A PROBLEM ON INVERSE MAPPING SYSTEMS[1]

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Let A be a set of indices $\alpha, \beta, \cdots$ directed by an ordering relation $<$; that is, for each α, β there is a γ such that $\alpha<\gamma$ and $\beta<\gamma$. By an *inverse-mapping-onto-system of type* A is meant a family of nonempty disjoint sets M_α together with functions Π_α^β, for each $\alpha<\beta$, mapping M_β onto M_α. It is required that these functions be such that if $\alpha<\beta<\gamma$ then $\Pi_\alpha^\beta(\Pi_\beta^\gamma x)=\Pi_\alpha^\gamma x$ for each $x \in M_\gamma$. The *inverse limit* of such a system consists of those functions f defined over A such that $f(\alpha)=x_\alpha \in M_\alpha$ and if $\alpha<\beta$ then $\Pi_\alpha^\beta x_\beta=x_\alpha$. If every inverse-mapping-onto-system of type A has a nonempty inverse limit then the directed set A is called *special.*

THEOREM. *The necessary and sufficient condition for A to be special is that it possess either a maximal element or a simple cofinal sequence.*

To show sufficiency in the case where A has a maximal element is trivial. If $\alpha_1<\alpha_2< \cdots$ is a simple cofinal sequence in A, an element of the inverse limit may be constructed by choosing x_{α_1} arbitrarily in M_{α_1}, and selecting $x_{\alpha_{i+1}}$ to satisfy $\Pi_{\alpha_i}^{\alpha_{i+1}}\, x_{\alpha_{i+1}}=x_{\alpha_i}$—possible since $\Pi_{\alpha_i}^{\alpha_{i+1}}$ is always *onto*. Finally, in case β is not one of the α_i, it is easily shown that x_β is uniquely defined by the formula $x_\beta=\Pi_\beta^{\alpha_i}x_{\alpha_i}$, where any α_i such that $\beta<\alpha_i$ is employed. (There must be one since $\{\alpha_i\}$ is cofinal.) This simple argument is contained in Tukey's paper.[1]

To show necessity we assume that A is special and construct the following inverse-mapping-onto-system of type A.

A *point* is a finite sequence $x=(\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}, \alpha_{2n})$ consisting of an even number of indices from A which satisfy the following conditions:

(i) $\alpha_1<\alpha_2$,

(ii) $\alpha_{2i-1}<\alpha_{2i+2}$ for $0<i<n$,

(iii) $\alpha_{2i+1}<\alpha_{2i+2}$ and $\alpha_{2i+1}\not<\alpha_{2j+1}$ for $0\leqq j<i<n$, where $\alpha \not< \beta$ holds when neither $\alpha<\beta$ nor $\alpha=\beta$.

We define *index* $x=\alpha_{2n-1}$, *order* $x=\alpha_{2n}$, *length* $x=n$.

Let M_α consist of all points of index α. Given $\alpha<\beta$, we define the mapping Π_α^β by prescribing its image for an arbitrary point

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[1] The problem treated and notation used in this paper are taken from J. W. Tukey, *On denumerability in topology*. The theorem proved is one of two alternative conjectures made there about the problem.

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$x=(\alpha_1, \alpha_2, \cdots, \alpha_{2n})$ of M_β (so that $\alpha_{2n-1}=\beta$) as follows. In case $\alpha \leqq \alpha_1$ let $\Pi_\alpha^\beta x=(\alpha, \alpha_2)$; since x is a point we have $\alpha_1<\alpha_2$, whence $\alpha \leqq \alpha_1$ implies $\alpha<\alpha_2$ so that (α, α_2) is a point. In case $\alpha \nleq \alpha_1$ we can find (since $\alpha<\beta=\alpha_{2n-1}$) a least j such that $\alpha \leqq \alpha_{2j+1}$ (whence $\alpha \nleq \alpha_{2k+1}$ for $k<j$). In that case we set $\Pi_\alpha^\beta x=(\alpha_1, \alpha_2, \cdots, \alpha_{2j}, \alpha, \alpha_{2j+2})$, which is easily seen to satisfy conditions i, ii, and iii if x does.

These sets M_α and mappings Π_α^β form an inverse-mapping-onto-system of type A. It is a simple matter to check the transitivity of the mappings: $\Pi_\alpha^\beta \Pi_\beta^\gamma=\Pi_\alpha^\gamma$ for $\alpha<\beta<\gamma$. To see that Π_α^β is onto (where $\alpha<\beta$), let $x=(\alpha_1, \alpha_2, \cdots, \alpha_{2n})$ be any point of M_α (so that $\alpha_{2n-1}=\alpha$). Choose $\gamma>\beta$ and consider the sequence $y=(\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \beta, \gamma)$. Since x is a point and $\alpha_{2n-1}=\alpha<\beta<\gamma$, it is only necessary to verify that $\beta \nleq \alpha_{2j+1}$, $j=0, 2, \cdots, n-1$, in order to conclude that y is a point. But this is certainly the case since $\beta \leqq \alpha_{2j+1}$ and $\alpha<\beta$ imply $\alpha_{2n-1}=\alpha<\alpha_{2j+1}$ contrary to the fact that x is a point and so satisfies iii. It is now easily seen that y is in M_β and $\Pi_\alpha^\beta y=x$.

Since we are assuming that A is special it follows that the above inverse-mapping-onto-system has a non-empty inverse limit. Let f be a function defined over A such that $f(\alpha)=x_\alpha \in M_\alpha$, where $\alpha<\beta$ implies $\Pi_\alpha^\beta x_\beta=x_\alpha$. The set of orders of points x_α is cofinal in A, since for any α we have $\alpha=\text{index } x_\alpha<\text{order } x_\alpha$ (by i and iii). Hence we can prove our theorem by showing that the orders of points x_α either possess a maximal element or form a simple sequence.

To see this last fact it is only necessary to observe that if length $x_\alpha=\text{length } x_\beta$ then order $x_\alpha=\text{order } x_\beta$. For choose γ so that $\alpha<\gamma$ and $\beta<\gamma$. Hence we must have $\Pi_\alpha^\gamma x_\gamma=x_\alpha$ and $\Pi_\beta^\gamma x_\gamma=x_\beta$. But from the definition of the mappings Π it then follows that the orders of x_α and x_β are certain elements α_{2i} and α_{2j} in the sequence x_γ. Since $i=\text{length } x_\alpha=\text{length } x_\beta=j$ we have order $x_\alpha=\alpha_{2i}=\alpha_{2j}=\text{order } x_\beta$.

Thus if the lengths of the points x_α are unbounded there will be a simple sequence of orders β (one for each length), which is cofinal in A. In the contrary case the orders form a finite cofinal subset of A and so one of them must be maximal.

This completes the proof of our theorem.

COROLLARY I. *If A is special and B is a cofinal subset of A then B is special.*

COROLLARY II. *Without the axiom of choice we can construct, for each directed set A, an inverse-mapping-onto-system of type* A *which has a nonempty inverse limit if and only if A is special.*

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