"CHARACTERISTIC DIRECTIONS" IN THREE-DIMENSIONAL SUPERSONIC FLOWS

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1. **Introduction.** It is well known from H. Lewy's work [1][1] that a hyperbolic quasi-linear second order partial differential equation in two independent variables can be replaced by a canonical system of first order partial differential equations with the aid of the two families of characteristic curves. Recently, R. Courant and K. O. Friedrichs showed [2; 3] that a system of n-totally hyperbolic quasi-linear first order partial differential equations in two independent variables can be replaced by another system of such equations with the property that each of the latter equations involves differentiation in only one direction. The n-directions which possess this property are called "characteristic directions" by Courant and Friedrichs. The "characteristic directions" of this type are used by these authors in formulating a finite difference method for treating problems such as steady, non-isentropic, two-dimensional supersonic flows.

It will be shown that no such "characteristic directions" exist for the following steady, three-dimensional supersonic flows: (1) non-isentropic, (2) isentropic, irrotational. In other words, any linear combinations of the governing first order equations for these flows must involve derivatives with respect to (at least) two directions. Further, two such directions are sufficient for the second type of flow. It is shown that any two directions lying in the planes which envelope the bicharacteristic cone are permissible.

2. **The flow equations—"characteristic directions."** We shall use tensor notation although the x^λ $(\lambda=1, 2, 3)$ coordinate system will be considered to be Euclidean orthogonal. The components of the velocity vector are denoted by $v^\lambda(\lambda=1, 2, 3)$; the metric tensor by $g^{\lambda\mu}$; the permutation tensor density by $\epsilon^{\lambda\mu\beta}$. Further, the pressure, p, will be considered to be a known function of the density, ρ, and the entropy, s. Hence, we shall introduce the quantities c^2, b defined by

$$(2.1)\qquad c^2 = \frac{\partial p}{\partial \rho}, \qquad b = \frac{\partial p}{\partial s}.$$

For steady motion, the equation of conservation of mass, the laws of

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[1] Numbers in brackets refer to the references cited at the end of the paper.

motions, and the condition that the entropy of a given gas particle be constant are

$$\rho g^{\lambda\mu}\frac{\partial v_\mu}{\partial x^\lambda} + v^\mu\frac{\partial \rho}{\partial x^\mu} = 0, \tag{2.2}$$

$$v^\lambda\frac{\partial v_\mu}{\partial x^\lambda} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x^\mu} + \frac{b}{\rho}\frac{\partial s}{\partial x^\mu} = 0, \tag{2.3}$$

$$v^\mu\frac{\partial s}{\partial x^\mu} = 0. \tag{2.4}$$

The conditions for irrotational motion are

$$\epsilon^{\alpha\lambda\mu}\frac{\partial v_\lambda}{\partial x^\mu} = 0. \tag{2.5}$$

If we multiply the equation (2.2) by one, (2.3), (2.5) by the undetermined vectors l^λ, w_α respectively, and (2.4) by an undetermined scalar σ, we obtain, upon adding the resulting equations,

$$\begin{aligned}(\rho g^{\lambda\mu} + l^\mu v^\lambda + w_\alpha\epsilon^{\alpha\lambda\mu})&\frac{\partial v_\mu}{\partial x^\lambda}\\ &+\left(v^\mu + \frac{c^2}{\rho}l^\mu\right)\frac{\partial \rho}{\partial x^\mu} + \left(\frac{bl^\mu}{\rho} + \sigma v^\mu\right)\frac{\partial s}{\partial x^\mu} = 0.\end{aligned} \tag{2.6}$$

For non-isentropic flows, $w_\alpha = 0$, since (2.5) cannot be used; for isentropic, irrotational flows, $b = 0$, $\sigma = 0$, since (2.4) is identically satisfied. With this understanding, (2.6) represents the most general linear combination of (2.2), (2.3), (2.4), (2.5). Let i^λ denote a tangent vector to a congruence of space curves. That is,

$$i^\lambda = \frac{dx^\lambda}{ds}, \tag{2.7}$$

where s is some parameter along any curve of the congruence. The condition that (2.6) involve differentiation with respect to s leads to the equations

$$\rho g^{\lambda\mu} + l^\mu v^\lambda + w_\alpha\epsilon^{\alpha\lambda\mu} = p^\mu i^\lambda, \tag{2.8}$$

$$v^\mu + \frac{c^2}{\rho}l^\mu = ki^\mu, \tag{2.9}$$

$$\sigma v^\mu + \frac{bl^\mu}{\rho} = mi^\mu, \tag{2.10}$$

where p^μ is an additional undetermined vector and k, m are additional undetermined scalars. For non-isentropic flows $(w_\alpha = 0)$, we solve (2.9) for l^μ and substitute this result into (2.8). This leads us to the equation

$$\frac{\rho}{c^2}(c^2 g^{\lambda\mu} - v^\lambda v^\mu) + \frac{k\rho}{c^2} i^\mu v^\lambda = p^\mu i^\lambda. \tag{2.11}$$

Let

$$\underset{j}{i}{}^\lambda \qquad (j = 1, 2)$$

denote two independent vector fields orthogonal to i^λ. Forming the scalar product of (2.11) with

$$\underset{k}{i}{}_\lambda \underset{j}{i}{}_\mu,$$

we obtain

$$c^2 \underset{j}{i}{}^\lambda \underset{k}{i}{}_\lambda = (v^\lambda \underset{j}{i}{}_\lambda)(v^\lambda \underset{k}{i}{}_\lambda), \qquad j, k = 1, 2. \tag{2.12}$$

Since

$$\underset{j}{i}{}^\lambda$$

consists of two independent vector fields, equation (2.12) implies a contradiction. Hence, we conclude that *"characteristic directions" do not exist for non-isentropic, three-dimensional flows.* For isentropic, irrotational flows, $b = \sigma = 0$ but w_α need not vanish. As in the previous case, we solve (2.9) for l^μ and substitute the result into (2.8). We obtain

$$\frac{\rho}{c^2}(c^2 g^{\lambda\mu} - v^\lambda v^\mu) + \frac{\rho k}{c^2} i^\lambda v^\mu + w_\alpha \epsilon^{\alpha\lambda\mu} = p^\mu i^\lambda. \tag{2.13}$$

Forming the scalar product of (2.13) with

$$\underset{j}{i}{}_\mu \underset{j}{i}{}_\lambda,$$

we find

$$c^2 = (v^\mu \underset{j}{i}{}_\mu)^2, \qquad j = 1, 2. \tag{2.14}$$

The equation (2.14) states that

$$\underset{j}{i}{}^\lambda, \qquad (j = 1, 2)$$

belong to the characteristic normal cone. Since

$$\underset{i}{i^\lambda}$$

may be taken to be any two distinct vectors in the plane perpendicular to i^λ at any point P, the normal characteristic cone must degenerate to a plane at any such point. But this implies a contradiction since the normal cone for isentropic, irrotational flows is a right circular cone with axis along the velocity vector. Thus, *"characteristic directions" do not exist for isentropic, irrotational three-dimensional flows.*

3. **"Characteristic pairs of directions."** In this section, we consider isentropic, irrotational three-dimensional flows. If two distinct congruences of curves with tangent vectors i^λ, j^λ exist such that linear combinations of the governing partial differential flow equations can be expressed as linear combinations of derivatives along these two congruences, then we say that a "characteristic pair of directions" exists. Some simplicity is obtained by eliminating ρ between (2.2), (2.3), where $b=0$ since $s=$constant throughout the flow field. Thus, (2.2), (2.3) may be replaced by

$$a^{\lambda\mu}\frac{\partial v_\lambda}{\partial x^\mu} = 0, \qquad a^{\lambda\mu} = c^2 g^{\lambda\mu} - v^\lambda v^\mu. \tag{3.1}$$

Our problem reduces to the question of determining when three vector fields w_α, p_α, q_α and two congruences of curves i_α, j_α exist such that

$$a^{\lambda\mu} + w_\alpha \epsilon^{\alpha\lambda\mu} = p^\lambda i^\mu + q^\lambda j^\mu. \tag{3.2}$$

If k^μ represents a vector field orthogonal to i^μ, j^μ and r^μ some unknown vector field, then the tensor $a^{\lambda\mu}+w_\alpha\epsilon^{\alpha\lambda\mu}$ can always be expressed in the form

$$a^{\lambda\mu} + w_\alpha \epsilon^{\alpha\lambda\mu} = p^\lambda i^\mu + q^\lambda j^\mu + r^\lambda k^\mu. \tag{3.3}$$

It is easily verified that

$$r^\lambda = (a^{\lambda\mu} + w_\alpha \epsilon^{\alpha\lambda\mu})k_\mu. \tag{3.4}$$

Similar expressions involving i^μ, j^μ (instead of k^μ) are valid for p^λ, q^λ (when i^λ, j^λ are orthogonal). The necessary and sufficient condition that (3.3) reduce to (3.2) is that $r^\lambda=0$. That is,

$$w_\alpha(\epsilon^{\alpha\lambda\mu}k_\mu) = -\,a^{\lambda\mu}k_\mu. \tag{3.5}$$

Since $\epsilon^{\alpha\lambda\mu}k_\mu$ is an antisymmetric tensor, the rank of this tensor is equal to zero or two. If k_μ is not identically zero, the rank is two. Evidently, (3.5) will possess ∞^1 solutions for w_α if and only if the rank of the augmented matrix of (3.5) is two. Direct computation or use of the $\epsilon^{\alpha\lambda\mu}$ symbols shows that

$$a^{\lambda\mu}k_\lambda k_\mu = 0 \tag{3.6}$$

is the necessary and sufficient condition for this to be valid. But (3.6) implies that k_μ lies in the normal characteristic cone. Hence i^λ, j^λ *must lie in a plane tangent to the bicharacteristic cone.* It is evident that if (3.6) is satisfied, then ∞^1 vectors w_α can be obtained satisfying (3.5). For any such vector w_α, equations similar to (3.4), for appropriately chosen i^λ, j^λ, determine p^λ, q^λ so that (3.2) is satisfied.

The above results furnish some additional insight into the method used by C. L. Dolph and the author in their study of three-dimensional supersonic flows. In their study, it was found that a characteristic system can be constructed from the geometry of the characteristic surfaces. However, the equations of the system contain derivatives with respect to two directions. This is essentially different from the two-dimensional case, where only one directional derivative enters into each equation of the characteristic system. But in view of the above results for three-dimensional flows, it is seen that no smaller number of derivatives than two can enter.

References

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