A DERIVATION OF ZORAWSKI'S CRITERION FOR PERMANENT VECTOR-LINES[1]

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The classical hydrodynamic theorems of Helmholtz[2] state that in any motion of an inviscid, incompressible fluid subject to conservative extraneous force the vortex-tubes are material tubes whose strength is constant in space and time. They were extended to barotropic motions of compressible fluids by Kelvin[3] and Nanson.[4] Underlying these theorems of hydrodynamics are three purely kinematical questions:

1. What is a necessary and sufficient condition that the strength of the vector-tubes of a continuously differentiable vector field $\mathbf{c}$ be the same at all cross-sections?

2. Given a continuously differentiable velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$, what is a necessary and sufficient condition that the strength of the vector-tubes of a second continuously differentiable field $\mathbf{c}(\boldsymbol{r}, t)$ remain constant throughout the motion?

3. Given a continuously differentiable velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$, what is a necessary and sufficient condition that the vector-tubes of a second continuously differentiable field $\mathbf{c}(\boldsymbol{r}, t)$ be material tubes?

The answer to the first question was given by Kelvin,[5] prior to the work of Helmholtz, the required condition being

$$\operatorname{div} \mathbf{c} = 0. \tag{1}$$

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[1] This investigation was carried out under contract No. 53–47, *Mechanics of deformable continua*, from the Office of Naval Research to the Naval Ordnance Laboratory, while the authors were employed by the latter institution.

[2] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, J. Reine Angew. Math. vol. 55 (1858) pp. 25–55; reprinted in *Wissenschaftliche Abhandlungen von Hermann Helmholtz*, vol. 1, Leipzig 1882, pp. 101–134. See §2.

[3] W. Thomson, *On vortex motion*, Transactions of the Royal Society of Edinburgh vol. 25 (1869) pp. 217–260; reprinted *Mathematical and physical papers of Lord Kelvin*, vol. 4, Cambridge, 1910, pp. 13–66. See §§59–60(r).

[4] E. J. Nanson, *Note on hydrodynamics*, Messenger of Mathematics vol. 3 (1874) pp. 120–121. The derivations of Helmholtz and Nanson are reproduced and criticised in H. Lamb, *Hydrodynamics*, 6th ed., Cambridge, 1932, pp. 203–206.

[5] W. Thomson, *A mathematical theory of magnetism*, Philosophical Transactions of the Royal Society of London vol. 141 (1851) pp. 243–245; reprinted *Papers on electrostatics and magnetism*, §§432–523. See §74 of former, §513 of latter.

The answer to the second question was given by Zorawski,[6] the required condition being

$$\frac{D\mathbf{c}}{Dt} - \mathbf{c}\cdot\mathbf{grad}\, v + \mathbf{c}\operatorname{div} v = 0, \tag{2}$$

where D/Dt is the symbol of material differentiation. This result is really an immediate consequence of a formula given earlier by Lamb.[7] The answer to the third question was also given by Zorawski,[8] the required condition being

$$\mathbf{c}\times\left[\frac{D\mathbf{c}}{Dt} - \mathbf{c}\cdot\mathbf{grad}\, v + \mathbf{c}\operatorname{div} v\right] = 0, \tag{3}$$

which was implicit in the infinitesimal analysis of Helmholtz and Nanson. The purpose of the present note is to give a simple vectorial derivation of the criterion (3).

Let a single material line be given by $\boldsymbol{r}=\boldsymbol{r}(\theta, t)$, where θ is a parameter along the curve and t is the time; the same coordinate is associated with each material point at all times, so that θ and t are independent variables. Then

$$\begin{aligned}\frac{D}{Dt}\left(\frac{\partial \boldsymbol{r}}{\partial\theta}\times\mathbf{c}\right) &= \frac{\partial v}{\partial\theta}\times\mathbf{c} + \frac{\partial \boldsymbol{r}}{\partial\theta}\times\frac{D\mathbf{c}}{Dt},\\ &= \left(\frac{\partial \boldsymbol{r}}{\partial\theta}\cdot\mathbf{grad}\, v\right)\times\mathbf{c} + \frac{\partial \boldsymbol{r}}{\partial\theta}\times\frac{D\mathbf{c}}{Dt}.\end{aligned} \tag{4}$$

Now if at the instant $t=0$ the material curve in question is a vector-line of $\mathbf{c}$, we shall have

$$\frac{\partial \boldsymbol{r}}{\partial\theta}\times\mathbf{c} = 0, \quad \text{or} \quad \frac{\partial \boldsymbol{r}}{\partial\theta} = \lambda\mathbf{c}, \tag{5}$$

where λ is a scalar function. Then (4) becomes

[6] K. Zorawski, *Ueber die Erhaltung der Wirbelbewegung*, Bulletin de l'Académie des Sciences de Cracovie, Comptes Rendus (1900) pp. 335–342. Preferable proofs are given by G. Jaumann, *Die Grundlagen der Bewegungslehre von einem modernen Standpunkte aus*, Leipzig, 1905, see §383; J. Spielrein, *Lehrbuch der Vektorrechnung nach den Bedürfnissen in der technischen Mechanik und Elektrizitätslehre*, Stuttgart, 1916, see §29; E. Lohr, *Vektor- und Dyadenrechnung für Physiker und Techniker*, Berlin, 1939, Part III, chap. 5, §f; C. A. Truesdell and R. C. Prim, *Zorawski's kinematic theorems*, Naval Ordnance Laboratory Memorandum, 9354, 1947.

[7] H. Lamb, *Note on a theorem in hydrodynamics*, Messenger of Mathematics vol. 7 (1877–1878) pp. 41–42.

[8] Loc. cit.

$$\frac{D}{Dt}\left(\frac{\partial \boldsymbol{r}}{\partial \theta} \mathbf{X}\, \mathbf{c}\right) = \lambda \mathbf{c}\, \mathbf{X} \left[\frac{D\mathbf{c}}{Dt} - \mathbf{c}\cdot \mathbf{grad}\ \boldsymbol{v}\right]. \tag{6}$$

A necessary and sufficient condition that the material line remain a vector-line of $\mathbf{c}$ is that (5) hold at all times. Since $\mathbf{c}\mathbf{X}\mathbf{c}=0$, from (6) we may then deduce the necessity of (3). Conversely, if (3) be satisfied, then $\partial\boldsymbol{r}/\partial\theta\mathbf{X}\mathbf{c}$ at each point on the material line is a quantity initially zero whose time derivative is always zero, and hence itself remains zero, so that (3) is also sufficient.

Since (3) is a consequence of (2), but (2) is not generally a consequence of (3), in order for the vector-tubes of $\mathbf{c}$ to be of strength constant in time it is necessary, but not sufficient, for the vector-tubes to be material tubes. The criterion (1) has not been used in the deduction of (2) or (3); thus even if the vector-tubes are of strength constant in time at each material cross-section they are not generally of equal strength at all cross-sections.

If $\mathbf{c}=\mathbf{curl}\ \boldsymbol{v}$, (1) is satisfied and (2) and (3) become respectively[9]

$$\mathbf{curl}\ \boldsymbol{a} = 0, \tag{7}$$

$$\mathbf{curl}\ \boldsymbol{v}\ \mathbf{X}\ \mathbf{curl}\ \boldsymbol{a} = 0, \tag{8}$$

where $\boldsymbol{a}$ is the acceleration. It is easy to show that (7) is a necessary and sufficient condition that the circulation around an arbitrary closed material curve be constant throughout the motion. From (8) it follows that the constancy of circulation is sufficient but not necessary in order to ensure the permanence of the vector-tubes.

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[9] The formula (8) and an incorrect substitute for (7) were stated without proof by M. Levy, *L'hydrodynamique moderne et l'hypothèse des actions à distance*, Révue Générale des Sciences Pures et Appliquées vol. 1 (1890) pp. 721–728. Proofs are given by H. Poincaré, *Théorie des Tourbillons*, Paris, 1893, §§5–6, pp. 150–151. See also Jaumann, op. cit. §386; E. Vessiot, *Sur les transformations infinitésimales et la cinématique des milieux continus*, Bull. Sci. Math. (2) vol. 35, part 1, pp. 233–244, §4; P. Appell, *Traité de mécanique rationnelle*, vol. 3, 3d ed., Paris, 1921, chap. 25, ex. 5.