ON FAITHFUL REPRESENTATIONS OF LIE GROUPS

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Let G and H be two connected Lie groups and ϕ a continuous homomorphism of H into the group of automorphisms of G. Then we define a new group $G\times_\phi H$ as follows. The elements of $G\times_\phi H$ are pairs (g, h) $(g\in G,\ h\in H)$ and group multiplication is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1(\phi(h_1)g_2), h_1h_2).$$

Topologically $G\times_\phi H$ is taken to be just the Cartesian product of G and H. It is then easily proved that under this topology $G\times_\phi H$ is a Lie group. It is called the semidirect product of G and H under ϕ. The object of this note is to prove the following theorem.

THEOREM. *Let G be a connected, simply connected solvable Lie group and H a connected Lie group which has a faithful representation. Let ϕ be any continuous homomorphism of H into the group of automorphisms of G. Then $G\times_\phi H$ has a faithful representation.*

The special case of this theorem when H is semisimple is due to Cartan.[1]

Let R be the field of real numbers and K the field of either real or complex numbers. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$ and θ a representation of G over K of degree $\dot{d}$. Then we denote by $d\theta$ the representation of $\mathfrak{g}$ given by[2]

$$d\theta(x) = \lim_{t\to 0} \frac{\theta(\exp tX) - I}{t} \qquad (X \in \mathfrak{g})$$

where $t\in R$ and I is the unit matrix of degree $\dot{d}$. Let $GL(K, \dot{d})$ denote the group of all nonsingular matrices of degree $\dot{d}$ with coefficients in K. Any subgroup of $GL(K, \dot{d})$ will be called a linear group of degree $\dot{d}$. Let θ be the identity representation of a linear Lie group G with the Lie algebra $\mathfrak{g}$ so that $\theta(x)=x$ $(x\in G)$. Then $d\theta$ is a faithful representation of $\mathfrak{g}$ and $\exp d\theta(X)=\theta(\exp X)=\exp X$ for any $x\in\mathfrak{g}$. Hence we may identify $\mathfrak{g}$ with $d\theta(\mathfrak{g})$ under $d\theta$. $\mathfrak{g}$ can therefore be regarded as a linear Lie algebra. In particular the Lie algebra of $GL(K, \dot{d})$ then

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[1] Cartan, J. Math. Pures Appl. vol. 17 (1938) pp. 1–12. See also Malcev, C. R. (Doklady) Acad. Sci. URSS. vol. 40 (1943) pp. 87–89.

[2] For the precise definitions of the terms used in this paper see Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

consists of all matrices of degree $\dot{d}$ with coefficients in K. We denote it by $\mathfrak{gl}(K, \dot{d})$. Given any subalgebra $\mathfrak{h} \subset \mathfrak{gl}(K, \dot{d})$, by the linear Lie group generated by $\mathfrak{h}$ we mean the analytic subgroup of $GL(K, \dot{d})$ corresponding to $\mathfrak{h}$.

Let us call a matrix subtriangular if it has zeros on and below the the diagonal. First we state the following two well known lemmas.

LEMMA 1.[3] *Let $\mathfrak{N}$ be the Lie algebra of all subtriangular $\dot{d} \times \dot{d}$ matrices over K and let G be the linear Lie group generated by $\mathfrak{N}$. Then $X \rightarrow \exp X$ is a topological mapping of $\mathfrak{N}$ onto G. Also G consists of all matrices which have zeros below the diagonal and* 1 *everywhere on the diagonal.*

LEMMA 2.[4] *Let G be a connected, simply connected solvable Lie group. Then every analytic subgroup of G is closed and simply connected.*

From now on I adhere strictly to the notation of my paper, *Faithful representations of Lie algebras*,[5] which will be quoted as FRL.

LEMMA 3. *Let $\mathfrak{L}$, $\mathfrak{N}$, $\mathfrak{D}$ be as in Lemma* 1 *of* FRL. *We construct the faithful representation θ of $\mathfrak{L}+\mathfrak{D}$ as described there. Then for any $Y_1, \cdots, Y_s \in \mathfrak{L}$ and $D_1, \cdots, D_s \in \mathfrak{D}$,*

$$(1) \quad \exp \theta(Y_1) \cdots \exp \theta(Y_s) \neq I \text{ unless } \exp Y_1 \cdots \exp Y_s = I'',$$

$$(2) \quad \exp \theta(D_1) \cdots \exp \theta(D_s) = I$$

$$\text{if and only if } \exp D_1 \cdots \exp D_s = I',$$

where I, I', and I'' are unit matrices of suitable degrees.

If we use the notation of the proof of Lemma 1 of FRL, (1) follows immediately from the fact that $\mathfrak{A}/\mathfrak{X} \cong \mathfrak{A}^*/\mathfrak{X}^*$ where $\mathfrak{X}^* = \mathfrak{X}/\mathfrak{X}_0$. Now we prove (2). Put $\omega^*(X) = (\omega(X))^*$ for $X \in \mathfrak{L}$. Then it is easily proved by induction on s that for any $D \in \mathfrak{D}$

$$\{\theta(D)\}^s \omega^*(X) = \omega^*(D^s X), \qquad s \geqq 1.$$

Hence $(\exp \theta(D))\omega^*(X) = \omega^*((\exp D)X)$. Also since $\theta(D) = d_D^*$ is a derivation of $\mathfrak{A}^* = \mathfrak{A}/\mathfrak{X}_0$, $\exp \theta(D)$ is an automorphism of $\mathfrak{A}^*$. Put $Y_X = (\exp D_1) \cdots (\exp D_s)X$ $(X \in \mathfrak{L})$. Then if

$$\exp \theta(D_1) \cdots \exp \theta(D_s) = I,$$

$\omega^*(Y_X) - \omega^*(X) = 0$ for every $X \in \mathfrak{L}$. Hence $\omega(Y_X - X) \in \mathfrak{X}_0 \subset \mathfrak{X}$. Therefore $\pi\omega(Y_X - X) = Y_X - X = 0$. Since this is true for every X,

[3] Birkhoff, Ann. of Math. vol. 38 (1937) pp. 526–532.

[4] Chevalley, Ann. of Math. vol. 42 (1941) pp. 668–675.

[5] Harish-Chandra, Ann. of Math. vol. 50 (1949) pp. 68–76.

$(\exp D_1) \cdots (\exp D_s) = I'$. The converse is obvious. Hence the lemma is proved.

LEMMA 4. *Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{N}$ be the maximal nilpotent ideal of $\mathfrak{g}$. Then G has a faithful representation ψ such that $d\Psi(X)$ is nilpotent for every $X \in \mathfrak{N}$.*

By Corollary 1 of FRL we can find a faithful representation ρ_0 of $\mathfrak{g}$ such that $\rho_0(X)$ is nilpotent for every $X \in \mathfrak{N}$. We can therefore choose, if necessary, a new base in our representation space such that with respect to this base the matrix representing $\rho_0(X)$ is subtriangular for every $X \in \mathfrak{N}$. Now consider the factor algebra $\mathfrak{g}/\mathfrak{N}$ which is abelian and hence nilpotent. By the same corollary it follows that $\mathfrak{g}/\mathfrak{N}$ has a faithful representation by nilpotent matrices. Hence $\mathfrak{g}$ has a representation ρ_1 such that the kernel of ρ_1 is $\mathfrak{N}$ and $\rho_1(X)$ is nilpotent for all $X \in \mathfrak{g}$. We can again arrange that $\rho_1(X)$ is subtriangular for all X. Put $\rho = \rho_0 \dotplus \rho_1$, where $\dotplus$ denotes direct sum. Since G is simply connected, there exist representations ψ_0 and ψ_1 of G such that $d\psi_0 = \rho_0$, $d\psi_1 = \rho_1$. Put $\psi = \psi_0 \dotplus \psi_1$. Then $d\psi = \rho$. Let N be the analytic subgroup of G corresponding to $\mathfrak{N}$. Then from Lemma 2, N is a closed invariant subgroup. Consider $\psi(N)$. It is clear that $d\psi(Y) = \rho_0(Y) \dotplus \rho_1(Y)$ is subtriangular for all $Y \in \mathfrak{N}$. Hence from Lemmas 1 and 2 it follows that the linear Lie group $\psi(N)$ generated by $d\psi(\mathfrak{N})$ is simply connected. Since $d\psi$ is an isomorphism, $\psi(N)$ is locally isomorphic to N. Therefore since $\psi(N)$ is simply connected, ψ maps N isomorphically. Similarly we prove that $\psi_1(G)$ is simply connected. It is clear that the kernel of ψ_1 contains N. Hence ψ_1 defines a representation ψ^* of G/N given by $\psi^*(x^*) = \psi_1(x)$ where $x \to x^*$ is the natural homomorphism of G onto $G/N = G^*$. Since $d\psi^*$ is an isomorphism, the kernel of $d\psi_1 = \rho_1$ being $\mathfrak{N}$, it follows from the simple connectivity of $\psi^*(G^*) = \psi_1(G)$ that ψ^* is an isomorphism. Hence the kernel of ψ_1 is exactly N. Let D be the kernel of ψ. Then D is contained in the kernel of ψ_1 which is N. Also since ψ is faithful on N, $D \cap N = \{e\}$ where e is the unit element of N. Hence $D = \{e\}$ and ψ is a faithful representation. Also $d\psi(X)$ is nilpotent for every $X \in \mathfrak{N}$.

Now we come to the proof of the theorem. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{N}$ the maximal nilpotent ideal of $\mathfrak{g}$. By Lemma 4, G has a faithful representation. Hence we may assume that G is a linear Lie group such that every element of $\mathfrak{N}$ is nilpotent. We keep to the notation of Lemma 3 except that $\mathfrak{L}$ is replaced by $\mathfrak{g}$. Let $\mathfrak{h}$ be the Lie algebra of H. Define a homomorphism $d\tau$ of $\mathfrak{h}$ into $\mathfrak{D}$ as follows. Let

Aut (G) be the group of automorphisms of G. Then Aut (G) is a Lie group with a Lie algebra $\mathfrak{A}$. It is well known that there exists an isomorphism λ of $\mathfrak{A}$ onto $\mathfrak{D}$ such that

$$(\exp A) \exp X = \exp ((\exp \lambda(A))X)$$

for any $A \in \mathfrak{A}$ and $X \in \mathfrak{g}$. We put $d\tau = \lambda \circ d\phi$ where $d\phi$ is the homomorphism of $\mathfrak{h}$ into $\mathfrak{A}$ induced by ϕ. Then for any $P_i \in \mathfrak{h}$, $1 \leqq i \leqq r$,

$$\phi(\exp P_1 \cdots \exp P_r) \exp X = \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X).$$

Let e and e' denote the identity elements of G and H respectively. Suppose $\exp P_1 \cdots \exp P_r = e'$. Then clearly

$$\begin{aligned} \exp X &= \phi(\exp P_1 \cdots \exp P_r) \exp X \\ &= \exp ((\exp d\tau(P_1) \cdots \exp d\tau(P_r))X). \end{aligned}$$

Since this is true for every $X \in \mathfrak{g}$,

$$\exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'.$$

Hence we can define a representation τ of H by the rule

$$\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).$$

Put $d\psi = \theta \circ d\tau$. Then $d\psi$ is a representation of $\mathfrak{h}$. Suppose $\exp P_1 \cdots \exp P_r = e'$ $(P_1, \cdots, P_r \in \mathfrak{h})$. Then

$$\tau(\exp P_1 \cdots \exp P_r) = \exp d\tau(P_1) \cdots \exp d\tau(P_r) = I'$$

and from Lemma 3

$$\exp d\psi(P_1) \cdots \exp d\psi(P_r) = I.$$

Hence we can again define a representation ψ of H by putting

$$\psi(\exp P_1 \cdots \exp P_r) = \exp d\psi(P_1) \cdots \exp d\psi(P_r) \quad (P_1, \cdots, P_r \in \mathfrak{h}).$$

Also since G is simply connected there exists a representation χ of G such that $d\chi(X) = \theta(X)$ for every $X \in \mathfrak{g}$. Hence

$$\chi(\exp Y_1 \cdots \exp Y_r) = \exp \theta(Y_1) \cdots \exp \theta(Y_r) \quad (Y_i \in \mathfrak{G}, 1 \leqq i \leqq r).$$

From Lemma 3 it follows that χ is faithful.

Consider the mapping μ of $G \times_\phi H$ defined by $\mu(g, h) = \chi(g)\psi(h)$. We claim that μ is a representation. For any $P \in \mathfrak{h}$ and $X \in \mathfrak{g}$ consider

$$\begin{aligned} \psi(\exp P)\chi(\exp X)(\psi(\exp P))^{-1} &= \exp d\psi(P) \exp \theta(X) \exp (-d\psi(P)) \\ &= \exp \theta(D) \exp \theta(X) \exp (-\theta(D)) \end{aligned}$$

where $D = d\tau(P)$. Now for any two elements $A, B \in \mathfrak{gl}(K, d)$,

$$\exp A \exp B \exp(-A) = \exp((\exp \operatorname{ad} A)B)$$

where ad A is defined as in FRL. Since $[\theta(D), \theta(Y)] = \theta([D, Y]) = \theta(DY)$ for any $Y \in \mathfrak{g}$, it follows immediately that

$$\begin{aligned}\exp \theta(D) \exp \theta(X) \exp(-\theta(D)) &= \exp \theta((\exp D)X)\\ &= \exp \theta(\tau(\exp P)X)\\ &= \chi(\exp \tau(\exp P)X)\\ &= \chi(\phi(\exp P) \exp X).\end{aligned}$$

Since any $h \in H$ can be written in the form $\exp P_1 \cdots \exp P_r$, $P_i \in \mathfrak{h}$, $1 \leqq i \leqq r$, $r \geqq 1$, we get

$$\psi(h)\chi(\exp X)(\psi(h))^{-1} = \chi(\phi(h) \exp X).$$

Similarly since every $g \in G$ can be written as $\exp Y_1 \cdots \exp Y_r$, $Y_i \in \mathfrak{g}$, $1 \leqq i \leqq r$, $r \geqq 1$, we have

$$\psi(h)\chi(g)(\psi(h))^{-1} = \chi(\phi(h)g).$$

Therefore

$$\begin{aligned}\mu((g_1, h_1)(g_2, h_2)) = \mu(g_1\phi(h_1)g_2, h_1h_2) &= \chi(g_1\phi(h_1)g_2)\psi(h_1h_2)\\ &= \chi(g_1)\chi(\phi(h_1)g_2)\psi(h_1)\psi(h_2)\\ &= \chi(g_1)\psi(h_1)\chi(g_2)\psi(h_2) = \mu(g_1, h_1)\mu(g_2, h_2).\end{aligned}$$

Since μ is clearly a continuous mapping it is a representation of $G \times_\phi H$. By hypothesis H has a faithful representation ν_0. Define a representation ν of $G \times_\phi H$ by $\nu(g, h) = \nu_0(h)$ and put $\xi = \mu \dotplus \nu$. Suppose (g, h) belongs to the kernel of ξ. Then since $\xi(g, h) = \mu(g, h) \dotplus \nu_0(h) = \chi(g)\psi(h) \dotplus \nu_0(h)$ and since ν_0 is faithful on H, $h = e'$. Hence g belongs to the kernel of χ. But as χ is faithful on G, $g = e$. Therefore $(g, h) = (e, e')$ and ξ is faithful on $G \times_\phi H$.

COROLLARY (MALCEV).[1] *A connected solvable Lie group G has a faithful representation if and only if $G = NA$, where N is a closed, connected, simply connected invariant subgroup and A is a connected, compact abelian subgroup such that $N \cap A = \{e\}$.*

Suppose $G = NA$. For any $a \in A$ let $\phi(a)$ denote the automorphism of N given by $\phi(a)n = ana^{-1}$ $(n \in N)$. Then it is easily seen that $(n, a) \rightarrow na$ is an isomorphism of $N \times_\phi A$ onto G. Since A is compact, it has a faithful representation. Hence by the above theorem it follows immediately that G has a faithful representation.

In order to establish the converse we make use of the following lemma which follows easily from the results of Chevalley.[4]

LEMMA 5.[6] *If G is a connected solvable Lie group and N a closed, connected invariant subgroup such that G/N is compact, then there exists a compact connected abelian subgroup A of G such that $G=AN$ and $A\cap N$ is finite.*

Returning to the corollary, suppose G is linear. Since $\mathfrak{g}$ is solvable, we deduce in the usual way that every element $X\in[\mathfrak{g}, \mathfrak{g}]=\mathfrak{g}'$ is nilpotent and therefore may be assumed to be subtriangular. Therefore by Lemmas 1 and 2 the group G' generated by $\mathfrak{g}'$ is simply connected. Let $\dot{d}$ be the degree of G and G_0 the group of all matrices in $GL(K, \dot{d})$ which have zero below the diagonal and 1 everywhere on the diagonal. By Lemma 2, G' is closed in G_0. However, since G_0 is clearly closed in $GL(K, \dot{d})$, G' is closed in $GL(K, \dot{d})$ and therefore in G. Let $x\to x^*$ denote the natural homomorphism of G onto $G/G'=G^*$. Since G^* is abelian, $G^*=T^*V^*$ where T^* and V^* are connected subgroups, T^* being compact and V^* simply connected and $T^*\cap V^*=\{e^*\}$. Let N be the complete inverse image of V^* in G. Since $N/G'=V^*$ and G' are both simply connected, N is simply connected and $G/N\cong T^*$ is compact. Therefore by Lemma 5, $G=AN$ where A is compact, connected, and abelian and $A\cap N$ is finite. Let $\sigma\in A\cap N$. Then $\sigma^r=e$ for some $r\geqq 1$. Then $\sigma^*\in V^*$ and $(\sigma^*)^r=e^*$. Since V^* is simply connected and abelian, $\sigma^*=e^*$. Hence $\sigma\in A\cap G'$. Since $X\to\exp X$ is a topological mapping of $\mathfrak{g}'$ onto G', it follows that $\sigma\in G'$, $\sigma^r=e$ implies $\sigma=e$. Hence $A\cap N=\{e\}$. The corollary is therefore proved.

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[6] This lemma was pointed out to me by Dr. G. D. Mostow.