LOCAL CONNECTION IN LOCALLY COMPACT SPACES

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It was proved by Hurewicz[1] that a compact space which is both LC^1 and lc^n is LC^n. In the present paper the corresponding result for locally compact spaces is proved, (a) for uniform local connection, and (b) for relative local connection.[2] The extension of Hurewicz's theorem to locally compact spaces is included in (b). The main difficulty in extending Hurewicz's methods is that his "Satz 6," on the passage from ϵ-homotopy to true homotopy, cannot be carried over to locally compact spaces without substantial modification, even when uniform local connection is assumed. To overcome this a stronger form of the lc^p and LC^p conditions is used, namely (for lc^p), the existence of a function $\zeta(\delta, x)$ such that, given a compact set F in the neighbourhood $U(x, \zeta(\delta, x))$ of any point x, there is a compact subset F' of $U(x, \delta)$ such that every q-cycle in F bounds in F', for $0 \leqq q \leqq p$; and analogously for LC^p. It is shown that these are equivalent to the ordinary lc^p and LC^p properties in locally compact (metric) spaces.

1. **Definitions.** It is assumed once for all that the containing space X is locally compact, and has metric ρ.[3] Homologies are relative to integral coefficients; cycles in X are Vietoris-cycles (but finite cycles are chains on some simplicial complex with vertices in X). The statement that Γ bounds in E means that Γ bounds in a compact subset of E. p denotes an integer not less than 0.

The letter F, with various suffixes, always denotes a compact set. If G is open, the statement "$F \subseteq G$ with a margin α" means that $\alpha > 0$, and $\mathrm{Cl}(U(F, \alpha))$ is compact and contained in G.[4] The existence of margins for every such F and G is ensured by the local compactness of X.

A set E_1 is *ac^p rel. E_2* ("acyclic up to p rel. E_2") if every q-cycle in E_1 bounds in E_2, for $0 \leqq q \leqq p$. E_1 is *as^p rel. E_2* ("aspherical up to p rel. E_2") if every mapping of the q-sphere S^q into E_1 is null-homotopic in E_2, for $0 \leqq q \leqq p$. The set E_1 is *strongly* ac^p (or *strongly* as^p) rel. E_2 if, given any F in E_1 there is an F' in E_2 such that F is ac^p (or as^p) rel. F'.

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[1] Hurewicz [4], denoted hereafter by H. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Problem 4 of Eilenberg and Wilder [3] is thereby settled affirmatively.

[3] Use is made at one point (Theorem 1, (B)) of a local separability condition.

[4] Cl. (X) denotes the closure of X.

Among the various definition of lc^p and ulc^p that are in the field we choose those which impose the heaviest conditions on the bounding cycles, and therefore the lightest conditions on the space. If $E_1 \subseteq E_2 \subseteq X$, E_1 is lc^p rel. E_2 if there is a positive function $\eta(x, \delta)$ such that $E_1 U(x, \eta(x, \epsilon))$ is ac^p rel. $E_1 U(x, \epsilon)$ for all points x of E_2 and any positive ϵ;[5] and X is (absolutely) lc^p if it is lc^p rel. X. The space X is ulc^p if there is a positive function $\eta(\epsilon)$ such that for all points x of X, $U(x, \eta(\epsilon))$ is ac^p rel. $U(x, \epsilon)$.

2. **Homology.** We consider *chain-realisations* of (abstract simplicial) complexes, in the sense of Lefschetz [7] and Begle [1]. The complexes realised all have their vertices in X, and every realisation t of a complex K is to satisfy $t(\sigma^0) = \sigma^0$, for all vertices σ^0 of K. If C is a finite chain on a complex in X, a *realisation of* C means a chain $t(C)$, where t is a realisation of the carrier complex $\|C\|$.[6]

If X is connected, a realisation of any complex can, by an arbitrarily small displacement of the vertices, be so modified that accidental clashes are avoided, that is, any common vertex of $\|t(\sigma)\|$ and $\|t(\sigma')\|$ belongs to some $\|t(\sigma'')\|$, where σ'' is a common sub-cell of σ and σ'. It will be assumed that this is always arranged. There is, then, for any vertex x of $\|t(C^p)\|$, a unique simplex σ of lowest dimension in $\|C^p\|$, such that $x \in \|t(\sigma)\|$. This σ may be called the C^p-carrier of x.

THEOREM 1. *If X is lc^p and $\alpha > 0$, and if F is a compact subset of X, there exists a finite set of p-cycles $\Gamma_1^p, \Gamma_2^p, \cdots, \Gamma_k^p$ in $U(F, \alpha)$ such that every Γ^p in $F \sim \sum_{i=1}^k n_i \Gamma_i^k$ in $U(F, \alpha)$, for suitable integers n_i.*[7]

Since X is 0-lc its components are open sets, and therefore the compact set F meets only a finite number of them. It is clearly sufficient to prove the theorem for each separate component meeting F, that is, we may assume X to be connected.

The theorem is proved by combining the following results.

(A). *Given a compact set F_1 there exists a positive function $\lambda(\epsilon)$*

[5] Cf. Eilenberg and Wilder [3] for the corresponding homotopy property.

[6] Note that the definition of a *partial* realisation t of K requires $t(\sigma^0)$ to be defined for all 0-cells σ^0 of K. The *norm* of a full or partial realisation t of K is max $\rho(x, y)$ for $x \in \|t(\sigma_i)\|$, $y \in \|t(\sigma_j)\|$, where σ_i and σ_j are subcells of the same cell σ of K. The *mesh* of t is max $\Delta\|\sigma'\|$ for all simplexes σ' of chains $t(\sigma)$. (ΔE = diameter of E.)

[7] Wilder [8] (see also Begle [2, Corollary 2.3]) has proved that when homologies are mod m, the conditions of Theorem 1 imply that at most a finite number of cycles of F are independent in $U(F, \alpha)$. The analogous result with integral coefficients is not strong enough for present purposes, since it would allow, for example, an infinite base (Γ_i^p) with $\Gamma_i^p \sim 2\Gamma_{i+1}^p$.

such that any partial chain-realisation t_0 *of a complex of dimension not greater than* $p+1$ *of norm less than* $\lambda(\epsilon)$ *in* F_1 *can be extended to a full realisation in* X *of norm less than* ϵ; *and there is a positive function* $\kappa(\delta, \epsilon)$ *such that if mesh* $t_0 < \kappa(\delta, \epsilon)$, *mesh* t *can be made less than* δ.

This is Begle [2], Theorem 2.1.[8] (Note that when t_0 is defined only for vertices the κ-condition is automatically satisfied.)

The α of the enunciation of Theorem 1 may be supposed such that $\mathrm{Cl}(U(F, \alpha))$ is compact. Let $\sum_0^\infty \epsilon_n$ be a positive series with sum less than α, such that $\epsilon_{n+1} < \epsilon_n/3$ and, taking $F_1 = \mathrm{Cl}(U(F, \alpha))$ in (A), $\epsilon_{n+1} < \lambda(\epsilon_n)$.

(B) *Every finite cycle* C_0^p *in* F *of mesh less than* ϵ_1 *is the first member of a projection-cycle* $\{C_n^p\}$ *in* $U(F, \alpha)$, *the projection* ϕ_n: $C_{n+1}^p \to C_n^p$ *being an* ϵ_n-*projection.*[9]

Let $\eta_n = \sum_0^n \epsilon_r$, and make the inductive hypothesis that a finite cycle C_n^p of mesh less than ϵ_{n+1} is defined in $U(F, \eta_{n-1})$ (for $n=0$, in F). Since $U(F, \eta_{n-1}) \subseteq U(F, \alpha) \subseteq F_1$, there is by (A) a chain-realisation C_{n+1}^p of C_n^p, of mesh less than ϵ_{n+2} and norm less than ϵ_n, and hence contained in $U(F, \eta_{n-1} + \epsilon_n) = U(F, \eta_n)$. This justifies the recursive definition of C_n^p. For each vertex x of C_{n+1}^p take $\phi_n(x)$ to be any vertex of the C_n^p-carrier of x (defined above). Then ϕ_n is an ϵ_n-projection.

Let the sequence (ϵ_n) satisfy the conditions of (B), and also $\epsilon_{n+3} < \kappa(\epsilon_{n+2}, \epsilon_n)$ (F_1 being as before).

(C) *Let* $\Gamma_r^p = (Z_{rn}^p)$ *be, for* $r = 1, 2$, *a* p-*cycle in* X. *Sufficient conditions for* $\Gamma_1^p \sim \Gamma_2^p$ *in* $U(F, \alpha)$ *are*

$$(1) \qquad Z_{rn}^p \sim_{\epsilon_{n+3}} Z_{r,n+1}^p \quad in \quad U(F, \eta_n) \qquad for\ n \geqq 0$$

and

$$(2) \qquad Z_{10}^p \sim_{\epsilon_2} Z_{20}^p \quad in \quad U(F, \epsilon_0).$$

Let $Z_{r,n+1}^p - Z_{r,n}^p = \beta Y_{rn}^{p+1}$ (=boundary of Y_{rn}^{p+1}), where Y_{rn}^{p+1} is a chain of mesh less than ϵ_{n+3} in $U(F, \eta_n)$ (whence mesh $Z_{rn}^p < \epsilon_{n+3}$); and let $Z_{10}^p - Z_{20}^p = \beta D_0^{p+1}$, where D_0^{p+1} is of mesh less than ϵ_2, and $D_0^{p+1} \subseteq U(F, \epsilon_0)$. Assume inductively that for some $n \geqq 0$, a finite chain D_n^{p+1} has been defined so that $\beta D_n^{p+1} = Z_{1n}^p - Z_{2n}^p$, mesh $D_n^{p+1} < \epsilon_{n+2}$, and $D_n^{p+1} \subseteq U(F, \eta_n)$. Then $Q_n^{p+1} = D_n^{p+1} + Y_{1n}^{p+1} - Y_{2n}^{p+1}$ is a $(p+1)$-chain with boundary $Z_{1,n+1}^p - Z_{2,n+1}^p$, and $\|Q_n^{p+1}\| \subseteq U(F, \eta_n) \subseteq F_1$. A partial realisation, t_0, of $\|Q_n^{p+1}\|$ is determined by putting $t_0(\sigma) = \sigma$ if[10] $\sigma \in \|Q_n^0\|$

[8] Our definitions are slightly different, but the proof is almost exactly similar.

[9] Cf. Begle [1, Lemma 2.4]. The property asserted of ϕ_n means that the projection-prism has mesh less than ϵ_n, and hence mesh $C_n^p < \epsilon_{n+1}$.

[10] $\|Q_n^0\|$ = set of vertices of $\|Q_n^{p+1}\|$, and in general K^m = set of cells of K of dimensions not greater than m.

$\cup\|Z^p_{1,n+1}\|\cup\|Z^p_{2,n+1}\|$. Mesh $t_0<\epsilon_{n+4}<\kappa(\epsilon_{n+3}, \epsilon_{n+1})$, and norm $t_0\leqq$ mesh $\|\phi^{p+1}_n\|<\epsilon_{n+2}<\lambda(\epsilon_{n+1})$. Hence t_0 can be extended to give a realisation D^{p+1}_{n+1} of Q^{p+1}_n, of mesh less than ϵ_{n+3}, and norm less than ϵ_{n+1}. Hence

$$D^{p+1}_{n+1}\subseteq U(F, \eta_n+\epsilon_{n+1}) = U(F, \eta_{n+1});$$

and

$$\beta D^{p+1}_{n+1} = t(\beta Q^{p+1}_n) = t(Z^p_{1,n+1}-Z^p_{2,n+1}) = Z^p_{1,n+1}-Z^p_{2,n+1}.$$

The recursive definition of D^{p+1}_n is justified, and (C) is therefore proved.

Proof of Theorem 1. Let $\{U(x_i, \epsilon_4/6\}$ be a finite covering of F, with $x_i\in F$, and let N be the nerve of the covering $\{U(x_i, \epsilon_4/2)\}$, with the points x_i as vertices. Choose a p-dimensional basis of homology $C^p_{(i)}$ $(i=1, 2, \cdots, k)$ of N. Since mesh $N<\epsilon_4$, (B) is applicable, with the series $(\epsilon_0, \epsilon_1, \cdots)$ replaced by $(\epsilon_3, \epsilon_4, \cdots)$, to give a projection-cycle Γ^p_i in $U(F, \alpha)$ with first member $C^p_{(i)}$. *The cycles* Γ^p_i *are the required set.* For let Γ^p be any p-cycle in F, and (Z^p_n) $(n=0, 1, \cdots)$ a subsequence of its members satisfying

$$\text{(a)}\qquad Z^p_{n+1}\sim_{\epsilon_{n+3}} Z^p_n \text{ in } F, \qquad\qquad \text{(b)}\qquad \text{mesh } Z^p_0<\epsilon_4/6.$$

If, for each x of F, $\theta(x)$ is a vertex of N in $U(x, \epsilon_4/6)$, and if x and y are vertices of the same cell of Z^p_0, $\rho(\theta x, \theta y)<\epsilon_4/6+\rho(x, y)+\epsilon_4/6<\epsilon_4/2$, and hence θ is a simplicial mapping of Z^p_0 into N.

Let $\theta(Z^p_0)=Z'^p\sim\sum_1^k n_iC^p_{(i)}$ in N, for some n_i. *The pair of cycles* Γ^p *and* $\sum_1^k n_i\Gamma^p_i$ *satisfy the conditions* (1) *and* (2) *of* (C). Let $\Gamma^p_i=\{C_{0i}, C_{1i}, C_{2i}, \cdots\}$. *Condition* (1). For Γ^p this follows from (a) above. The chain $\sum n_i\Gamma^p_i$ is a projection-cycle for which ϕ_n is an ϵ_{n+3}-projection, and all vertices of $\|\sum n_iC_{ni}\|$ belong to $\|\sum n_iC_{n+1,i}\|$.[11] Hence condition (1) is satisfied if $\|C_{n+1,i}\|\subseteq U(F, \eta_n)$. This is so, since $\|C_{n+1,i}\|\subseteq U(F, \sum_3^{n+3}\epsilon_r)$ (proof of (B), ϵ_{r+3} replacing ϵ_r), and $\sum_3^{n+3}\epsilon_r<\sum_0^n\epsilon_r=\eta_n$. *Condition* (2). The θ-prism joining to Z^p_0 Z'^p has mesh less than $\epsilon_4/2<\epsilon_2$, whence $Z^p_2\sim_{\epsilon_2}Z'^p$ in F; and $Z'^p\sim\sum n_iC^p_{(i)}$ in N, a complex of mesh less than ϵ_2 in F.

Theorem 2. *If X is lc^p, an open set G_1 which is ac^p rel. an open set G_2 is also strongly ac^p rel. G_2.*

Suppose $F\subseteq G_1$ with a margin α. For $0\leqq q\leqq p$, let $\Gamma^q_1, \Gamma^q_2, \cdots, \Gamma^q_{k_q}$ be a basis of q-cycles in $U(F, \alpha)$ constructed as in Theorem 1. Let Γ^q_i

[11] By the general rule that $t(\sigma^0)=\sigma^0$, above.

bound in the subset F_{qi} of G_2. Then every q-cycle in F bounds in the compact subset

$$F' = \mathrm{Cl}(U(F, \alpha)) \cup \bigcup_{q=0}^{p} \bigcup_{i=1}^{k_q} F_{qi}$$

of G_2.

3. **Homotopy.** The relation of homotopy is denoted by $\simeq$ and ϵ-homotopy[12] by $\simeq_\epsilon$. S^p is the sphere $\sum_0^p \xi_r^2 = 1$ in R^{p+1}; S^r is, for $r < p$, the intersection of S^{r+1} with $\xi_{r+1} = 0$; c_0 is the point $(1, 0, \cdots, 0)$ of R^{p+1}. A set E_1 is ϵ-as^p *rel.* E_2 if, for $0 \leqq q \leqq p$, every mapping $f: S^q \to E_1 \simeq_\epsilon 0$ in E_2 rel. c_0.

THEOREM 3. *Let the open set G be LC^{p-1} rel. X[13] and suppose that a positive function $\eta_0(\delta, x)$ exists with this property: to any point x, and any compact F in $GU(x, \eta_0(\delta, x))$ there corresponds a compact F' in $GU(x, \delta)$ such that F is ϵ-as^p rel. F' for every positive ϵ. Then $GU(x, \eta_0(\delta, x))$ is strongly as^p rel. $GU(x, \delta)$; and therefore G is LC^p rel. X. (The LC^{p-1} condition is vacuously satisfied if $p = 0$.)*

COROLLARY. *If $G = X$ and $\eta_0(\delta, x) = \eta_0(\delta)$, independent of x, X is p-ULC.*

This theorem replaces "Satz 6" of Hurewicz [4] in locally compact spaces. Although our proof follows his closely, the many changes of detail make it necessary to give the full proof, which depends on the following lemmas D, E, and F.

(D) ($p \geqq 1$) *Given a compact set F in a locally compact LC^{p-1} space X, and a positive number ϵ, there exists a positive $\eta_2 = \eta_2(\epsilon, F)$ with the following property: if P is a polyhedron, and Q a subpolyhedron[14] of P^{p-1}, and if f_0, f_1 map P into F and satisfy $\rho(t_0, t_1) < \eta_2$, there exists a mapping $f_0': P \to X$, agreeing with f_0 on Q, and deformable into f_1 within ϵ, in X.*[15]

The proof of (D) is omitted, since only obvious changes are needed in the proofs of H, Sätze 1–3.

Let G be as in Theorem 3. Let $F_0 \subseteq G$ with a margin α_0, and let $0 < \delta \leqq \alpha_0$. There exists a finite covering $\{U_i\} = \{U(x_i, \eta_0(\delta, x_i))/2\}$

[12] See H, p. 477: $f_1 \simeq_\epsilon f_2$ if f_1 and f_2 are connected by an ϵ-chain of points in the space of mappings.

[13] This is the homotopy-local-connection introduced by Lefschetz [5]. For relative local connection see Eilenberg and Wilder [3].

[14] No distinction is made in terminology between a polyhedral complex P and the polyhedron which is its locus; but the corresponding abstract simplicial complex determined by the vertices of P is to be distinguished from P. It is denoted by $\|P\|$.

[15] That is, f_0' and f_1 are connected in the space X^P by an arc of diameter less than ϵ.

of the compact set F_0, with $x_i \in F_0$. Since $\eta_0(\delta, x_i) \leqq \delta \leqq \alpha_0$, $\bigcup U_i \subseteq G$. Let $\eta_3 = \min_i \eta_0(\delta, x_i)$.

(E) ($p \geqq 1$) *If f_1, f_2 map the p-element E^p (with boundary S^{p-1}) into F_0, and if $f_1 | S^{p-1} = f_2 | S^{p-1}$ and $\Delta f_r(E^p) < \eta_3/2$, for $r = 1, 2$, then $f_1 \simeq_\epsilon f_2$ with fixed S^{p-1}, in $U(x_i, \delta)$, for some i.*

If $y \in f_1(S^{p-1})$, one of the points x_i satisfies $\rho(x_i, y) < \eta_0(\delta, x_i)/2$ and therefore $f_r(E^p) \subseteq U(x_i, \eta_3/2 + \eta_0(\delta, x_i)/2) \subseteq U(x_i, \eta_0(\delta, x_i))$ for $r = 1, 2$. Hence $f_1 \simeq_\epsilon f_2$ in $U(x_i, \delta)$, by the conditions of Theorem 3 and H, Satz 4.

We suppose that for each compact F_0 and each positive δ, such a covering $\{U_i\}$ is chosen, and denote the corresponding $\min_i \eta_0(\delta, x_i)$ by $\eta_3(\delta, F_0)$. Let $\mu(x, y)$ be defined as in H.[16]

(F) *Given any compact F_0 in G there exists $\eta_4(\delta, F_0) > 0$ such that if f_1, f_2 map S^p into F_0 and $\rho(f_1, f_2) < \eta_4(\delta, F_0)$, then $\mu(f_1, f_2) < \delta$.* (The distances $\rho(f_1, f_2)$ and $\mu(f_1, f_2)$ are in the function space X^{S^p}.)

First suppose $p > 0$. Let α be a margin of F_0 rel. G. We define:

$$\delta' = \min(\delta, \alpha), \qquad F_1 = \mathrm{Cl}\,(U(F_0, \alpha)),$$

$$\xi = \frac{1}{6}\eta_3(\frac{1}{4}\delta', F_1), \qquad \eta_4(\delta, F_0) = \eta_2(\xi, F_0),$$

the functions η_2, η_3 having the meanings given above. Let two mappings f_1, f_2 of S^p into F_0 be given, satisfying $\rho(f_1, f_2) < \eta_4(\delta, F_0)$. Let S^p be simplicially subdivided into a polyhedron Σ^p, so finely that $\Delta f_r(\sigma^p) < \xi$ for each (continuous) p-simplex σ^p of Σ^p, $r = 1, 2$. Then by (D), a mapping $f_1' : S^p \to X$ exists, agreeing with f_1 on Σ^{p-1}, and such that $f_2 \simeq f_1'$ within ξ in G. Hence $\mu(f_1', f_2) < \xi < \delta/2$. Also, since $\delta' \leqq \alpha$ $f_1'(S^p) \subseteq U(F_0, \alpha/2) \subseteq F_1$; and $\rho(f_1', f_2) \leqq \mu(f_1', f_2) < \xi$. Therefore $\Delta f_1'(\sigma^p) < 3\xi = \eta_3(4^{-1}\delta', F_1)/2$ for every σ^p of Σ^p. By Lemma (E) it follows that $f_1' | \sigma^p \simeq_\epsilon f_1 | \sigma^p$, with fixed $\beta\sigma^p$, in a set of diameter $\delta/2$, for every positive ϵ. Since this holds for every σ^p of Σ^p, $\mu(f_1', f_1) < \delta/2$. Thus $\mu(f_1, f_2) \leqq \mu(f_1, f_1') + \mu(f_1', f_2) < \delta$.

If $p = 0$ let $\eta_4(\delta, F_0) = \eta_3(\delta', F_0)/2$. (The definition of η_3 remains significant when $p = 0$.) It is sufficient in this case to show that if $x, y \in F_0$ and $\rho(x, y) < \eta_4(\delta, F_0)$, then x and y are joined by an ϵ-chain of points of G, of diameter less than δ. There exists a point x_i such that $\rho(x, x_i) < \eta_0(\delta', x_i)/2$. Since $\rho(x, y) < \eta_3/2 \leqq \eta_0(\delta', x_i)/2$, x and y are both in $U(x_i, \eta_0(\delta', x_i)) \subseteq U(x_i, \alpha) \subseteq G$. Hence the required chain exists, by the conditions of Theorem 3.

Theorem 3 can now be proved. Let a positive δ, a point x of X, and a compact F in $GU(x, \eta_0(\delta, x))$ be given, and let F' be a set as in the

[16] In any metric space R, $\mu(x, y)$ is the greatest lower bound of diameters of subsets of R in which x and y are ϵ-connected for every positive ϵ.

enunciation of Theorem 3. Then F *is* as^p *rel.* $F_2=\mathrm{Cl}(U(F',\beta))$, where β is any margin of F' rel. $GU(x, \delta)$. Let $\delta_n=\eta_4(\beta/2^n, F_2)$, let x_0 be a point of F, and let f map S^p into F. By the conditions of the theorem, the points f and $[x_0]$ (=constant function x_0) of F^{S^p} are joined by a δ_1-chain, L_1, of points, all lying in F'^{S^p}. Assume inductively that $L_1, \cdots, L_n$ have been determined, L_r being formed by joining each consecutive pair of points of L_{r-1} by a δ_r-chain; and that the mappings which are the "points" of L_n all map S^p into $U(F', \sum_1^{n-1}\beta/2^r) \subseteq F_2$ (into F' when $n=1$). Two consecutive points f_r, f_s of L_n satisfy $\rho(f_r, f_s)<\delta_n=\eta_4(\beta/2^n, F_2)$; and hence by Lemma (F), $\mu(f_r, f_s)<\beta/2^n$, that is, f_r and f_s can be joined by a δ_{n+1}-chain of total diameter less than $\beta/2^n$ in the function space. This justifies the recursive definition of L_n.

It now follows, exactly as in H, p. 481, that $\mathrm{Cl}(\bigcup_1^\infty L_n)$ is the locus of a continuous path joining f to $[x_0]$ in $F_2^{S^p}$. The proof of Theorem 3 is thus completed.

4. **Homology and homotopy.** Theorem 4, general case ($p\geqq 2$). *If G is LC^1 rel. X and lc^p rel. X there is a positive function $\zeta(\delta, x)$ such that $GU(x, \zeta(\delta, x))$ is strongly as^p rel. $GU(x, \delta)$ for all x of X.*

Case 0: *Put $p=0$ and omit "LC^1 rel. X and."*

Case 1: *Put $p=1$ and omit "and lc^1 rel. X."*

Immediate corollaries of this theorem are

THEOREM 4.1 ($p\geqq 2$). *If G is LC^1 and lc^p it is LC^p, all rel. X.*

(When $G=X$ this is the generalisation of Hurewicz's theorem to locally compact spaces.)

THEOREM 4.2 ($p\geqq 0$). *If G is LC^p rel. X there exists a positive function $\zeta(\delta, x)$ such that $GU(x, \zeta(\delta, x))$ is strongly as^p rel. $GU(x, \delta)$ for all x of X.*

PROOF OF THEOREM 4, CASE 0. Let $\eta(\delta, x)$ be such that $GU(x, \eta(\delta, x))$ is ac^0 rel. $GU(x, \delta)$, and let $F\subseteq GU(x, \eta(\delta, x))$. By Theorem 2 there is an F' in $GU(x, \delta)$ such that F is ac^0 rel. F'. This implies that for any positive ϵ any two points of F are joined by an ϵ-chain in F', that is, that F is ϵ-as^0 rel. F' for every positive ϵ. Thus the conditions of Theorem 3 (0) are satisfied, if η_0 is replaced by η.

PROOF OF THEOREM 4, CASE 1. This is contained in the following theorem.

THEOREM 5. *If X is LC^1, any open set G_1 which is as^1 rel. an open set G_2 is also strongly as^1 rel. G_2.*

Let $F \subseteq G_1$ with margin α_0. Then there exist positive α_1 and α_2 such that, for $r=0, 1$, if $x \in F$, $U(x, \alpha_{r+1})$ is as[1] rel. $U(x, \alpha_r/3)$. Let $\{U(x_i, \alpha_2/6)\}$ be a finite covering of F $(x_i \in F)$, and let N be the nerve of the covering $\{U(x_i, \alpha_2/2)\}$ realised in X, with the x_i as vertices. If then P is a polyhedron abstractly isomorphic with N, the mapping g_0 of P^0 into N^0 determined by the isomorphism can be extended to a mapping g_1 of P^1 into $U(F, \alpha_1/3)$.

Let f be a re-entrant path in F, that is, $f: \langle 0, 1\rangle \rightarrow F$ with $f(0)=f(1)=x_0=g_0(z_0)$, say; and let the points $0<\tau_1<\tau_2< \cdots <\tau_k=1$ divide f into sub-paths s_j of diameter less than $\alpha_2/6$. Now each $f(\tau_j)$ is in some $U(y_j, \alpha_2/6)$, where $y_j \in N^0$ $(y_0=y_k=x_0)$, and a path γ_j of diameter less than $\alpha_1/3$ therefore runs from y_j to $f(\tau_j)$. By the usual process of extruding "tails" the path $f=\sum s_j$ is deformable in $U(F, \alpha_1/3)$ into

$$\sum_{j=1}^{k} (\gamma_{j-1} + s_j - \gamma_j)$$

the $\sum$ and $+$-signs denoting the usual path-summation. Now $\rho(y_{j-1}, y_j) < \alpha_2/6 + \Delta s_j + \alpha_2/6 < \alpha_2/2$ and therefore y_{j-1}, y_j are the g_1-images of the ends of a 1-cell σ_j^1 of P^1. Since $g_1(\sigma_j^1)$ and $\gamma_{j-1}+s_j-\gamma_j$ are both in $U(y_j, \alpha_1)$, the path $\gamma_{j-1}+s_j-\gamma_j$ is deformable in $U(y_j, \alpha_0)$, with fixed end points, into the path $g_1|\sigma_j^1$. Thus $f \simeq g_1(s)$ in $U(F, \alpha_0)$, where s is a path on P with $s(0)=s(1)=z_0$.

Let $P_1, P_2, \cdots, P_l$ be the components of P, and let the paths $a_{1r}, a_{2r}, \cdots, a_{m_r r}$ in P_r^1 be representatives of a base of the fundamental group of P_r^1. The path s lies in one component, say P_r^1, and

$$g_1 s \simeq \pm g_1 a_{n_1 r} \pm g_1 a_{n_2 r} \pm \cdots \pm g_1 a_{n_k r}$$

on $g_1(P_r^1) \subseteq U(F, \alpha_1) \subseteq G_1$. By hypothesis, for each i and r, $g_1 a_{ir} \simeq 0$ in G_2, and therefore in a compact set F_{ir} in G_2. Hence

$$f \simeq 0 \text{ in } \mathrm{Cl}(U(F, \alpha_0)) \cup \bigcup_{1}^{l} \bigcup_{1}^{m_r} F_{ir}$$

a compact subset of G_2 independent of f.

Proof of Theorem 4, general case ($p \geqq 2$). We make the inductive assumption that 4($p-1$) is proved, and may therefore, by 4.1 ($p-1$) and 4.2 ($p-1$), assume that G is LC^{p-1} rel. X. Let $\zeta'(\delta, x)$ be the function corresponding to ζ in the dimension $p-1$, and let $\eta(\delta, x)$ be such that $GU(x, \eta(\delta, x))$ is ac^p rel. $GU(x, \delta)$ for every x. *Then* $\zeta(\delta, x)$ *may be put equal to* $\eta(\zeta'(\delta, x), x)$. This choice will be justified by Theorem 3 if it is shown that the condition of that theorem is satisfied, with η_0 replaced by ζ.

Let $F \subseteq GU(x, \zeta(\delta, x))$. By Theorem 2, there is an $F_1 \subseteq GU(x, \zeta'(\delta, x))$ (with a margin α) such that F is ac^p rel. F_1; and by Theorem 4$(p-1)$ there is an $F' \subseteq GU(x, \delta)$ such that $\mathrm{Cl}(U(F_1, \alpha))$ is as^{p-1} rel. F'. *This is the set F' required in Theorem* 3.

Since X is LC^{p-1} and F_1 is compact, there is[17] a positive function $\eta_1(\xi)$ such that, given a polyhedron P^p, and any subpolyhedron Q containing all its vertices, any mapping $f_0: Q \to F_1$ whose continuous norm[18] is less than $\eta_1(\xi)$ can be extended to a mapping $f_1: P^p \to X$ of continuous norm less than ξ. Let a mapping $f: S^p \to F$ and a positive $\epsilon < \alpha$ be given. Divide S^p simplicially into a polyhedron Σ^p, and let C^p be a fundamental p-cycle on $\|\Sigma^p\|$.[14] The simplicial division is to be so fine that (a) $\Delta f(\sigma) < \eta_1(\epsilon/2)$ for every (continuous) simplex σ of Σ^p; and (b) there exist an abstract complex K^{p+1} containing $\|\Sigma^p\|$ as a subcomplex, a chain C^{p+1} on K^{p+1} with boundary C^p, and a mapping $f_1: K^{p+1} \to F_1$ of mesh less than $\eta_1(\epsilon/2)$, with $f_1 | \Sigma^0 = f | \Sigma^0$. That this is possible follows from the definition of F_1. Let P^{p+1} be a polyhedron such that $\|P^{p+1}\| = K^{p+1}$ and Σ^p is a subpolyhedron of P^{p+1}. The combination of f in S^p and f_1 at the vertices of K^{p+1} determines a continuous mapping of the subpolyhedron $\Sigma^p \cup P^0$ of P^p into F_1, of continuous norm less than $\eta_1(\epsilon/2)$. It can therefore be extended to a mapping $g_1: P^p \to X$ of continuous norm less than $\epsilon/2$. Thus g_1 is a mapping into $U(F_1, \epsilon/2) \subseteq \mathrm{Cl}(U(F_1, \alpha))$. From the definition of F' it follows that if $y_0 = g_1(c_0)$[19] $g_1 | P^{p-1} \simeq [y_0]$ in F'. Hence[20] there exists a mapping $g_2: P^p \to F'$ such that $g_1 \simeq g_2$ in F', and $g_2 | P^{p-1} = [y_0]$.

From this point on, the proof that $f \simeq_\epsilon 0$ (rel. c_0) in F' proceeds exactly like the remainder of the proof in H (pp. 484 and 485) that $f \simeq_\epsilon 0$ (rel. x_0) in U. The proof of Theorem 4(p) is thereby completed.

From the definition of ζ it is clear that if ζ' and η are independent of x, so also is ζ. The case $G = X$ is then of most interest, and gives the following theorems.

THEOREM 6. *If X is ULC^1 and ulc^p it is ULC^p, if $p \geqq 2$; and if X is ulc^0 it is ULC^0.*

THEOREM 6.1. $(p \geqq 0)$. *If X is ULC^p there exists a positive function $\zeta(\delta)$ such that $U(x, \zeta(\delta))$ is strongly as^p rel. $U(x, \delta)$ for all x of X.*

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[17] Lefschetz [6, p. 120] = H, Satz (1a). The modifications needed to allow for X being only locally compact are obvious in view of the compactness of F_1.

[18] *Continuous norm of f_0* = least upper bound of $\rho(f_0(x), f_0(y))$ for x, y in the same cell of P^p.

[19] c_0 is the point $(1, 0, \cdots, 0)$ (cf. §3).

[20] H, Satz 2.

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