A THEOREM ON THE ACCESSIBILITY OF BOUNDARY PARTS OF AN OPEN POINT SET

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Let Ω be a bounded open set in ordinary n-space. Denote by p a point of that space. A closed subset of the boundary of Ω is called accessible from within Ω if Ω contains a semi-open Jordan curve $p(t)$, $0 \leqq t < \infty$, whose points of accumulation, for $t \to \infty$, precisely constitute that boundary part. In this note we prove the following theorem.

THEOREM 1. *If Ω is a bounded open set and if Ω' is a connected*[1] *open subset Ω' of Ω whose boundary has at least a point in common with the boundary of Ω, then the set C of all of these common boundary points contains a set accessible from within Ω. If C is the sum of two disjoint closed sets then each of these sets contains a set accessible from within Ω.*

The theorem was established and proved in order to bridge a gap in the proof of a classical theorem on surfaces $z(x, y)$ of nonpositive Gaussian curvature. Theorem 1 might, however, be of interest in other respects. The completed proof of the differential geometric theorem is given in the subsequent paper. The special case of Theorem 1 needed for the completion is that where the boundaries of the open set Ω and of the open and connected sets $\Omega' \subset \Omega$ have only one or two points in common. None of these points need, of course, be accessible from Ω'. Theorem 1, however, ascertains that each of these points is accessible from Ω.

The conclusion of the theorem remains valid if the open set $\Omega' \subset \Omega$, instead of being connected, is merely supposed to be connected to the vicinity of C. We call an open set Ω' connected to the vicinity of a closed set S if there is a neighborhood N^* of S (open set containing S) with the following property. For every neighborhood N of S there is a Jordan arc whose interior lies in Ω' and whose end points lie in N and on the boundary of N^*, respectively. The generalized Theorem 2 is found to be more easily proved than Theorem 1. It is convenient to extend the notion of accessibility of a closed set S from an open set Ω to arbitrary closed sets (not necessarily parts of the boundary of Ω).

THEOREM 2. *Let Ω and $\Omega' \subset \Omega$ be two bounded open sets and denote by S a closed set which contains all points common to the boundaries of Ω*

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[1] Throughout this note the notion of connectedness is meant in the sense of arcwise connectedness.

and Ω'. Let Ω' be connected to the vicinity of S. Then S contains a set accessible from within Ω. If S splits into two disjoint closed sets the same is true for each part.

PROOF OF THEOREM 2. Recall the set N^* in the definition of vicinity-connectedness. Let N_1 and N_2 be two arbitrary neighborhoods of S subject to the conditions

$$N_1 \subset N^*, \text{ closure of } N_2 \subset N_1. \tag{1}$$

Let

$$N_1\Omega = \sum C_\nu \tag{2}$$

be the decomposition of the open set N_1 into disjoint connected components. The open sets

$$C_n' = C_n N_2 \Omega' \tag{3}$$

are disjoint and satisfy, according as $N_2 \subset N_1$ and $\Omega' \subset \Omega$, the relation

$$N_2\Omega' = \sum C_\nu'. \tag{4}$$

N_2 contains S and S in turn contains, according to hypothesis, boundary points of Ω'. The open set N_2 contains, therefore, points of Ω' and the sets C_ν' cannot all be empty.

We first show that a boundary point of C_n is either a boundary point of Ω or a boundary point of N_1. In fact, a boundary point p of C_n is a point of accumulation of N_1. If p is not in N_1 it must lie on the boundary of N_1. Suppose that $p \subset N_1$. As a boundary point of the open set C_n, p cannot belong to C_n. Furthermore, p cannot belong to any other C_m. Otherwise a whole neighborhood of p would belong to C_m and $C_n C_m$ would not be empty since the neighborhood contains points of C_n. p therefore is not in $\sum C_\nu = N_1\Omega$ and, as $p \subset N_1$, not in Ω. On the other hand, every neighborhood of p contains points of C_n, and $C_n \subset \Omega$. p must be a boundary point of Ω.

The boundary of C_n and the boundary of the subset C_n' of C_n cannot have common points except on S. In fact, such a point must be a boundary point of Ω since, according as $C_n' \subset N_2$ and as the second inequality (1) holds, it cannot lie on the boundary of N_1. It must also be a boundary point of Ω' since it is a point of accumulation for $C_n' \subset \Omega' \subset \Omega$. According to the hypothesis, a common boundary point of Ω and Ω' can only lie on S.

The main point of the proof is to show that among the sets C_n' there is at least one that is connected to the vicinity of S. To this purpose we first consider the "special" C_n', that is, those C_n' which have a boundary point, say p_n, on the boundary of N_2. We prove that the special C_n' must be finite in number. If there were infinitely many, the corresponding points p_n would have a point accumulation p^* on the boundary of N_2. Every sufficiently small neighborhood V of p^* would, according to the second inequality (1), contain no points of the boundary of N_1. We can always assume V to be connected. V would contain points p_n, p_m belonging to two disjoint open sets C_n, C_m and, therefore, all the points of a suitable Jordan arc joining them. This arc, and therefore V, must contain a boundary point of C_n. This point is either a boundary point of Ω or of N_1. The latter case cannot occur in V. Consequently, every neighborhood of p^* contains a boundary point of Ω, and, therefore, p^* is itself a boundary point of Ω. From $p_n \subset \Omega'$ and $\Omega' \subset \Omega$ it is obvious that p^* is also a boundary point of Ω'. This is a contradiction, since a common boundary point of Ω and Ω' must, according to the hypothesis, lie on $S \subset N_2$ and, therefore, cannot lie on the boundary of N_2. Hence the special C_n' must be finite in number. Now we make use of the hypothesis that the open set Ω' is connected to the vicinity of S. For any neighborhood N of S,

$$N \subset N_2,$$

there is a Jordan arc leading from some point of the boundary of N^* entirely through Ω' to some point of N. On this arc there is, according to (1), a last point of entry into N_2. This is the initial point of a subarc which thus joins some boundary point of N_2 entirely through $N_2\Omega' = \sum C_\nu'$ to some point of N. Since the C_ν $(C_\nu' \subset C_\nu)$ are mutually disconnected, the subarc must lie wholly in one of the C_ν'. This must, obviously, be one of the finitely many special C_ν'. If we let N run through a sequence of neighborhoods of S closing down on S, and if we attach an arc and a subarc to each of these N, we see that one of the C_ν', say C_n', must contain infinitely many of these subarcs. C_n' is evidently connected to the vicinity of S.

The result obtained is this. Given any two neighborhoods $N_1 N_2$ of S satisfying (1), there exist two open sets Ω_1 and $\Omega_1' \subset \Omega_1$,

$$\Omega_1 \subset \Omega, \qquad \Omega_1 \subset N_1, \qquad \Omega_1' \subset N_2,$$

with the following properties. The common points of the boundaries of Ω_1 and Ω_1' lie on S. Ω_1 is connected. Ω_1' is connected to the vicinity of S.[2] The three sets Ω_1, Ω_1', and S thus have precisely the properties

[2] The neighborhood from whose boundary the arcs are drawn is N_2. In the case of Ω' it was N^*, which could, of course, be replaced by N_1.

supposed about Ω, Ω', and S with the important additional information that Ω_1 is connected.

We now start from a sequence of neighborhoods $N_1, N_2, N_3, \cdots$ of S closing down on S. We choose them such that

$$(\text{closure of } N_{n+1}) \subset N_n.$$

In the same way as before, starting this time from Ω_1 and Ω_1', we can construct two open sets Ω_2 and $\Omega_2' \subset \Omega_2$,

$$\Omega_2 \subset \Omega_1, \qquad \Omega_2 \subset N_2, \qquad \Omega_2' \subset N_3,$$

such that Ω_2, Ω_2', and S have the same properties as Ω_1, Ω_1', and S.[3] Continuing in this manner we obtain a sequence of connected open sets with the properties

$$\Omega \supset \Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots, \quad \Omega_n \subset N_n.$$

The Ω_n' are no longer needed. In each Ω_n we choose a point q_n. As Ω_n is connected, q_n and q_{n+1} can be joined by an arc $\alpha_n \subset \Omega_n \subset N_n$. The curve $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ has the desired properties, except perhaps simplicity. Every point of accumulation, as $t \to \infty$, lies in the common part of the N_n, which is S since S is closed. By the well known procedure of "omitting loops" we obtain a simple Jordan curve with the same properties.

The additional concluding statement of Theorem 1 is proved in the following way. Let $C = C_1 + C_2$, where the parts are closed and disjoint. There exist neighborhoods N_1 of C_1 and N_2 of C_2 without common points. The boundaries of the sets $\Omega + N_2$ and Ω' obviously have precisely C_1 as their common part. According to Theorem 1, C_1 is therefore accessible from $\Omega + N_2$. The Jordan curve in $\Omega + N_2$ that converges towards C_1 must, from a certain point on, lie in N_1 and thus outside of N_2, in other words in Ω. C_1 is, therefore, accessible from Ω.

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[3] N^3 is the neighborhood from whose boundary the arcs are drawn through Ω_2 to the vicinity of S.