|f|_{p,\infty}$$

holds for all $f \in L^{p,\infty}$. It is worth mentioning that $L^{p,\infty}$ belongs (as a limiting case) to the class of *Lorentz spaces* $L^{p,q}$, $q \in [1, \infty]$ (see, e.g., [1, sect. 1.6] and references cited in this book).

Another generalization of $L^{p,\infty}$, known as the Marcinkiewicz space M_φ, is defined with the help of a nonnegative, concave function $\varphi \in C[0, \infty)$. This Banach space consists of all (equivalence classes of) measurable functions for which the norm

$$\|f\|_\varphi = \sup_{t>0} \frac{1}{\varphi(t)} \int_0^t f^*(s) \, ds$$

is finite. Here f^* denotes the nonincreasing rearrangement of f, i.e.,

$$f^*(s) = \inf_{\lambda>0}\{\lambda : m(\{x : |f(x)| > \lambda\}) \leq s\} \quad \text{for } s \geq 0,$$

and so is nonnegative and right-continuous. Moreover, its distribution function $m(\{x : |f^*(x)| > \lambda\})$ coincides with that of f. If $\varphi(t) = t^{1-1/p}$, then the corresponding Marcinkiewicz space is $L^{p,\infty}$, whereas $\varphi(t) \equiv 1$ and $\varphi(t) = t$ give L^1 and L^∞, respectively.

The Marcinkiewicz interpolation theorem for bounded linear operators. This kind of continuous operator is usually considered as mapping one normed space to another one, in which case the operator's norm is an important characteristic. However, the latter can be readily generalized for a mapping of L^p to $L^{p,\infty}$. Indeed, if $|Tf|_{p,\infty} \leq C\|f\|_p$, then it is natural to introduce the norm (or quasinorm) of T as the infimum over all possible values of C. Now we are in a position to formulate the following.

Theorem 2. *Let p_0, q_0, p_1, $q_1 \in [1, \infty]$ satisfy the inequalities $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$, and let p, $q \in [1, \infty]$ be such that $p \leq q$ and the equalities*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{1}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{1}{q_1}$$

hold for some $\theta \in (0, 1)$. If T is a linear operator that maps L^{p_0} into $L^{q_0,\infty}$ and its norm is N_0 and simultaneously $T : L^{p_1} \to L^{q_1,\infty}$ has N_1 as its norm, then T maps L^p into L^q and its norm N satisfies the estimate

$$N \leq CN_0^{1-\theta}N_1^{\theta}, \tag{1}$$

with C depending on p_0, q_0, p_1, q_1, and θ.

The convexity inequality (1) is a characteristic feature of the interpolation theory. The general form of this theorem (it is valid for quasi-additive operators, whose special case are subadditive ones described prior to Theorem 1) is proved in [23, ch. XII, sect. 4]. In particular, it is shown that one can take

$$C = 2\left(\frac{q}{|q-q_0|} + \frac{q}{|q-q_1|}\right)^{1/q} \frac{p_0^{(1-\theta)/p_0} p_1^{\theta/p_1}}{p^{1/p}};$$

see [23, Vol. II, p. 114, formula (4.18)], where, unfortunately, the notation differs from that adopted here. Special cases of Theorem 2 and diagrams illustrating them can be found in [9, pp. 155–156]. It should be emphasized that the restriction $p \leq q$ is essential; indeed, as early as 1964, R. A. Hunt [6] constructed an example demonstrating that Theorem 2 is not true without it. For a description of this example see, e.g., [1, pp. 16–17].

It was Marcinkiewicz himself who proposed an extension of his interpolation theorem to other function spaces; namely, the so-called diagonal case (when $p_0 = q_0$ and $p_1 = q_1$) of his theorem is formulated for Orlicz spaces in [12]. References to papers containing further results on interpolation in these and other spaces (e.g., Lorentz and M_{φ}) can be found in [1, pp. 128–129] and [9, pp. 163–166].

Applications of the interpolation theorems. (1) In his monograph [23], Zygmund gave a detailed study of the one-dimensional Fourier transform

$$F(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp\{-i\xi x\} dx, \qquad \xi \in \mathbb{R}.$$

See Vol. II, ch. XVI, sects. 2 and 3, where, in particular, it is demonstrated that F, originally defined on a dense set in L^p, $p \in [1, 2]$, is extensible to the whole space as a bounded operator $F : L^p \to L^{p'}$, $p' = p/(p-1)$, and so the integral converges in $L^{p'}$. To prove this assertion and its n-dimensional analogue one can use Theorem 2. Indeed, $F : L^1 \to L^\infty$ is bounded (this is straightforward to see), and by Plancherel's theorem F is bounded on L^2, and so this theorem is applicable. On the other hand, the Riesz–Thorin theorem, which has no restriction $p \leq q$, yields a more complete result valid for the inverse transform F^{-1} as well. The latter operator acting from $L^{p'}$ to L^p is bounded; here $p' \in [2, \infty)$, and so $p = p'/(p'-1) \in (1, 2]$.

(2) In studies of conjugate Fourier series, the singular integral operator (the periodic Hilbert transform)

$$H(f)(s) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon \leq |t| \leq \pi} f(s-t) \cot\frac{t}{2} dt$$

plays an important role. Indeed, by linearity it is sufficient to define H on a basis in $L^2(-\pi, \pi)$, and the relations

$$H(\cos nt) = \sin ns \text{ for } n \geq 0, \ H(\sin nt) = -\cos ns \text{ for } n \geq 1$$

show that it expresses passing from a trigonometric series to its conjugate. Moreover, these formulae show that H is bounded on $L^2(-\pi, \pi)$ and its norm is equal to one.

In the mid-1920s, Marcel Riesz obtained his celebrated result about this operator; first, he announced it in a brief note in the *Comptes rendus de l'Académie des Sciences Paris*, and three years later published his rather long proof that H is bounded on $L^p(-\pi, \pi)$ for $p \in (1, \infty)$; i.e., for every finite $p > 1$ there exists $A_p > 0$ such that

$$\|H(f)\|_p \leq A_p\|f\|_p \quad \text{for all } f \in L_p(-\pi, \pi). \tag{2}$$

However, (2) does not hold for $p = 1$ and ∞; see [23, Vol. I, ch. VII, sect. 2] for the corresponding examples and a proof of this inequality.

There are several different proofs of this theorem; the original proof of M. Riesz was reproduced in the first edition of Zygmund's monograph [23], which appeared in 1935. In the second edition published in 1959, this proof was replaced by that of Calderón obtained in 1950. Let us outline another proof based on the Marcinkiewicz interpolation theorem analogous to Theorem 1 but involving L^p-spaces on $(-\pi, \pi)$ instead of the spaces on $\mathbb{R}$.

First we notice that it is sufficient to prove (2) only for $p \in (1, 2)$. Indeed, assuming that this is established, then for $f \in L_p$ and $g \in L_{p'}$ we have

$$\int_{-\pi}^{\pi} [H(f)(s)] g(-s) ds \leq A_p\|f\|_p \|g\|_{p'}$$

by the Hölder inequality (as above $p' = p/(p-1)$, and so

$p' \geq 2$ when $p \leq 2$). Since

$$\int_{-\pi}^{\pi} [H(f)(s)] \, g(-s) \, ds = \int_{-\pi}^{\pi} f(-s) \, [H(g)(s)] \, ds,$$

the inequality $\|H(g)\|_{p'} \leq A_p^{-1} \|g\|_{p'}$ is a consequence of the assertion converse to the Hölder inequality.

It was mentioned above that H is bounded in L^2. Hence, in order to apply Theorem 1 for $p \in (1, 2]$, it is sufficient to show that this operator is of weak type $(1, 1)$, and this is an essential part of Calderón's proof; see [23, Vol. I, ch. IV, sect. 3]. Moreover, an improvement of the latter proof allowed S. K. Pichorides [16] to obtain the least value of the constant A_p in (2). It occurs that $A_p = \tan \pi/(2p)$ and $\cot \pi/(2p)$ is this value for $p \in (1, 2]$ and $p \geq 2$, respectively.

There are many other applications of interpolation theorems in analysis; see, e.g., [1, ch. 1], [23, ch. XII], and references cited in these books.

Further development of interpolation theorems. Results constituting the interpolation space theory were obtained in the early 1960s and are classical now. This theory was created in the works of Nachman Aronszajn, Alberto Calderón, Mischa Cotlar, Emilio Gagliardo, Selim Grigorievich Krein, Jacques-Louis Lions, and Jaak Peetre, to list a few. We leave aside several versions of complex interpolation spaces developed from the Riesz–Thorin theorem (see, e.g., [1, ch. 4]) and concentrate on "espaces de moyennes" introduced by Lions and Peetre in their celebrated article [8]. These "real method interpolation spaces," usually denoted $(A_0, A_1)_{\theta, p}$, are often considered as "descendants" of the Marcinkiewicz interpolation theorem.

Prior to describing these spaces, it is worth mentioning another germ of interpolation theory originating from Lwów. Problem 87 in the *Scottish Book* [14] posed by Banach demonstrates his interest in nonlinear interpolation. Presumably, it was formulated during Marcinkiewicz's stay in Lwów. Indeed, he solved problems 83 and 106 in [14], which were posed before and after, respectively, Banach's problem on interpolation. A positive solution of the latter problem (due to L. Maligranda) is presented in [14, pp. 163–170].

Let us turn to defining the family of spaces $\{(A_0, A_1)_{\theta, p}\}$ involved in the real interpolation method; here $\theta \in (0, 1)$ and $p \in [1, \infty]$. In what follows, we write $A_{\theta, p}$ instead of $(A_0, A_1)_{\theta, p}$ for the sake of brevity. Let A_0 and A_1 be two Banach spaces, both continuously embedded in some (larger) Hausdorff topological vector space. Then for a pair (θ, p) the space $A_{\theta, p}$ with $p < \infty$ consists of all $a \in A_0 + A_1$

for which the norm

$$\|a\|_{\theta, p} = \left\{ \int_0^\infty \left[t^{-\theta} K(t, a) \right]^p \frac{dt}{t} \right\}^{1/p}$$

is finite. Here $K(t, a)$ is defined on $A_0 + A_1$ for $t \in (0, \infty)$ by

$$\inf_{a_0, a_1} \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} : a_0 \in A_0, \ a_1 \in A_1 \text{ and } a_0 + a_1 = a \}.$$

This K-functional was introduced by Peetre. If $p = \infty$, then the expression $\sup_{t > 0} \{ t^{-\theta} K(t, a) \}$ gives the norm $\|a\|_{\theta, \infty}$ when finite.

Every $A_{\theta, p}$ is an intermediate space with respect to the pair (A_0, A_1), i.e.,

$$A_0 \cap A_1 \subset A_{\theta, p} \subset A_0 + A_1.$$

Moreover, if $A_0 \subset A_1$, then

$$A_0 \subset A_{\theta_0, p_0} \subset A_{\theta_1, p_1} \subset A_1,$$

provided either $\theta_0 > \theta_1$ or $\theta_0 = \theta_1$ and $p_0 \leq p_1$. For any p, it is convenient to put $A_{0,p} = A_0$ and $A_{1,p} = A_1$. Now we are in a position to explain what the interpolation of an operator is in terms of the family $\{A_{\theta, p}\}$ and another family of spaces $\{B_{\theta, p}\}$ constructed by using some Banach spaces B_0 and B_1 in the same way as A_0 and A_1.

Let $T : A_0 + A_1 \to B_0 + B_1$ be a linear operator such that its norm as the operator mapping A_0 (A_1) to B_0 (B_1) is equal to M_0 (M_1). Then the operator $T : A_{\theta, p} \to B_{\theta, p}$ is also bounded, and its norm is less than or equal to $M_0^{1-\theta} M_1^\theta$. Along with the method based on the K-functional, there is an equivalent method (also developed by Peetre) involving the so-called J-functional. Further details concerning this approach to interpolation theory can be found in [1, chs. 3 and 4].

The Marcinkiewicz Function

In the *Annales de la Société Polonaise de Mathématique*, volume 17 (1938), Marcinkiewicz published two short papers. Two remarkable integral operators were considered in the first of these notes (see [10] and [13, pp. 444–451]); they and their numerous generalizations became indispensable tools in analysis. One of these operators is always called the "Marcinkiewicz integral"; see [23, ch. IV, sect. 2] for its definition and properties. In particular, it is used for investigation of the structure of a measurable set near an "almost arbitrary" point; see [18, sects. 2.3 and 2.4], whereas further references to papers describing some of its generalizations can be found in the monographs [18] and [23]. The second operator is usually referred to as the "Marcinkiewicz function" (see, e.g., [9, pp. 192–194]), but it also appears as the "Marcinkiewicz integral." Presumably, the mess with names began as early as 1944, when Zygmund published the extensive article [22], section 2 of

which was titled "On an Integral of Marcinkiewicz." In fact, this 14-page section is devoted to a detailed study of the Marcinkiewicz function μ, whose properties were just outlined by Marcinkiewicz himself in [10]. It is not clear whether Zygmund had already received information about Marcinkiewicz's death when he decided to present in detail the results from [10] (the discovery of mass graves in the Katyń forest was announced by the Nazi government in April 1943).

Zygmund begins his presentation with a definition of the Littlewood–Paley function $g(\theta; f)$, which is a nonlinear operator applied to an integrable, 2π-periodic f. The purpose of introducing $g(\theta; f)$ was to provide a characterization of the L^p-norm $\|f\|_p$ in terms of the Poisson integral of f. After describing some properties of $g(\theta)$, Zygmund notes:

> It is natural to look for functions analogous to $g(\theta)$ but defined without entering the interior of the unit circle.

After a reference to [10], Zygmund continues:

> Marcinkiewicz had the right idea of introducing the function

$$\mu(\theta) = \mu(\theta; f)$$
$$= \left\{ \int_0^\pi \frac{[F(\theta + t) + F(\theta - t) - 2F(\theta)]^2}{t^3} dt \right\}^{1/2}$$
$$= \left\{ \int_0^\pi t \left[\frac{F(\theta + t) + F(\theta - t) - 2F(\theta)}{t^2} \right]^2 dt \right\}^{1/2}$$

where $F(\theta)$ is the integral of f,

$$F(\theta) = C + \int_0^\theta f(u) \, du.$$

More generally, he considers the functions

$$\mu_r(\theta) = \left\{ \int_0^\pi \frac{|F(\theta + t) + F(\theta - t) - 2F(\theta)|^r}{t^{r+1}} dt \right\}^{1/r}$$
$$= \left\{ \int_0^\pi t^{r-1} \left| \frac{F(\theta + t) + F(\theta - t) - 2F(\theta)}{t^2} \right|^r dt \right\}^{1/r},$$

so that $\mu_2(\theta) = \mu(\theta)$. He proves the following facts which are clearly analogues of the corresponding properties of $g(\theta)$.

These facts are the estimates

$$\|\mu_q\|_q \le A_q \|f\|_q \quad \text{and} \quad \|f\|_p \le A_p \|\mu_p\|_p$$

valid for $q \ge 2$ and $1 < p \le 2$, respectively, where f has the zero mean value in the second inequality and the assertion: *For every $p \in (1, 2]$ there exists a continuous, 2π-periodic function f such that $\mu_p(\theta; f) = \infty$ for almost every θ.*

Furthermore, Marcinkiewicz conjectured that for $p > 1$ the inequalities

$$A_p \|f\|_p \le \|\mu\|_p \le B_p \|f\|_p \tag{3}$$

hold, where again f must have the zero mean value in the second inequality. Moreover, he foresaw that it would not be easy to prove these inequalities; indeed, the proof given by Zygmund in his article [22] is more than 11 pages long.

The first step towards generalization of the Marcinkiewicz function was made by Daniel Waterman; his paper [21] was published seven (!) years after presentation of the work to the AMS. However, its abstract appeared in the *Proceedings of the International Congress of Mathematicians* held in 1954 in Amsterdam. Waterman considered the μ-function

$$\mu(\tau; f) = \left\{ \int_0^\infty \frac{[F(\tau + t) + F(\tau - t) - 2F(\tau)]^2}{t^3} dt \right\}^{1/2},$$

where $\tau \in (-\infty, \infty)$ and F is a primitive of $f \in L^p(-\infty, \infty)$, $p > 1$. His proof of inequalities (3) for $\mu(\tau; f)$ heavily relies on the M. Riesz theorem about conjugate functions on $\mathbb{R}^1$ (see [21, p. 130] for the formulation), and its proof involves the Marcinkiewicz interpolation theorem described above.

Another consequence of inequalities (3) for $\mu(\tau; f)$ is a characterization of the Sobolev space $W^{1,p}(\mathbb{R})$, $p \in (1, \infty)$. Indeed, putting

$$M(\tau; f) = \left\{ \int_0^\infty \frac{[f(\tau + t) + f(\tau - t) - 2f(\tau)]^2}{t^3} dt \right\}^{1/2}$$

for $f \in W^{1,p}(\mathbb{R})$, we have that $M(\tau; f) = \mu(\tau; f')$. Then (3) can be written as

$$A_p \|f'\|_p \le \|M(\cdot; f)\|_p \le B_p \|f'\|_p,$$

which implies the following assertion. *Let $p \in (1, \infty)$. Then $f \in W^{1,p}(\mathbb{R})$ if and only if $f \in L^p(\mathbb{R})$ and $M(\cdot; f) \in L^p(\mathbb{R})$.*

Stein extended these results to higher dimensions in the late 1950s and early 1960s (it is worth mentioning that μ is referred to as the Marcinkiewicz integral in his paper [17]). For this purpose he applied the real-variable technique used in the generalization of the Hilbert transform

$$\text{P.V.} \int_0^\infty \frac{f(x+t) - f(x-t)}{t} dt$$

to higher dimensions. Indeed, this can be written as

$$\int_0^\infty \frac{F(x+t) + F(x-t) - 2F(x)}{t^2} dt,$$

which resembles the expression for $\mu(\tau; f)$, and so Stein, in

his own words, was

> guided by the techniques used by A. P. Calderón and A. Zygmund [2] in their study of the n-dimensional generalizations of the Hilbert transform; connected with this are some earlier ideas of Marcinkiewicz.

The definition of singular integral given in [2], to which Stein refers, involves a function $\Omega(x)$ defined for $x \in \mathbb{R}^n$ and assumed (i) to be homogeneous of degree zero, i.e., to depend only on $x' = x/|x|$; (ii) to satisfy the Hölder condition with exponent $\alpha \in (0,1]$; and (iii) to have the zero mean value over the unit sphere in $\mathbb{R}^n$. Then

$$S(f)(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y) \, dy$$

exists almost everywhere provided $f \in L^p(\mathbb{R}^n)$, $p \in [1,\infty)$. Furthermore, this singular integral operator is bounded in $L^p(\mathbb{R}^n)$ for $p > 1$; i.e., the inequality $\|S(f)\|_p \le A_p \|f\|_p$ holds with A_p independent of f.

Moreover, in the section dealing with background facts, Stein notes that μ is a nonlinear operator and writes (see [17, p. 433]):

> An "interpolation" theorem of Marcinkiewicz is very useful in this connection.
>
> In quoting the result of Marcinkiewicz, [...] we shall not aim at generality. For the sake of simplicity we shall limit ourselves to the special case that is needed.

After that the required form of the interpolation theorem (see Theorem 1 above) is formulated and used later in the paper, thus adding one of the first items in the now long list of its applications. Since the term interpolation was novel, quotation marks are used by Stein in the quoted piece. Indeed, Zygmund's proof of the Marcinkiewicz theorem had appeared in 1956, just two years earlier than Stein's article.

Stein begins his generalization of the Marcinkiewicz function $\mu(\tau; f)$ with the case when $f \in L^p(\mathbb{R}^n)$, $p \in [1,2]$. Realizing the analogy described above, he puts

$$F_t(x) = \int_{|y| \le t} \frac{\Omega(y')}{|y|^{n-1}} f(x-y) \, dy, \qquad x \in \mathbb{R}^n, \quad (4)$$

where Ω satisfies conditions (i)–(iii), and notes that if $n = 1$ and $\Omega(y) = \text{sign } y$, then

$$F_t(x) = F(x+t) + F(x-t) - 2F(x) \quad \text{with } F(x) = \int_0^x f(s) \, ds.$$

Therefore, it is natural to define the n-dimensional Marcinkiewicz function as follows:

$$\mu(x; f) = \left\{ \int_0^\infty \frac{[F_t(x)]^2}{t^3} \, dt \right\}^{1/2}. \quad (5)$$

Stein begins his investigation of properties of this function by proving that $\|\mu(\cdot; f)\|_2 \le A \|f\|_2$, where A is independent of f, and his proof involving Plancherel's theorem is not elementary at all. Even less elementary is his proof that $\mu(\cdot; f)$ is of weak type $(1,1)$. Then the Marcinkiewicz interpolation theorem (see Theorem 1 above) implies that $\|\mu(\cdot; f)\|_p \le A \|f\|_p$ for $p \in (1,2]$ provided $f \in L^p(\mathbb{R}^n)$. For all $p \in (1,\infty)$ this inequality is proved in [17] with assumptions (i)–(iii) changed to the following ones: $\Omega(x')$ is absolutely integrable on the unit sphere and is odd there, i.e., $\Omega(-x') = -\Omega(x')$. A few years later, A. Benedek, A. P. Calderón, and R. Panzone demonstrated that for a C^1-function Ω, condition (iii) implies the last inequality for all $p \in (1,\infty)$.

In another note, Stein obtained the following generalization of the one-dimensional result.

Let $p \in (2n/(n+2), \infty)$ and $n \ge 2$. Then f belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and

$$\left\{ \int_{\mathbb{R}^n} \frac{[f(\cdot + y) + f(\cdot - y) - 2f(\cdot)]^2}{|y|^{n+2}} dy \right\}^{1/2} \in L^p(\mathbb{R}^n).$$

For $n > 2$ this does not cover $p \in (1, 2n/(n+2)]$ and so is weaker than the assertion formulated above for $n = 1$.

In the survey article [9, pp. 193–194], one finds a list of papers concerning the Marcinkiewicz function. In particular, further properties of μ were considered by A. Torchinsky and S. Wang [19] in 1990, whereas T. Walsh [20] proposed a modification of the definition (4), (5) in 1972.

Multipliers of Fourier Series and Integrals

During his stay in Lwów, Marcinkiewicz collaborated with Stefan Kaczmarz and Juliusz Schauder,[2] who had awakened his interest in *multipliers* of orthogonal series. Studies in this area of analysis were initiated by Hugo Steinhaus in the 1920s; in its general form, the problem of multipliers is as follows. Let B_1 be a Banach space with a Schauder basis $\{g_n\}_{n=1}^\infty$. The (linear) operator T is called a *multiplier* when there is a sequence $\{m_n\}_{n=1}^\infty$ of scalars of this space and T acts as follows:

$$B_1 \ni f = \sum_{n=1}^\infty c_n g_n \quad \to \quad Tf \sim \sum_{n=1}^\infty m_n c_n g_n.$$

Here $\sim$ means that the second sum assigned as Tf can belong to the same space B_1 or be an element of another Banach space B_2; this depends on properties of the sequence. Multipliers of Fourier series are of paramount interest, and this was the topic of the remarkable paper [11] published by Marcinkiewicz in 1939.

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[2] *Both perished in World War II. Being in the reserve, Kaczmarz was drafted and killed during the first week of war; the circumstances of his death are unclear. Schauder was in hiding in occupied Lwów, and the Gestapo killed him in 1943 while he was trying to escape arrest.*

Not long before Marcinkiewicz's visit to Lwów started, Kaczmarz investigated some properties of multipliers in the function spaces (mainly $L^p(0,1)$ and $C[0,1]$) under rather general assumptions about the system $\{g_n\}_{n=1}^\infty$. Further results about multiplier operators were obtained in the joint paper [7] of Kaczmarz and Marcinkiewicz. It was submitted to the *Studia Mathematica* in June 1937; i.e., their collaboration lasted for another year after Marcinkiewicz left Lwów. This paper has the same title as that of Kaczmarz and concerns the case when $L^p(0,1)$ with $p \neq \infty$ is mapped to $L^q(0,1)$, $q \in [1,\infty]$; it occurs that the case $q = \infty$ is the simplest one. In this paper, it is assumed that every function g_n is bounded, whereas the sequence $\{g_n\}_{n=1}^\infty$ is closed in $L^1(0,1)$. In each of four theorems that differ by the ranges of p and q involved, certain conditions are imposed on $\{m_n\}_{n=1}^\infty$, and these conditions are necessary and sufficient for the sequence to define a multiplier operator $T : L^p \to L^q$.

After returning to Wilno, Marcinkiewicz kept on his studies of multipliers initiated in Lwów, and in May 1938, he submitted (again to the *Studia Mathematica*) the seminal paper [11], in which the main results are presented in a curious way. Namely, Theorems 1 and 2, concerning multipliers of Fourier series and double Fourier series, are formulated in the reverse order. Presumably, the reason for this is the importance of multiple Fourier series for applications and generalizations. Let us formulate Theorem 1 in a slightly updated form.

Let $f \in L^p(0,2\pi)$, $p \in (1,\infty)$, be a real-valued function and let its Fourier series be

$$a_0/2 + \sum_{n=1}^\infty A_n(x), \text{ where } A_n(x) = a_n \cos nx + b_n \sin nx.$$

If a bounded sequence $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{R}$ is such that

$$\sum_{n=2^k}^{2^{k+1}} |\lambda_n - \lambda_{n+1}| \leq M \text{ for all } k = 0,1,2,\dots, \quad (6)$$

where M is a constant independent of k, then the mapping $f \mapsto \sum_{n=1}^\infty \lambda_n A_n$ is a bounded operator in $L^p(0,2\pi)$.

It is well known that for $p = 2$ this theorem is true with condition (6) omitted, but this is not mentioned in [11]. The assumptions that f is real-valued and $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{R}$ were not stated in [11] explicitly but used in the proof. This was noted by Solomon Grigorievich Mikhlin [15], who extended this theorem to complex-valued multipliers and functions. Also, he used the exponential form of the Fourier expansion:

$$f(x) = \sum_{n=-\infty}^\infty c_n \exp inx.$$

The trigonometric form was used by Marcinkiewicz for double Fourier series as well, and his sufficient conditions on bounded real multipliers $\{\lambda_{mn}\}$ look rather awkward. Now, the restrictions on $\{\lambda_{mn}\} \subset \mathbb{C}$ are usually expressed in a rather condensed form by using the so-called dyadic intervals; see, e.g., [18, sect. 5.1]. Applying these conditions to multipliers acting on the expansion

$$\sum_{m,n=-\infty}^\infty c_{mn} \exp i\{mx + ny\}$$

of $f \in L^p((0,2\pi)^2)$, $p \in (1,\infty)$, one obtains an updated formulation of the multiplier theorem; see, e.g., [9, p. 201].

A simple corollary derived by Marcinkiewicz from this theorem is as follows (see [11, p. 86]). The fractions

$$\frac{m^2}{m^2 + n^2}, \quad \frac{n^2}{m^2 + n^2}, \quad \frac{|mn|}{m^2 + n^2} \quad (7)$$

provide examples of multipliers in L^p for double Fourier series. The reason to include these examples was to answer a question posed by Schauder, and this is specially mentioned in a footnote. Moreover, after remarking that his Theorem 2 admits an extension to multiple Fourier series, Marcinkiewicz added a straightforward generalization of formulae (7) to higher dimensions, again referring to Schauder's question. This is evidence that the question was an important stimulus for Marcinkiewicz in his work.

A natural way to generalize Marcinkiewicz's theorems is to consider multipliers of Fourier integrals. Study of these operators was initiated by Mikhlin in 1956; see note [15], in which the first result of that kind was announced. Several years later, Mikhlin's theorem was improved by Lars Hörmander [5], and since then it has been widely used for various purposes. To formulate this theorem we need the n-dimensional Fourier transform

$$F(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \exp\{-i\,\xi \cdot x\}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n,$$

defined for $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $p \in (1,\infty)$. It is clear that any bounded measurable function Λ on $\mathbb{R}^n$ defines the mapping

$$T_\Lambda(f)(x) = F^{-1}[\Lambda(\xi)F(f)(\xi)](x), \qquad x \in \mathbb{R}^n,$$

such that $T_\Lambda(f) \in L^2(\mathbb{R}^n)$. If $T_\Lambda(f)$ is also in $L^p(\mathbb{R}^n)$ and T_Λ is a bounded operator, i.e.,

$$\|T_\Lambda(f)\|_p \leq B_{p,n}\|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n) \quad (8)$$

with B_p independent of f, then Λ is called a multiplier for L^p.

The description of all multipliers for L^2 is known as well for L^1 and L^∞ (it is the same for these two spaces); see [18, pp. 94–95]. However, the question about characterization of the whole class of multipliers for other values of p is far

from resolved. The following assertion gives widely used sufficient conditions.

Theorem (Mikhlin, Hörmander). *Let Λ be a function of the C^k-class in the complement of the origin of $\mathbb{R}^n$. Here k is the least integer greater than $n/2$. If there exists $B > 0$ such that*

$$|\xi|^\ell \left| \frac{\partial^\ell \Lambda(\xi)}{\partial \xi_{j_1} \partial \xi_{j_2} \cdots \partial \xi_{j_\ell}} \right| \le B, \qquad 1 \le j_1 < j_2 < \cdots < j_\ell \le n,$$

for all $\xi \in \mathbb{R}^n$, $\ell = 0, \ldots, k$, and all possible ℓ-tuples, then inequality (8) *holds; i.e., Λ is a multiplier for L^p.*

In various versions of this theorem, different assumptions are imposed on the differentiability of Λ. In particular, Hörmander [5, pp. 120–121] replaced the pointwise inequality for weighted derivatives of Λ by a weaker one involving certain integrals (see also [18, p. 96]). Recently, Loukas Grafakos and Lenka Slavíková [4] obtained new sufficient conditions for Λ in the multiplier theorem, thus improving Hörmander's result. Their conditions are optimal in a certain sense explicitly described in [4].

Corollary. *Every function that is smooth everywhere except at the origin and is homogeneous of degree zero is a Fourier multiplier for L^p.*

Its immediate consequence is the Schauder estimate

$$\left\| \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}} \right\|_p \le C_{p,n} \|\Delta u\|_p, \qquad 1 \le j_1, j_2 \le n,$$

valid for u belonging to the Schwartz space of rapidly decaying infinitely differentiable functions. For this purpose one has to use the equality

$$F\left(\frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}} \right)(\xi) = \frac{\xi_{j_1} \xi_{j_2}}{|\xi|^2} F(\Delta u)(\xi), \qquad 1 \le j_1, j_2 \le n,$$

and the fact that the function $\xi_{j_1} \xi_{j_2}/|\xi|^2$ is homogeneous of degree zero.

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