THE MAXIMAL REGULAR IDEAL OF A RING

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1. Introduction. An element a of a ring R is said to be *regular* if and only if there exists an element x of R such that $axa=a$. The ring R is *regular* if and only if each element of R is regular. The concept of a regular ring was introduced by von Neumann [5, 6][1] who, however, required also that a regular ring have a unit element.

Unless otherwise stated, the word *ideal* shall mean two-sided ideal, and an ideal in R will be said to be regular if and only if it consists entirely of regular elements of R. It is easy to see that a regular ideal A in R is itself a regular ring. For if $a \in A$, there exists an element x of R such that $axa=a$. It follows that $axaxa=a$ and $xax \in A$, so a is regular in the ring A.

We shall show that the join of all regular ideals in an arbitrary ring R is a regular ideal, and hence that there exists a unique maximal regular ideal $M=M(R)$ in R. The purpose of this note is to establish a few fundamental properties of $M(R)$. Among these are the following "radical-like" properties: (i) $M(R/M(R))=0$, (ii) if B is an ideal in R, then $M(B)=B\cap M(R)$, (iii) if R_n is the complete matrix ring of order n over R, then $M(R_n)=(M(R))_n$. A special case of this last result is that R_n is regular if and only if R is regular. This was proved by von Neumann [6], but we shall include a very simple proof of this fact.

It is well known that every regular ring has zero (Jacobson) radical J. For the rest of the introduction it is assumed that R is a ring such that R/J is regular. We note that this condition is satisfied if, for example, the right ideals of R satisfy the descending chain condition. Thus $M=R$ if and only if $J=0$, and hence, in some sense, M may be considered as an "anti-radical." It is shown in §4 that $M=0$ if and only if R is bound to its radical J in the sense of Marshall Hall [2]. Moreover, in §5 it is proved that, under the descending chain condition for right ideals, R is expressible as a direct sum

$$R = M + M^*,$$

where M^* is the ideal consisting of all elements a of R such that $aM=Ma=0$. It follows that M is semi-simple and M^* is bound to its radical, and thus this direct sum decomposition coincides with one

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

obtained by Marshall Hall [2].

2. **Existence and simple properties of $M(R)$.** Let R be an arbitrary ring, and a an element of R. The following lemma plays a central role in several of our proofs:

LEMMA 1. *If y is an element of R such that $a-aya$ is regular, then a is regular.*

PROOF. If $a-aya$ is regular, there exists an element z of R such that

$$(a - aya)z(a - aya) = a - aya.$$

If we set $x=z-zay-yaz+yazay+y$, a simple calculation shows that $axa=a$, and thus a is regular, which completes the proof.

We shall indicate by (a) the principal ideal in R generated by a. We now prove the following theorem.

THEOREM 1. *If M is the set of all elements a of R such that (a) is regular, then M is an ideal in R.*

PROOF. If $z\in M$ and $t\in R$, then $zt\in M$ since $(zt)\subseteq(z)$. Similarly, $tz\in M$. If z, $w\in M$ and $a\in(z-w)$, then $a=u-v$ for some u in (z) and v in (w). Since (z) is regular, $u=uru$ for some element r of R. Then

$$a - ara = u - v - (u - v)r(u - v) = - v + urv + vru - vrv.$$

Since $v\in(w)$, this shows that $a-ara\in(w)$ and is therefore regular. Lemma 1 now implies that a is regular, and hence $z-w\in M$. This completes the proof of the theorem.

It is clear that M, being the join of all regular ideals in R, and being itself regular, is the unique maximal regular ideal in R. It may be remarked that the proof of the above theorem is analogous to the proof of Theorem 1 in Brown and McCoy [1].

We shall next prove the following theorem.

THEOREM 2. *If R is any ring, $M(R/M(R))=0$.*

PROOF. Let $\bar{a}$ denote the residue class modulo $M(R)$ which contains the element a of R. If $\bar{b}\in M(R/M(R))$ and $a\in(b)$, then $\bar{a}\in(\bar{b})$. Since $(\bar{b})$ is a regular ideal in $R/M(R)$, $\bar{a}$ is regular. If $\bar{a}=\bar{a}\bar{x}\bar{a}$, $a-axa\in M(R)$, therefore $a-axa$ is regular and Lemma 1 implies that a is regular. This shows that every element of (b) is regular, and hence $b\in M(R)$. Thus $\bar{b}=0$, completing the proof.

Suppose now that B is an ideal in R, and let b be an element of B which generates a regular ideal $(b)'$ in the ring B. Let (b) be the ideal in R generated by the element b, and let

$$c = nb + rb + bs + \sum r_i b s_i \qquad (n \text{ an integer}; r, s, r_i, s_i \text{ in } R)$$

be any element of (b). Since b is regular in B we have $b = bb_1b$ for some b_1 in B. Hence

$$c = nb + (rbb_1)b + b(b_1bs) + \sum (r_ibb_1)b(b_1bs_i),$$

and thus $c \in (b)'$; therefore (b) is regular since it coincides with $(b)'$. This shows that if $b \in M(B)$, then $b \in B \cap M(R)$. Conversely, if $b \in B \cap M(R)$, then b is an element of B which is regular in R, and it is easy to see that b is therefore regular in the ring B. Since $B \cap M(R)$ is a regular ideal in the ring B, it follows that $B \cap M(R) \subseteq M(B)$. We have therefore proved the following theorem.

THEOREM 3. *If B is an ideal in R, then $M(B) = B \cap M(R)$.*

3. The maximal regular ideal of a complete matrix ring. In this section we shall prove the following theorem.

THEOREM 4. *If R_n is the complete matrix ring of order n over R, then*

$$M(R_n) = (M(R))_n.$$

First we give an elementary proof of the special case of this result in which R itself is regular, and therefore $R = M(R)$. This result, under the assumption that R has a unit element, is due to von Neumann [6].

LEMMA 2. *If R is a regular ring, then R_n is a regular ring.*

The proof of this is in two steps, the first being the proof for $n = 2$, and the second the extension to arbitrary n. If $r \in R$, let us denote by r' an element of R such that $rr'r = r$. Now let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary element of R_2. If we set

$$X = \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix},$$

and denote $A - AXA$ by B, a simple calculation shows that

$$B = \begin{pmatrix} g & 0 \\ h & i \end{pmatrix}$$

for suitable choice of elements g, h, i of R. If

$$Y = \begin{pmatrix} g' & 0 \\ 0 & i' \end{pmatrix},$$

then

$$C = B - BYB = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix},$$

for some element k of R. Finally, if

$$Z = \begin{pmatrix} 0 & k' \\ 0 & 0 \end{pmatrix},$$

we see that

$$C - CZC = 0.$$

This means that C is regular and hence, by Lemma 1, B is regular. Again applying Lemma 1, we see that A is regular, and this completes the proof for $n=2$.

Since $(R_2)_2 \cong R_4$, it follows from the case just proved that R_4 is regular, and similarly R_{2^k} is regular for any positive integer k. If now n is an arbitrary positive integer, choose k so that $2^k \geqq n$. If $A \in R_n$, let A_1 be the matrix of R_{2^k} with A in the upper left-hand corner and zeros elsewhere. Now, as an element of R_{2^k}, A_1 is regular, that is, there exists an element

$$X = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \qquad (B \in R_n),$$

of R_{2^k} such that $A_1XA_1 = A_1$. However, this implies that $ABA = A$, and hence A is regular. The proof of the lemma is therefore complete.

By the lemma just proved, $(M(R))_n$ is a regular ideal in R_n, and hence $(M(R))_n \subseteq M(R_n)$. Conversely, let A be a matrix in $M(R_n)$, and let a_{ij} be a fixed element of A. Since (A) is a regular ideal, there exists an element X of R_n such that $A = AXA = AXAXA$, and therefore

$$a_{ij} = \sum_{p,q} t_{pq} a_{pq} s_{pq}$$

for suitable elements t_{pq}, s_{pq} of R. But it is easy to see[2] that there exists a matrix of (A) with $t_{pq}a_{pq}s_{pq}$ in (1, 1) position and zeros elsewhere, and hence an element of (A) with a_{ij} in (1,1) position and zeros elsewhere. Now if b is any element of the principal ideal in R

[2] See, for example, Lemma 5 of [1].

generated by a_{ij}, it is clear that there exists an element B of (A) with b in the (1,1) position and zeros elsewhere. Furthermore, we have $BYB=B$ for suitable choice of Y in R_n since (A) is regular. But this implies that $by_{11}b=b$, and hence b is regular. This shows that $a_{ij} \in M(R)$, and hence that $M(R_n)\subseteq(M(R))_n$, completing the proof of the theorem.

4. **Some additional properties of $M(R)$.** By the *annihilator* B^* of an ideal B in a ring R is meant the ideal consisting of all elements a of R such that $aB=Ba=0$.

THEOREM 5. *If M is the maximal regular ideal of a ring R and J is the Jacobson radical of R, then $M\cap J=0$, $J\subseteq M^*$, $M\subseteq J^*$, and $M\cap M^*=0$. Furthermore, J is the radical of the ring M^* and M is the maximal regular ideal of the ring J^*.*

PROOF. Since J contains no nonzero idempotent element [3, p. 305], $M\cap J=0$. From this it follows that $MJ=JM=0$, so $J\subseteq M^*$ and $M\subseteq J^*$. If $a\in M\cap M^*$, then $a=axa$ for some x. But $a\in M$ and $xa \in M^*$, hence $a(xa)=0$ and $M\cap M^*=0$. The last sentence of the theorem follows from the observation of Perlis [4] that if B is any ideal in R, the radical of the ring B is just $B\cap J$, and from the analogous Theorem 3.

Following Marshall Hall [2], we may say that a ring R is *bound* to its radical J if and only if $J^*\subseteq J$.

The next theorem gives, for a class of rings including all those whose right ideals satisfy the descending chain condition, a necessary and sufficient condition that the maximal regular ideal be the zero ideal.

THEOREM 6. *If R is a ring such that R/J is regular, then $M=0$ if and only if R is bound to J.*

PROOF. If R is bound to J, it follows that $M=0$ even without the condition that R/J be regular. For $M\cap J=0$, and this implies, as in Theorem 5, that $M\subseteq J^*\subseteq J$. Hence $M=0$.

Conversely, let R/J be regular and $M=0$. We show first by induction that $J\cap J^{*2}=0$. Suppose that $j\in J$ and that $j=\sum_{i=1}^n a_ib_i$ where a_i, b_i are in J^*. It must be proved that $j=0$. In the regular ring R/J, $\bar{a}_i$ is regular, so R contains x_i such that $a_i-a_ix_ia_i=j_i\in J$. Since $b_i\in J^*$, we have

$$j = \sum_{i=1}^{n} (a_ix_ia_i + j_i)b_i = \sum_{i=1}^{n} a_ix_ia_ib_i. \tag{1}$$

If $n=1$, this implies that $j=a_1x_1a_1b_1=a_1x_1j=0$ since $a_1\in J^*$. If $n\neq 1$, then

$$a_n b_n = j - \sum_{i=1}^{n-1} a_i b_i.$$

Thus by (1)

$$j = \sum_{i=1}^{n-1} a_i x_i a_i b_i + a_n x_n\left(j - \sum_{i=1}^{n-1} a_i b_i\right) = \sum_{i=1}^{n-1} (a_i x_i - a_n x_n) a_i b_i.$$

But the induction hypothesis asserts that if $j = \sum_{i=1}^{n-1} c_i d_i$ and c_i, d_i are in J^*, then $j=0$. Since $(a_i x_i - a_n x_n)a_i$ and b_i are in J^*, it follows that $j=0$, and we have proved that $J \cap J^{*2} = 0$. This implies, however, that J^{*2} is a regular ideal. For if $a \in J^{*2}$, then in the regular ring R/J, the element $\bar{a}$ is regular, that is, for some x, $a - axa \in J \cap J^{*2} = 0$, so a is regular. Hence $J^{*2} \subseteq M = 0$, from which it follows that $J^* \subseteq J$ since the radical contains all nil ideals [3, p. 304]. Thus R is bound to J and the proof is complete.

5. **A decomposition theorem.** In this section we point out the role played by the maximal regular ideal M in a theorem of Hall [2], and incidentally give a new proof of his result.

LEMMA 3. *If an ideal B in a ring R has a unit element e, then*

$$R = B \dotplus B^*.$$

PROOF. The existence of a unit element in B implies that $B \cap B^* = 0$. If $x \in R$, then $ex + xe \in B$ and hence $(ex+xe)e = e(ex+xe)$, from which it follows that $xe = ex$ and e is in the center of R. Thus the Peirce decomposition

$$x = ex + (x - ex)$$

expresses each element x of R as a sum of elements ex of B and $x - ex$ of B^*, and the desired result is established.

We remark that a right ideal I in the ring M is a right ideal in R. For if $a \in I$, $r \in R$, then $ar \in M$, hence for some element y of R, $aryar = ar$. But $ryar \in M$, so $ar \in I$. Thus I is a right ideal in R.

From this remark, it follows that if the descending chain condition for right ideals holds in R, it holds also in M. In the presence of this chain condition, regularity is equivalent to semi-simplicity. Hence M has a unit element, and the first sentence of the following theorem is implied directly by Lemma 3.

THEOREM 7. *If a ring R satisfies the descending chain condition for right ideals, then*

$$R = M \dotplus M^*.$$

The ring M is semi-simple and the ring M^* is bound to its radical.

The semi-simplicity of M is implied by the regularity of M. Since the maximal regular ideal of M^* is zero by Theorem 3, and the chain condition holds in M^*, it follows from Theorem 6 that M^* is bound to its radical.

Hall has shown that a ring R satisfying the descending chain condition for right ideals can be represented in a unique way as the direct sum of a semi-simple ring and a ring which is bound to its radical. The result just established shows that the semi-simple component is precisely the maximal regular ideal M of R, and the bound component is the annihilator of M.

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