NOTE ON MAXIMAL ALGEBRAS

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Introduction. It has been shown in a previous paper [3][1] that every algebra A with radical R, such that A/R is separable, is a homomorphic image of a certain maximal algebra which is determined to within an isomorphism by A/R, the A/R-module (two-sided) R/R^2, and the index of nilpotency of R. Furthermore, some indication was given of how the structure of maximal algebras can be determined in simple cases.

Here, we wish to give a further illustration by describing a rather wide class of maximal and primary algebras whose structure will be shown to resemble that of crossed products, in certain respects. In fact, we shall impose a certain normality condition and then trace the consequences of a few simple facts of the noncommutative Galois theory.

An algebra B over the field F, with radical R, is called primary if it has an identity element and if B/R is simple. As is well known,[2] B is then isomorphic with a Kronecker product $F_m \times C$, where F_m denotes the full matrix algebra of degree m over F, and where C is completely primary, in the sense that it has an identity element and that the quotient of C by its radical is a division algebra over F. We are concerned with primary algebras B for which this division algebra (which is determined to within an isomorphism by B) is normal over F, in the sense of the noncommutative Galois theory.[3] This will be the case if and only if the center Z of B/R is a separable normal extension field of F and every automorphism of Z over F is induced by an automorphism of B/R.

A completely primary algebra C with radical S will henceforth be called quasinormal if C/S is normal over F. If ϕ is an isomorphism of $F_m \times C$ onto B then ϕ maps the radical $F_m \times S$ onto the radical R of B, and B/R is isomorphic with $F_m \times C/S$. Therefore, if C is quasinormal over F, then B/R is automatically separable over F. By [3], B has then a maximal related extension B^* which is evidently primary. Moreover, it is easily seen that $B^* \approx F_m \times C^*$, where C^* is the maximal related extension of C, and is quasinormal over F. Finally, the natural extension to B^* of a homomorphism of C^* onto C is a homomorphism of B^* onto B. From these facts it is evident

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] See, for instance, [2, chap. 2, §9].

[3] See [1] and [5].

that, without loss of generality, we may confine our attention to quasinormal algebras.

1. **Representation in the radical.** Let B be quasinormal over F, and let R be its radical. By Wedderburn's theorem and the quasinormality we have $B=A+R$, where A is a normal division algebra over F. Moreover, it is easy to see that the identity element of A coincides with the identity element of B. Hence, if R is regarded in the natural fashion as a two-sided A-module, the identity element of A acts as the identity operator on either side.

As in [3, §4], we make a direct decomposition of the A-module R in the form $R=T+R^2$. The submodule T (which is isomorphic with the A-module R/R^2) is decomposed further into a direct sum of (two-sided) simple submodules. Now it is easily seen that every simple A-module over F is an operator homomorphic image of the Kronecker product $A\times A$ with respect to F, regarded in the natural fashion as a two-sided A-module. By applying either the noncommutative Galois theory, or the classical representation theory for simple algebras in conjunction with the theory of field composites (for $Z\times Z$, where Z is the center of A), we conclude that the simple components of the A-module $A\times A$ are of dimension 1 over A.[4] It follows that every simple A-module is of dimension 1 over A, and if u is a generator we have $u\cdot a=\sigma\{a\}\cdot u$, for every $a\in A$, where σ is an automorphism of A over F. Two such simple modules are operator isomorphic if and only if the corresponding automorphisms differ only by an inner automorphism of A.

It follows that there is a set of automorphisms $\sigma_1, \cdots, \sigma_s$ (not necessarily distinct) of A over F which are determined uniquely up to inner automorphisms,[5] and a corresponding decomposition, $T=A\cdot u_1+\cdots+A\cdot u_s$ of T into simple submodules, such that, for each index i and every $a\in A$, $u_i\cdot a=\sigma_i\{a\}\cdot u_i$.

2. **The standard maximal algebras.** Let B be as in §1. We construct a maximal related extension of B from A and T by the process described in [3, §4]. If B is already maximal the resulting "standard algebra" will be isomorphic with B.

The radical of the algebra to be constructed is the linearly direct sum of the "powers" $T^{(k)}$, $k=1, \cdots, n$, where n is the index of nil-

[4] A proof of this, in a terminology close to the present one, will be found in [4, §4].

[5] This because the simple components of a semisimple module are unique to within an automorphism of the whole module, as follows, for instance, from the evident fact that they are isomorphic images of the factors in a composition series.

potency of R. We have $T^{(1)}=T$, and generally $T^{(k+1)}$ is the Kronecker product $T^{(k)}\times T$, which means that in the elementary products $u\times v$, where $u\in T^{(k)}$ and $v\in T$, we identify $u\cdot a\times v$ with $u\times a\cdot v$ for every $a\in A$, and define the module operations such that $a\cdot(u\times v)=a\cdot u\times v$ and $(u\times v)\cdot a=u\times v\cdot a$. There is a natural multiplication $(T^{(k)}, T^{(l)})\to T^{(k+l)}$, for $k+l\leqq n$, and $(T^{(k)}, T^{(l)})\to(0)$ for $k+l>n$, by means of which the multiplication in the standard algebra is defined.

It is easy to see that the partial products $A\cdot u_i\times A\cdot u_j$ are simple A-modules and may be represented in the form $A\cdot u$, with $u\cdot a=\sigma_i\sigma_j\{a\}\cdot u$, for every $a\in A$. From this and §4 of [3], it can be seen immediately that the standard algebra may be described as follows:

Its elements are polynomials of total degree not greater than n, with coefficients in A, in the freely noncommuting "variables" $u_1, \cdots, u_s$. The elements are multiplied like ordinary polynomials, subject to the commutation rules $u_ia=\sigma_i\{a\}u_i$, and all the terms of degree greater than n are omitted. We denote this algebra by $A(u_1, \cdots, u_s)_n$. Our result is the following:

THEOREM. *Every quasinormal maximal algebra B with radical R is isomorphic with a standard "polynomial algebra" $B/R(u_1, \cdots, u_s)_n$, where s is the number of simple components of the B/R-module R/R^2, and n is the index of nilpotency of R. To each u_i there belongs an automorphism σ_i of B/R over F such that $u_ia=\sigma_i\{a\}u_i$ for every $a\in B/R$. These automorphisms are determined by B to within inner automorphisms of B/R. Two standard algebras with the same coefficient rings B/R are isomorphic if and only if their autmorphisms σ_i and τ_i can be so indexed that, for each i, $\sigma_i\tau_i^{-1}$ is an inner automorphism of B/R.*

As in [3, §6], we obtain the following corollary:

COROLLARY. *If B is quasinormal and if R/R^2 is a simple B/R-module, then B is maximal and is isomorphic with a standard algebra $B/R(u_\sigma)_n$. Two such algebras are isomorphic if and only if the corresponding automorphisms differ only by an inner automorphism of B/R.*

In general, every primary algebra satisfying the normality condition laid down in the introduction is a homomorphic image of an algebra of the type $F_m\times A(u_1, \cdots, u_s)_n$, the kernel of the homomorphism consisting of "polynomials" in which the "constant" terms and the terms of degree 1 in the u_i's are absent.

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ON THE RADICAL OF A LIE ALGEBRA

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Let $\mathfrak{L}$ be a Lie algebra over a field K of characteristic zero. For any $X \in \mathfrak{L}$ we denote, as usual, the linear mapping $Y \rightarrow [X, Y]$ of $\mathfrak{L}$ into itself by ad X. Let Γ be the radical of $\mathfrak{L}$. Consider the set $\mathfrak{N}$ consisting of all $N \in \Gamma$ such that ad N is nilpotent. It was shown in a recent paper[1] that $\mathfrak{N}$ is the unique maximal nilpotent ideal[2] of $\mathfrak{L}$. Further if D is a derivation of Γ then $D\Gamma \subset \mathfrak{N}$.

For any $X, Y, Z \in \mathfrak{L}$ put $B(X, Y) = sp(\text{ad } X \text{ ad } Y)$ and $T(X, Y, Z) = sp(\text{ad } [X, Y] \text{ ad } Z)$. Then $B(X, Y)$ is a symmetric bilinear form on $\mathfrak{L}$ while $T(X, Y, Z)$ is a skewsymmetric trilinear form. It is easily verified that they are both invariant under all derivations of $\mathfrak{L}$, that is,

$$B(DX, Y) + B(X, DY) = 0,$$
$$T(DX, Y, Z) + T(X, DY, Z) + T(X, Y, DZ) = 0$$

for any derivation D and $X, Y, Z \in \mathfrak{L}$.

An ideal $\mathfrak{M}$ in $\mathfrak{L}$ is called characteristic if $D\mathfrak{M} \subset \mathfrak{M}$ for every derivation D of $\mathfrak{L}$. Our first theorem may now be stated as follows:

THEOREM 1. *An element X of $\mathfrak{L}$ belongs to the radical Γ if and only if $T(X, Y, Z) = 0$ for all $Y, Z \in \mathfrak{L}$.*[3]

As an immediate corollary we get the following:

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[1] Ann. of Math. vol. 50 (1949) p. 68.

[2] My attention has been drawn to a paper by Malcev (Bull. Acad. Sci. URSS. vol. 9 (1945) pp. 329–356) where it is shown that $\mathfrak{N}$ is an ideal.

[3] Since $T(X, Y, Z) = -T(Z, Y, X)$ this condition is clearly equivalent to $B(X, Y) = 0$ for all $Y \in \mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}]$. Professor Jacobson has kindly brought it to my notice that this theorem is contained in Cartan's thesis p. 109.