SOME REMARKS ON ONE-SIDED INVERSES

N. JACOBSON

Let $\mathfrak{A}$ be an arbitrary ring with an identity 1, and suppose that $\mathfrak{A}$ contains a pair of elements u, v such that

$$uv = 1 \text{ but } vu \neq 1. \tag{1}$$

We introduce the elements

$$e_{i,j} = v^{i-1}u^{j-1} - v^i u^j \tag{2}$$

for $i, j = 1, 2, 3, \cdots$, where it is understood that $u^0 = 1 = v^0$. It can be verified directly that the e_{ij} thus defined satisfy the multiplication table for matrix units:

$$e_{ij}e_{rs} = \delta_{jr}e_{is}. \tag{3}$$

In particular the elements $e_i = e_{ii}$ are orthogonal idempotent elements. No $e_{ij} = 0$. For by (3) the vanishing of one of the e_{ij} implies the vanishing of all; in particular, it implies that

$$0 = e_1 = 1 - vu$$

contrary to (1).

The existence of an infinite set of orthogonal idempotent elements in a ring $\mathfrak{A}$ is incompatible with mild chain conditions on the ring. If $\{e_i\}$ is such a set of idempotent elements and we set $f_k = \sum_1^k e_i$, then

$$f_1\mathfrak{A} \subset f_2\mathfrak{A} \subset f_3\mathfrak{A} \subset \cdots \tag{4}$$

is an infinite properly ascending chain of right ideals. The right annihilator of an idempotent element f is the set of elements $\{a - fa\}$. If $\mathfrak{A}$ has an identity, this right ideal is the principal right ideal $(1-f)\mathfrak{A}$. Even if $\mathfrak{A}$ does not have an identity, it is customary to denote the set $\{a - fa\}$ as $(1-f)\mathfrak{A}$. It is clear that the following is an infinite properly descending chain of annihilators

$$(1 - f_1)\mathfrak{A} \supset (1 - f_2)\mathfrak{A} \supset (1 - f_3)\mathfrak{A} \supset \cdots . \tag{5}$$

Our remarks imply the following theorem which includes a result due to Baer.[1]

THEOREM 1. *If $\mathfrak{A}$ is a ring with an identity that satisfies either the*

Received by the editors April 14, 1949.

[1] R. Baer, *Inverses and zero-divisors*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 630–638.

ascending or the descending chain condition for principal right ideals generated by idempotent elements, then $uv=1$ in $\mathfrak{A}$ implies $vu=1$.

If $e_{11}, e_{12}, e_{21}, e_{22}$ are elements of a ring satisfying (3), then $e_{11}\mathfrak{A}=e_{12}\mathfrak{A}$ and the mapping $x\rightarrow e_{12}x$ is an $\mathfrak{A}$-isomorphism of $e_{22}\mathfrak{A}$ onto $e_{11}\mathfrak{A}$. Hence if the e_{ii} are defined as above, then the right ideals $e_{ii}\mathfrak{A}$ are $\mathfrak{A}$-isomorphic. The right ideal $\mathfrak{B}=\sum e_{ii}\mathfrak{A}$ is a direct sum of the $e_{ii}\mathfrak{A}$. Hence we have the following result.

THEOREM 2. *If $\mathfrak{A}$ is a ring with an identity that contains two elements u and v such that $uv=1$, $vu\neq 1$, then $\mathfrak{A}$ contains a right ideal that is a direct sum of an infinite number of $\mathfrak{A}$-isomorphic right ideals.*

We note next a result that was proved first by Kaplansky (oral communication) using structure theory.

THEOREM 3. *If an element of a ring with an identity has more than one right inverse, then it has an infinite number of right inverses.*

PROOF. If v is one of the right inverses of the element u, then we have $uv=1$, $vu\neq 1$. If the e_{ij} are defined as above, then $ue_{11}=u(1-vu)=0$. Hence also $ue_{1k}=0$ for $k=1, 2, 3, \cdots$. If $e_{1k}=e_{1l}$ for $k\neq l$, then $e_{1k}e_{kk}=e_{1l}e_{kk}$ and $e_{1k}=0$. Hence the e_{1k} are all different and the elements $v_k=v+e_{1k}$ are all different. Evidently $uv_k=1$.

We assume next that $\mathfrak{A}$ is an algebra over a field Φ with an identity and that $\mathfrak{A}$ contains elements u, v satisfying (1). We wish to determine the structure of the algebra $\Phi[u, v]$ generated by u and v. For this purpose we introduce a vector space $\mathfrak{R}$ that has a denumerable basis $(x_1, x_2, x_3, \cdots)$ over Φ. Let $\overline{U}$ and $\overline{V}$, respectively, be the linear transformations in $\mathfrak{R}$ that have the matrices

$$U=\begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ & & & & \ddots \end{pmatrix}, \qquad V=\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & & 1 & \\ & & & \ddots \end{pmatrix} \tag{6}$$

relative to the given basis. We have the relations $UV=1$, $VU\neq 1$. Hence any element in $\Phi[U, V]$ is a linear combination of the elements V^iU^j, $i, j,=0, 1, 2, \cdots$. We shall now show that these elements are linearly independent; hence they form a basis. We can verify that

$$V^iU^i=\text{diag}\,\{0, 0, \cdots, 0; 1, 1, \cdots\}, \tag{7}$$

where there are i zeros. Hence V^iU^{i+k} has nonzero elements only in the kth super-diagonal and $V^{i+k}U^i$ has nonzero elements only in the kth sub-diagonal. Any relation $\sum\beta_{ij}\ V^iU^j=0$ therefore implies that

$$\sum_i \beta_{i,i+k}V^iU^{i+k} = 0, \qquad \sum_i \beta_{i+k,i}V^{i+k}U^i = 0.$$

Multiplication of the first of these equations on the right by V^k gives

$$\sum \beta_{i,i+k}V^iU^i = 0.$$

It is evident from (7) that the matrices $1=V^0U^0,\ V^1U^1, \cdots$ are linearly independent. Hence $\beta_{i,i+k}=0$. Similarly every $\beta_{i+k,k}=0$. This proves our assertion.

The matrices

$$E_{i,j} = (V^{i-1}U^{j-1} - V^iU^j) \tag{8}$$

are the usual matrix units. Hence $\Phi[U,\ V]$ contains every matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

where A is a finite square matrix. Using (8), we can express any V^iU^j in the form $\sum\beta_{rs}E_{rs}+\phi(U)+\psi(V)$, ϕ and ψ polynomials.

Now it is clear that the subalgebra of $\Phi[\overline{U},\ \overline{V}]$ corresponding to the algebra of matrices (9) is a dense algebra of linear transformations of finite rank.[2] Also it is easy to see that $\phi(\overline{U})+\psi(\overline{V})$ has infinite rank unless ϕ and ψ are 0. Hence the transformations with matrices (9) constitute the complete set of linear transformations of finite rank in $\Phi[U,\ V]$. It follows from known structure results that $\Phi[\overline{U},\ \overline{V}]$ (and $\Phi[U,\ V]$) is a primitive algebra that has minimal one-sided ideals.[3] Moreover, the subalgebra corresponding to (9) is the minimal two-sided ideal of this algebra. Any nonzero two-sided ideal contains this one, and in particular it contains the elements $1-\overline{V}\overline{U}$.

Since the linear transformations $\overline{V}^i\overline{U}^j$ are linearly independent it is clear that the mapping $\overline{U}\to u$, $\overline{V}\to v$ can be extended to a homomorphism of $\Phi[\overline{U},\ \overline{V}]$ onto $\Phi[u,\ v]$. Since $vu\neq 1$, the kernel of this homomorphism does not include $1-\overline{V}\overline{U}$. Hence it is 0 and our correspondence is an isomorphism. This completes the proof of the following theorem:

THEOREM 4. *Any two algebras* $\Phi[u_i,\ v_i]$, $i=1,\ 2$, *in which*

---

[2] Cf. the author's, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 47 (1945) p. 313.

[3] See the reference cited in footnote 2, p. 317.

$u_iv_i=1$, $v_iu_i\neq 1$ *are isomorphic under an isomorphism that pairs the* u_i *and the* v_i. *The algebras* $\Phi[u_i, v_i]$ *are primitive algebras that have minimal one-sided ideals.*

The minimal two-sided ideal $\mathfrak{B}$ of $\Phi[u, v]$ is the infinite matrix algebra with basis e_{ij}. Any element of $\Phi[u, v]$ is congruent mod $\mathfrak{B}$ to an element of the form $\phi(u)+\psi(v)$. It follows that $\Phi[u, v]/\mathfrak{B}$ is isomorphic to the group algebra of an infinite cyclic group.

Suppose now that $\mathfrak{R}$ is any vector space over Φ and that $\overline{U}$ and $\overline{V}$ are linear transformations in $\mathfrak{R}$ over Φ such that $\overline{U}\overline{V}=1$, $\overline{V}\overline{U}\neq 1$. Let $\overline{\mathfrak{B}}$ denote the minimal two-sided ideal in $\Phi[\overline{U}, \overline{V}]$. Then $\overline{\mathfrak{B}}$ has the basis $\overline{E}_{ij}$ defined as in (2). It follows easily that $\overline{\mathfrak{B}}$ is a direct sum of the right ideals $\overline{\mathfrak{J}}_k=\overline{E}_{kk}\Phi[\overline{U}, \overline{V}]$ and that the $\overline{\mathfrak{J}}_k$ are minimal: If $x\in\mathfrak{R}$, the subspace $x\overline{\mathfrak{J}}_k$ is either 0 or it is $\Phi[\overline{U}, \overline{V}]$-isomorphic to $\overline{\mathfrak{J}}_k$. In the latter case $x\overline{\mathfrak{J}}_k$ is irreducible. It follows that the subspace $\mathfrak{R}\overline{\mathfrak{B}}$ can be decomposed as a direct sum of subspaces that are invariant and irreducible relative to $\overline{U}$ and $\overline{V}$. It is easy to see that all of these spaces are isomorphic and that if suitable bases are chosen in these spaces, then the matrices U and V have the form (6). The factor space $\mathfrak{S}=\mathfrak{R}-\mathfrak{R}\overline{\mathfrak{B}}$ is annihilated by $\overline{\mathfrak{B}}$. Hence the induced transformations $\overline{U}$ and $\overline{V}$ in this space satisfy $\overline{U}\overline{V}=1=\overline{V}\overline{U}$.

Nearly all of our results hold also for quasi-inverses. In any ring $\mathfrak{A}$ we define $a\circ b=a+b-ab$. Then $\mathfrak{A}$ is a semigroup relative to this composition and 0 is the identity. An element b is a *right quasi-inverse* of a if $a\circ b=0$. If $\mathfrak{A}$ has an identity 1, $(1-a)(1-b)=1-a\circ b$, so that if $a\circ b=0$, then $(1-a)(1-b)=1$ and conversely. Now suppose that $\mathfrak{A}$ contains two elements a and b such that

$$a\circ b=0,\qquad b\circ a\neq 0. \tag{10}$$

If we define $x^{\circ k}=x^{\circ k-1}\circ x$, then we can verify that the elements

$$e_{i,j}=b^{\circ i}\circ a^{\circ j}-b^{\circ i-1}\circ a^{\circ j-1} \tag{11}$$

satisfy the multiplication for matrix units. All of the e_{ij} are nonzero. In particular, $\mathfrak{A}$ contains an infinite number of orthogonal idempotent elements. Then we see that if $\mathfrak{A}$ satisfies the ascending chain condition on principal ideals generated by idempotent elements, then $a\circ b=0$ in $\mathfrak{A}$ implies $b\circ a=0$.[4] This is the analogue of Theorem 1. Theorems 2, 3, and 4 carry over without change.

YALE UNIVERSITY

[4] Baer's results cited in footnote 1 have been extended to quasi-inverses by Andrunakievic in his paper *Semi-radical rings*, Izvestiya Akademii Nauk SSSR. Ser. Mat. vol. 12 (1948) pp. 129–178.