A NOTE ON PEANO SPACES

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In three-dimensional space set up a cylindrical coordinate system (r, θ, z). The Hahn-Mazurkiewicz theorem characterizes Peano spaces (locally connected metric (compact) continua) as the continuous images of the closed unit interval I on the z-axis. In this note we obtain an extension theorem for Peano spaces (henceforth called P-spaces) based upon this characterization.

We first define a dendrite L. To this end order the rationals in the interior of I into a sequence $\{(0, 0, r_i\}$. For each pair of positive integers i, j, let $L_{i,j}$ denote the closed line segment joining $(0, 0, r_i)$ and $(1/i+j, 1/i+j, r_i)$. Let $L_{-1,j}$, $L_{0,j}$ denote line segments from $(0, 0, 0)$ and $(0, 0, 1)$ parallel to and the same length as $L_{1,j}$ for every j. Then the dendrite L is defined to be the union of I and all the segments $L_{i,j}$. Let $a_{i,j}$ be the end point of $L_{i,j}$ which is not on I. We shall refer to $a_{i,j}$ as the free end of $L_{i,j}$.

Denote by D the sequence $\{(0, 0, d_i)\}$ consisting of the dyadic rational points interior to I enumerated in the usual way: $d_1=1/2$, $d_2=1/2^2$, $d_3=3/2^2$, $\cdots$. The following lemma is then easily established.

LEMMA. *Let Q^* be any finite or countable subset of I. Then there exists a homeomorphism $h(I)=I$ such that $h(0)=0$, $h(1)=1$, $h(Q^*-0-1) \subset D$.*

The result of this note may now be stated as follows.

THEOREM. *Let N be a P-space properly contained in the P-space P, and let $g(I)=N$ be any continuous transformation. Then there exists a homeomorphism $h(I)=I$ and a continuous transformation $f(L)=P$ with the following properties:*

(i) *on I, $f=gh$*

(ii) $f^{-1}(P-N)\subset L-I$.

In addition there exists a subset L^* of L, consisting of the union of I and a certain subcollection $L^*_{i,j}$ of the line segments $L_{i,j}$ such that

(iii) $f(\overline{L^*-I})=\overline{P-N}$,

and

(iv) *if $a^*_{i,j}$ is the free end of $L^*_{i,j}$ for each i, j while A^* is the union of the points $a^*_{i,j}$, then $f^{-1}(P-N)=L^*-I-A^*$.*

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PROOF. The proof will be given only for the nontrivial case in which N is nondegenerate. Let $T(I)=P$ be any continuous transformation. Then the open set $G=T^{-1}(P-N)$ is the union of a collection of open intervals $\{\omega_i\}$ having the respective left and right end points $\{q_i\}$ and $\{t_i\}$. It is convenient to assume that this sequence of intervals has been enumerated in order of decreasing diameters.

For each integer i for which a point q_i has been defined we choose a point q_i^* in $g^{-1}T(q_i)$. It does not follow that $q_i \neq q_j$ implies $q_i^* \neq q_j^*$. However, there are at most a countable number of q_i which have the same q_i^*. We denote these by $q_{i,1}, q_{i,2}, \cdots$ in order of decreasing diameters of the corresponding ω_i. Thus $q_i^* = q_{i,j}^*$ for all i. Denote by Q^* the set of all points q_i^* thus defined, and let $k(I)=I$ be the homeomorphism given by the lemma, and $h(I)=I$ the homeomorphism given by $h=k^{-1}$. For each pair of integers i, j for which a point $q_{i,j}$ has been defined, let $\omega_{i,j}$ be the component of G which has $q_{i,j}$ as its left end point. Set $d_i^* = k(q_i^*)$ for each i. Then, for each pair of integers i, j for which a point $q_{i,j}$ has been defined, denote by $L_{i,j}^*$ the line segment of L which has the point d_i^* as its foot, and the point $a_{i,j}^*$ as its free end.

The desired transformation $f(L)=P$ is now defined as follows:

(a) $f(I)=gh(I)=N$. Thus (i) is satisfied.

(b) $f(L_{i,j})=f(d_i)$ for every $L_{i,j}$ which is not an $L_{i,j}^*$.

(c) $f(L_{i,j}^*)=T(\bar{\omega}_{i,j})$ in such a way that $f(d_i^*)$ agrees with the definition in (a), $f(a_{i,j}^*)=T(r_{i,j})$, and $f(L_{i,j}^*-d_i^*-a_{i,j}^*)=T(\omega_{i,j})$. Here, of course, $t_{i,j}$ denotes the right end point of $\omega_{i,j}$.

It is easily seen that the transformation $f(L)=P$ satisfies all of the required conditions.

In the important special case where N is a simple arc we may choose $g(I)=N$ as a homeomorphism and obtain the following corollary.

COROLLARY. *If N is a simple arc in P, then $f(L)=P$ may be so defined that $f(I)=N$ is topological.*

This theorem may be restated as a decomposition theorem for the P-space P, where each of the sets $f(L_{i,j}^*)$ is regarded as a P-space. Recent results of O. G. Harrold[1] should not be overlooked in the light of this theorem.

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[1] See O. G. Harrold, Duke Math. J. vol. 6 (1940) pp. 750–752. Also Bull. Amer. Math. Soc. vol. 48 (1942) pp. 561–566.