A NOTE ON CONVERGENCE IN AREA

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Let T_n: $x=x_n(u, v)$, $y=y_n(u, v)$, $z=z_n(u, v)$, $n=0, 1, \cdots$, be a sequence of continuous transformations from the unit square S: $0\leqq u, v\leqq 1$, into E_3, Euclidean 3-space. For each point p on the unit sphere U: $x^2+y^2+z^2=1$, $E_2(p)$ will denote the plane through the origin which is perpendicular to the radius joining the origin and p; T_p will denote the transformation which projects E_3 perpendicularly onto $E_2(p)$. Then T_pT_n is a sequence of continuous transformations from S into $E_2(p)$. Assume that T_n, $n=1, 2, \cdots$, converges uniformly on S to T_0 and that the Lebesgue area $A(T_n)$ of the F-surface (see [1, II.3.44][1]) determined by the transformation T_n is finite for $n=0, 1, \cdots$. The purpose of the present note is to call attention to the fact that recent results of Helsel [2] and Radó [2, 3] imply the following theorem:

THEOREM. *Under the assumptions stated above, a necessary and sufficient condition for $A(T_n)\to A(T_0)$ is that $A(T_pT_n)\to A(T_pT_0)$ for every position of the point p on U.*

This theorem is the analogue for the Lebesgue area of a new result on convergence in length established by Ayer and Radó [4].

PROOF. The necessity of the condition has been proved by Radó [3]. To establish the sufficiency of the condition, use will be made of the following formula for the Lebesgue area of the F-surface determined by T_n (see [2]):

$$A(T_n) = 2\left[\frac{1}{4\pi}\int\int_U A(T_pT_n)d\sigma\right], \tag{1}$$

$d\sigma$ being the area element on U. In [2] it is shown that the Lebesgue area $A(T_n)$ is equal to twice the integral mean value over U of the lower area $a(T_pT_n)$ of the flat F-surface determined by the transformation T_pT_n; however, the Lebesgue area $A(T_pT_n)$ is equal to the lower area $a(T_pT_n)$ (see [5], [1, V.2.58], and [6]) so (1) follows. The assumption that $A(T_pT_n)\to A(T_pT_0)$ for every point p on U implies, in view of (1), that $A(T_n)\to A(T_0)$ if termwise integration of the sequence $A(T_pT_n)$ is permissible. To show that such is the case, a uniform bound for the functions $A(T_pT_n)$ will be displayed. First ob-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

serve that

$$A(T_pT_n) \leqq A(T_n). \tag{2}$$

Also, by a fundamental result of Cesari [5],

$$A(T_n) \leqq A(T_{p_1}T_n) + A(T_{p_2}T_n) + A(T_{p_3}T_n), \tag{3}$$

where p_1, p_2, p_3 are any three points on U such that the radii joining these points to the origin are mutually perpendicular. Regarding p_1, p_2, p_3 as fixed, the relations $A(T_{p_i}T_n) \to A(T_{p_i}T_0)$, $i=1, 2, 3$, imply the existence of a constant K such that $A(T_{p_i}T_n) \leqq K$ for $n=0, 1, \cdots$ and $i=1, 2, 3$. Hence, from (2) and (3), $A(T_pT_n) \leqq 3K$ for all $p \in U$ and $n=0, 1, \cdots$, which completes the proof of the sufficiency.

Bibliography

1. T. Radó, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948.

2. R. G. Helsel and T. Radó, *The Cauchy area of a Fréchet surface*, Duke Math. J. vol. 15 (1948) pp. 159–167.

3. T. Radó, *On convergence in area*, Duke Math. J. vol. 16 (1949) pp. 61–71.

4. M. C. Ayer and T. Radó, *A note on convergence in length*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 533–539.

5. L. Cesari, *Caratterizzazione analitica delle superficie continue di area finita secondo Lebesgue*, Annali della Reale Scuola Normale Superiore di Pisa (2) vol. 10 (1941) pp. 253–294 and vol. 11 (1942) pp. 1–42.

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