A PROBLEM OF P. A. SMITH

HSIEN-CHUNG WANG

In a paper in this Bulletin,[1] P. A. Smith has mentioned the problem whether in a non-abelian Lie group G there exists a non-countable proper subgroup everywhere dense in G. We can see that a negative answer to this problem is unlikely as the non-existence of such group implies the well known continuum hypothesis. It is the aim of the present short note to show that each separable, locally compact, nondiscrete metric group has a subgroup possessing the above properties.

Let G be an abstract group and S a subset of G. The least subgroup of G which contains S will be called the *group closure* of S and denoted by $\mathrm{gcl}(S)$. Evidently, $\mathrm{gcl}(S)$ consists of all the finite products of the elements of S and their inverses. It follows immediately that $S=\mathrm{gcl}(S)$ if and only if S forms a subgroup of G.

Suppose R to be a subset of G and p an element of G such that p does not belong to the group closure $\mathrm{gcl}(R)$ of R. Using Zorn's Theorem[2] which is equivalent to the Axiom of Choice, we can construct a subset H of G having the following properties:

(i) H forms a subgroup of G;

(ii) H contains R but does not contain p;

(iii) if t is an element of G not belonging to H, then p is contained in the group closure $\mathrm{gcl}(H\cup t)$ of the union $H\cup t$. In general, there exists more than one such subgroup H. We shall call each of them *a maximal subgroup including R but excluding p*.

Now let us consider a separable, locally compact, nondiscrete metric group G. We choose, in G, a countable everywhere dense subset R. The group closure $\mathrm{gcl}(R)$ of R is also countable. However, the group G, being nondiscrete, is a perfect space. Therefore, G must be non-countable, for otherwise it would be homeomorphic with the set of all rational numbers[3] which is not locally compact. It follows that there exists an element p of G which does not belong to $\mathrm{gcl}(R)$. We can construct a maximal subgroup H including R but excluding p. Evidently, H forms a proper, everywhere dense subgroup of G.

We shall show that H is non-countable. For this purpose, let us

Received by the editors October 27, 1948.

[1] P. A. Smith, *Everywhere dense subgroups of a Lie group*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 309–312.

[2] S. Leftschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942, p. 5.

[3] W. Sierpinski, *Introduction to general topology*, Toronto, 1934, p. 107.

consider equations of the form

$$p = z^{m_1} h_1 z^{m_2} h_2 \cdots h_s z^{m_{s+1}} \tag{1}$$

where m_i $(i=1, 2, \cdots)$ denote integers, h_i elements in H, and z the unknown. The set C of solutions of such an equation is closed in G, and from the maximal property of H, C belongs to the complement $G-H$ of H. Since H is everywhere dense, C is nowhere dense.

Suppose H to be countable. Then the aggregate of all equations of the form (1) is countable as well. Thus the union $Z=\bigcup C$ of all the possible C's is a set of the first category. Moreover, we can easily see that Z coincides with the complement $G-H$ of H. On the other hand, H is countable and G perfect, so that H is nowhere dense in G. It follows then that $G=H+Z$ is of the first category in itself. This contradicts the fact that G is a locally compact metric space. Therefore, H cannot be countable and we arrive at the following:

In any separable, locally compact, non-discrete metric group G, there always exists a non-countable proper subgroup filling G densely.[4]

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA

[4] The author wishes to express his sincere thanks to Professor D. Montgomery for his valuable suggestions which enable the author to prove the theorem under much weaker condition for G than a Lie group.