ON THE "COLLECTIVE HAUSDORFF METHOD"

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A sequence $\{s_n\}$ is said to be summable to S by the Hausdorff method $T\sim\mu_n$, if $\lim_{n\to\infty} t_n = S$, where $\{t_n\}$ is given by

$$(1)\qquad \Delta^n t_0 = \sum_{k=0}^{n} (-1)^k C_{n,k} t_k = \mu_n \Delta^n s_0 \qquad (n = 0, 1, 2, \cdots).$$

Hausdorff methods have been extensively investigated by F. Hausdorff and others [1, 3, 5].[1]

A method of summation A is said to be *stronger* than another method B if every sequence summed by B is also summed by A. Two methods of summation are *consistent* if any sequence which can be summed by both methods has the same limit assigned to it by both of them. I shall say that A *contains* B if A is stronger than B and also consistent with B. A method containing ordinary convergence is called *regular*. F. Hausdorff proved that any two regular Hausdorff methods are consistent [3]. This enabled R. P. Agnew to introduce the "collective Hausdorff method" $\mathfrak{H}$ by the definition: $\{s_n\}$ is summable $\mathfrak{H}$ to the sum S if $\{s_n\}$ is summable to S by any regular Hausdorff method [1]. He raised at the same time the question whether there is a matrix method of summation containing $\mathfrak{H}$. I shall now show that the answer is in the negative. More precisely I prove the following:

THEOREM. *There is no matrix $A = (a(m, n))$ such that*

(α) $t_m = \sum_{n=0}^{\infty} a(m, n)s_n$ $(m = 0, 1, 2, \cdots)$ *converges whenever $\{s_n\}$ is summable $\mathfrak{H}$ and*

(β) *if $\{s_n\}$ is summed to S by $\mathfrak{H}$, then $t_m \to S$ as $m \to \infty$.*[2]

To start with I list a few special sequences summed by $\mathfrak{H}$.

(a) Since ordinary convergence is the Hausdorff method $C\sim 1$, $\mathfrak{H}$ sums every convergent sequence to its ordinary limit. In other words, $\mathfrak{H}$ is regular.

(b) The sequence $s_n = (-c)^n$ $(c > 1)$ is summed to 0 by a suitable Euler method $E_p \sim p^{-n}$ $(p > 1)$.

(c) The sequence $s_n = C_{n,k}$ is summed to 0 by Mercer's method $M_k \sim (k-n)/k(n+1)$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] To avoid double subscripts I write $a(m, n)$ instead of the usual a_{mn}.

Euler methods have been extensively studied by Knopp [4]. The special result stated as (b) can easily be verified byt he use of (1). Mercer's methods are investigated in [5]; again (c) can be verified directly by means of (1).

Suppose now that A is a matrix satisfying the conditions (α) and (β) of the theorem. By (a) this matrix defines a regular method of summation and hence, by a well known theorem [6],

$$a(m, n) \to 0 \quad \text{as} \quad m \to \infty \qquad (n = 0, 1, \cdots) \tag{2}$$

and

$$\sum_{n=0}^{\infty} a(m, n) \to 1 \quad \text{as} \quad m \to \infty.$$

By (b) and condition (α) of the theorem

$$\sum_{n=0}^{\infty} | a(m, n) | c^n < \infty \tag{3}$$

for every c, since $\sum a(m, n) z^n$ is an integral function.

By (3) there is an integer $n(k)$ such that

$$\sum_{l \geqq n(k)} | a(k, l) | 4^l < 2^{-k}. \tag{4}$$

I show next that there is a matrix B satisfying the following conditions:

(i) $b(k, l) = 0$ except perhaps for $2 \leqq l < n(k)$.

(ii) Every sequence summed by A and satisfying $|s_n| \leqq 4^n$ for all large n is summed to the same sum by B.

It is an immediate consequence of (2) and (4) that the matrix $B = (b(k, l))$ with

$$\begin{aligned} b(k, l) &= a(k, l) \qquad && (\text{for } 2 \leqq l < n(k)) \\ &= 0 && (\text{otherwise}) \end{aligned}$$

satisfies the requirements (i) and (ii). In particular B defines a regular method of summation and therefore

$$\sum_{l=0}^{\infty} b(k, l) = \sum_{l=2}^{\infty} b(k, l) \to 1 \tag{5}$$

as $k \to \infty$.

If we write

$$\Delta^n s_0 = \sigma_n,$$

$$\sum b(k, l)s_l = \sum c(k, h)\sigma_h,$$

then

$$s_n = \Delta^n \sigma_0,$$

$$b(k, l) = (-1)^l \sum_{h \geqq l} c(k, h) C_{h,l}, \tag{6}$$

$$c(k, h) = (-1)^h \sum_{l \geqq h} b(k, l) C_{l,h}. \tag{7}$$

By condition (i) imposed on B

$$c(k, h) = 0 \qquad (h \geqq n(k)). \tag{8}$$

By condition (ii), (c), and (7)

$$c(k, h) \to 0 \qquad \text{as } k \to \infty. \tag{9}$$

I show now that for each $k > K$, where K is a sufficiently large number, there is at least one index h such that

$$| c(k, h) | > 3^{-h}. \tag{10}$$

For otherwise, by (6),

$$| b(k, l) | \leqq \sum_{h \geqq l} C_{h,l} 3^{-h} = 3^{-l}\left(1 - \frac{1}{3}\right)^{-l-1} = 3 \cdot 2^{-l-1}$$

and so

$$\left| \sum_{l=2}^{\infty} b(k, l) \right| \leqq \sum_{l=2}^{\infty} 3 \cdot 2^{-l-1} = \frac{3}{4}$$

in contradiction to (5).

By (10) we can choose $k_0 > K$, and h_0 such that $|c(k_0, h_0)| > 3^{-h_0}$. Because of (8) the element in the k_0th row of the matrix $(c(k, h))$ are 0 for $h \geqq m_0$, say. Because of (9) we can find an index $k_1 > K$ such that

$$\sum_{h < m_0} | c(k_1, h) | 3^h < \frac{1}{2} \cdot \tag{11}$$

Also, by (10), there is an index h_1 such that $|c(k_1, h_1)| > 3^{-h_1}$. By (11) we must have $h_1 \geqq m_0$. Again $c(k_1, h) = 0$ for $h \geqq m_1$, say. Proceeding in this way we find successively $k_2, h_2, m_2, k_3, h_3, m_3, \cdots$ such that

$$\sum_{h < m_{j-1}} | c(k_j, h) | 3^h < \frac{1}{2}, \tag{12}$$

$$| c(k_j, h_j) | > 3^{-h_j}, \tag{13}$$

$$h_j \geqq m_{j-1}, \tag{14}$$

$$c(k_j, h) = 0 \qquad (h \geqq m_j). \tag{15}$$

There is always a least possible choice of m_j, but we may choose a larger number, if desired, so that there is no loss of generality in supposing

$$\sum 1/m_j < \infty$$

which implies, by (14),

$$\sum 1/h_j < \infty. \tag{16}$$

We define now a sequence $\{s_n\}$ by

$$\Delta^n s_0 = \sigma_n = 0 \qquad (n \neq h_j), \tag{17}$$

$$\sum_l b(k_j, l)s_l = \sum_h c(k_j, h)\sigma_h = (-1)^j \frac{1}{2} \quad (j = 0, 1, 2, \cdots). \tag{18}$$

By (17), (14), and (15),

$$(-1)^j \frac{1}{2} = \sum_h c(k_j, h)\sigma_h = \sum_{i \leqq j} c(k_j, h_i)\tau_i \qquad (j = 0, 1, 2, \cdots),$$

where τ_i is written for σ with the index h_i. These equations can be solved successively for $\tau_0, \tau_1, \tau_2, \cdots$ and it is easily proved by means of (12) and (13) that $|\tau_i| < 3^{h_i}$. Together with (17) this shows that $|\sigma_h| < 3^h$ and, therefore,

$$|s_n| = |\Delta^n \sigma_0| \leqq \sum C_{n,h}\sigma_h < \sum C_{n,h}3^h = 4^n.$$

It is obvious from (18) that $\{s_n\}$ is not summed by B, and so, by condition (ii) imposed on B, $\{s_n\}$ is not summed by A. On the other hand it follows from (1) and (17) that $\{s_n\}$ is summed to 0 by any Hausdorff method for which

$$\mu_n = 0 \qquad (n = h_0, h_1, h_2, \cdots). \tag{19}$$

To complete the proof of the theorem it is necessary only to show that there is a regular Hausdorff method satisfying (19).

By a well known theorem [3] the method $T \sim \mu_n$ is regular if

$$\mu_n = T(n) = \int_0^1 t^n d\phi(t),$$

where $\phi(t)$ is of bounded variation in $\langle 0, 1 \rangle$,

$$\mu_0 = \phi(1) - \phi(0) = 1,$$

and

$$\phi(0) = \phi(0+).$$

It is an easy consequence of Mellin's formula that the function[3]

$$T_1(z) = (1 + z)^{-2} \prod_{j=0}^{\infty} \frac{h_j - z}{h_j + z}$$

can be written in the form $T_1(z) = \int_0^1 t^z d\phi(t)$ where $\phi(t)$ satisfies all the conditions just stated (details of the proof of this statement can be found in [2]). This proves the existence of the regular method $T_1 \sim \mu_n = T_1(n)$ satisfying (19) and completes the proof of the theorem.

This proof makes essential use of "unreasonable" Hausdorff methods for which the moment function $T(z)$ has zeros in the right half-plane. It remains an open problem whether there is a matrix method containing all reasonable Hausdorff methods, for which $T(z) \neq 0$ in $\Re z \geqq 0$.

References

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[3] The infinite product converges because of (16).