PROOF OF A CONJECTURE OF I. J. SCHOENBERG[1]

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Consider the class A of those functions $f(t)$ such that

$$(1)\qquad \begin{aligned} &f(t) \in C^{\infty} && (-\infty < t < \infty), \\ &f^{(n)}(\pm\infty) = 0 && (n = 0, 1, \cdots). \end{aligned}$$

By Rolle's theorem, if $f(t)$ belongs to A and is not identically zero, then $f^{(n)}(t)$ must have at least n changes of sign. This suggests the consideration of the subclass A' consisting of those nonzero functions of A for which $f^{(n)}(t)$ has not more than n and therefore exactly n changes of sign. Such functions correspond to our intuitive idea of the concept "bell-shaped." Many interesting functions belong to the class A', in particular all Pólya frequency functions belong to A'. See[3] [1] and [2]. There exist Pólya frequency functions vanishing identically in a neighborhood of $+\infty$ or of $-\infty$, but there does not exist one vanishing identically in neighborhoods of both $+\infty$ and $-\infty$, that is, outside an interval. This led I. J. Schoenberg[4] to conjecture that there does not exist a function of class A' vanishing outside an interval.

This conjecture is also connected with the theory established by Pólya and Wiener [3], and by Szegö [4], connecting the oscillation of the derivatives of a periodic function with its analytic character. It was, for example, proved in [4] that if B is the class of those infinitely differentiable functions, with period 2π, for which

$$(2)\qquad N_k(f(t)) = O\left[\frac{k}{\log k}\right] \qquad (k \to \infty),$$

where $N_k(f(t))$ is the number of changes of sign of $f^{(k)}(t)$ in a period, then every function of class B is the restriction to the real axis of an entire function. The validity of Schoenberg's conjecture implies that

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[3] Numbers in brackets refer to the bibliography at the end of the paper.

[4] This problem was proposed in connection with the Princeton Conference of 1946–1947.

if B' is the class of those infinitely differentiable functions, with period 2π, for which

$$N_k(f(t)) \leqq k \qquad (k = 0, 2, 4, \cdots), \tag{3}$$

then every function of class B' which vanishes with all its derivatives at a point is identically zero.

We shall now proceed to give a proof of this conjecture.

THEOREM. *If*

1. $f(t) \in C^\infty$ $(-\infty < t < \infty)$,
2. $f(t) \equiv 0$ $(|t| \geqq a)$,
3. $f^{(n)}(t)$ *has at most* n *changes of sign* $(n = 0, 1, \cdots)$,

then $f(t) \equiv 0$.

We define $\int_{-a}^{a} f(t) t^i dt = M_i$ $(i = 0, 1, 2)$. It is no restriction to suppose that $M_0 > 0$, $M_1 = 0$. Indeed if $M_0 = 0$, then $f(t) \equiv 0$, and if $M_0 \not> 0$, $M_1 \neq 0$, we may consider instead of $f(t)$ the function $g(t) = \epsilon f(t+b)$, where $b = M_1/M_0$, and ϵ is ± 1 so chosen that $0 \leqq g(t)$.

Since $f^{(n)}(t)$ has n changes of sign we can find n points, $z_1^{(n)}, \cdots, z_n^{(n)}$ between $-a$ and a such that

$$f^{(n)}(t) \prod_{i=1}^{n} (t - z_i^{(n)}) \tag{4}$$

is of constant sign. We define

$$s_n = -\sum_i z_i^{(n)}, \qquad r_n = \sum_{i<j} z_i^{(n)} z_j^{(n)}.$$

Integrating by parts n times one may show that

$$\int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) dt = (-1)^n n! M_0. \tag{5}$$

Using the fact that the function (4) is of constant sign, we see that

$$\left| \int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t dt \right| \leqq a n! M_0. \tag{6}$$

Now

$$\begin{aligned} \int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t dt &= (-1)^n [(n+1)! M_1 + n! s_n M_0], \\ &= (-1)^n n! s_n M_0. \end{aligned} \tag{7}$$

Combining (6) and (7), we have

$$|s_n| \leqq a. \tag{8}$$

Similarly

$$\left| \int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t^2 dt \right| \leqq a^2 n! M_0, \tag{9}$$

and

$$\int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t^2 dt = (-1)^n \left[\frac{1}{2} (n+2)! M_2 + n! r_n M_0 \right]. \tag{10}$$

Thus

$$M_2 \leqq \frac{2a^2 M_0}{(n+1)(n+2)} + \frac{2 |r_n| M_0}{(n+1)(n+2)} . \tag{11}$$

Now

$$r_n = [s_n^2 - \sum (z_i^{(n)})^2]/2$$

and hence

$$|r_n| \leqq (n/2)a^2. \tag{12}$$

It follows that

$$M_2 \leqq \frac{a^2 M_0}{(n+1)} . \tag{13}$$

Since n may be taken arbitrarily large, this implies that $M_2 = 0$ and thus that $f(t) \equiv 0$.

The function which is exp $[1/(x_2 - 1)]$ for $|x| < 1$ and 0 for $|x| \geqq 1$ is infinitely differentiable and its nth derivative has not more than $3n-2$ changes of sign for $n = 1, 2, \cdots$. Thus our theorem is no longer true if n is replaced by $3n$ in assumption 3. This example is due to D. V. Widder.

Bibliography

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