SOME THEOREMS ON MEROMORPHIC FUNCTIONS

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Let $f(z)$ be an integral function of finite order $\rho \geqq 0$, and let $n(r)$ denote the zeros of $f(z)$ in $|z| \leqq r$. Set

$$M(r, f) = \max_{|z| \leqq r} |f(z)|.$$

Pólya[1] proved

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r)} < \infty$$

if ρ is not an integer, and Shah[2] proved

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r)\phi(r)} = 0$$

if $f(z)$ is a canonical product of genus p such as $\rho = p$, and $\phi(x)$ is any positive continuous nondecreasing function of a real variable x such that $\int_a^\infty dx/x\phi(x)$ is convergent.

In this paper, some similar theorems will be obtained for meromorphic functions.

Let

$$F(z) = f(z)/g(z) \tag{1}$$

be a meromorphic function of finite order, where

$$\begin{aligned} f(z) &= \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p\right) \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{a_n}\right)^p\right), \\ g(z) &= \prod_{n=1}^{\infty} E\left(\frac{z}{b_n}, q\right) \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{z}{b_n}\right) \exp\left(\frac{z}{b_n} + \frac{1}{2}\left(\frac{z}{b_n}\right)^2 + \cdots + \frac{1}{q}\left(\frac{z}{b_n}\right)^q\right), \end{aligned}$$

these being canonical products of genera p, q and of orders ρ_1, ρ_2, respectively. Let s be the genus of (1), that is $s = \max(p, q)$, and de-

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[1] G. Pólya, Math. Ann. vol. 88 (1922) pp. 169–183.

[2] S. H. Shah, J. London Math. Soc. vol. 15 (1940) pp. 23–31.

note the number of the totality of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ in $|z| \leqq r$ by $N(r)$. Moreover set

$$M(r, F) = M(r) = \max_{|z| \leqq r} |F(z)|.$$

Then we have the following theorems:

THEOREM I. *If for any given meromorphic function* (1) *of finite order*

$$\max(\rho_1, \rho_2) = s \geqq 0,$$

then

$$\liminf_{r\to\infty} \frac{1}{rN(r)\phi(r)} \int_0^r \log^+ M(t)dt = 0, \tag{2}$$

where $\phi(x)$ is any positive continuous nondecreasing function of a real variable x such that $\int_a^\infty dx/x\phi(x)$ is convergent.

THEOREM II. *If for any given meromorphic function* (1) *of finite order* $\max(\rho_1, \rho_2)$ *is not an integer, then*

$$\liminf_{r\to\infty} \frac{1}{rN(r)} \int_0^r \log^+ M(t)dt < \infty. \tag{3}$$

To prove these theorems we establish the lemma:

LEMMA. *For any given meromorphic function* (1) *of genus s, let $c_1, c_2, \cdots$ denote the totality of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ in $|z| \leqq r$. Then*

$$\frac{1}{r}\int_0^r \log^+ M(t)dt \leqq h_1 \sum_{n=1}^\infty \frac{r^{s+1}}{|c_n|^s(|c_n| + r)}, \text{[3]} \tag{4}$$

when $s \geqq 1$; and for a number $k > 1$,

$$\frac{1}{r}\int_0^r \log^+ M(t)dt \leqq h_2 \sum_{n=1}^\infty \log\left(1 + \frac{kr}{|c_n|}\right), \tag{5}$$

when $s = 0$.

PROOF. Assume $s = \max(p, q) \geqq 1$. Since $s \geqq p$ and p is the genus of $f(z)$, for a certain polynomial $P(z)$ of degree $\leqq s$ we get

$$f(z) = e^{P(z)} \prod_{n=1}^\infty E\left(\frac{z}{a_n}, s\right),$$

[3] Hereafter we assume that $h_1, h_2, \cdots$ are all some suitable fixed positive constants independent of r.

whence

$$
\begin{aligned}
\log^+ M(r, f) &\leqq h_3 r^s + \log^+ \prod_{n=1}^{\infty} \left| E\left(\frac{z}{a_n}, s\right) \right| \\
&\leqq h_3 r^s + h_4 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|a_n|^s(|a_n| + r)} \qquad (6) \\
&\leqq h_5 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|a_n|^s(|a_n| + r)}.
\end{aligned}
$$

Similarly

$$
\log^+ M(r, g) \leqq h_6 \sum_{n=1}^{\infty} \frac{r^{s+1}}{|b_n|^s(|b_n| + r)}. \tag{7}
$$

Now, by the theory of meromorphic functions, we get

$$
\frac{1}{r} \int_0^r \log^+ M(t) dt < C(k) T(kr, F), \tag{8}
$$

where $T(r, F)$ is the characteristic function of $F(z)$, and $C(k)$ depends on a number $k > 1$ only, and also we have

$$
\begin{aligned}
T(kr, F) = T(kr, f/g) &\leqq T(kr, f) + T(kr, g) + O(1) \\
&\leqq \log^+ M(kr, f) + \log^+ M(kr, g) + O(1).
\end{aligned} \tag{9}
$$

Hence from (6), (7), (8), and (9) follows (4).

Next, assume $s = 0$. Then by the same process we get (5).

PROOF OF THEOREM I. By the lemma, when $s \geqq 0$ we get

$$
\frac{1}{r} \int_0^r \log^+ M(t) dt \leqq h_7 r^{s+1} \int_0^{\infty} \frac{N(t)}{t^{s+1}(t + r)} dt; \tag{10}
$$

for, when $s \geqq 1$, from (4) this is obviously seen, and when $s = 0$, from (5) we have

$$
\begin{aligned}
\frac{1}{r} \int_0^{\infty} \log^+ M(t) dt &\leqq h_2 \int_0^{\infty} \frac{krN(t)}{t(t + kr)} dt \\
&\leqq kh_2 \int_0^{\infty} \frac{rN(t)}{t(t + r)} dt.
\end{aligned}
$$

On the other hand the integral function

$$
G(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{c_n}, s\right)
$$

is of order $\rho=\max(\rho_1, \rho_2)$, since $c_1, c_2, \cdots$ are composed of $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$, whose exponents of convergence are ρ_1, ρ_2 respectively; and the genus of $G(z)$ is $s=\max(p, q)$, since $f(z)$ and $g(z)$ are of genera p and q respectively. Therefore by Shah[4]

$$\liminf_{r\to\infty} \frac{1}{N(r)\phi(r)}\, r^{s+1}\int_0^\infty \frac{N(t)}{t^{s+1}(t+r)}\,dt = 0$$

for $\rho=s\geqq 0$. Hence by (10) we get (2).

Proof of Theorem II. The order and genus of integral function $G(z)$ are $\rho=\max(\rho_1, \rho_2)$ and $s=\max(p, q)$, respectively, as already seen, so that the exponent of convergence of $|c_1|, |c_2|, \cdots$ is ρ, and $\rho>0$, $s=[\rho]$, since ρ is not an integer.

Now set $\Phi(x)=h_1x^{-(s+1)}/1+x^{-1}$ for $1<\rho$, and $\Phi(x)=h_2\log(1+k/x)$ for $0<\rho<1$. Then $\Phi(x)$ is a positive and decreasing function for $x>0$, and we have

$$\Phi(x) < x^{-\rho+\eta} \qquad (0<\eta<\rho-s)$$

for all values of $x>0$ near $x=0$, and

$$\Phi(x) < x^{-\rho-\eta} \qquad (0<\eta<s+1-\rho)$$

for all values of x sufficiently great. Therefore by Pólya[5]

$$\liminf_{r\to\infty} \frac{1}{N(r)}\sum_{n=1}^{\infty}\Phi\left(\frac{|c_n|}{r}\right) \leqq \int_0^\infty \Phi(x^{1/\rho})dx < \infty.$$

Hence from the lemma

$$\liminf_{r\to\infty} \frac{1}{rN(r)}\int_0^r \log^+ M(t)dt \leqq \liminf_{r\to\infty} \frac{1}{N(r)}\sum_{n=1}^{\infty}\Phi\left(\frac{|c_n|}{r}\right) < \infty.$$

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[4] Shah, loc. cit. pp. 26–30.

[5] Pólya, loc. cit., Theorem VI, pp. 173–174.