Thus, with respect to the coordinate system $w(\zeta)$, the product function of G is of class C^3 and hence G is a local Lie group.

PROOF OF THE THEOREM. When G has the discrete center, Lemma 5 shows that G is a local Lie group.

When G has the non-discrete center N, by Lemma 4, N is an abelian Lie group, and by Lemma 5 again, G/N is a local Lie group. Then we can introduce a canonical coordinate of the second kind by $x_1^{*\lambda_1} \cdots, x_m^{*\lambda_m}$, where $x_i^{*\lambda_i}$ $(i=1, 2, \cdots, m)$ are one-parameter subgroups of G/N. Take x_i of G from the coset x_i^* for each i, we can easily show that the set $M=\{y;\ y=x_1^{\lambda_1}\cdots x_m^{\lambda_m},\ |\lambda_i|\leqq 1\}$ satisfies the condition (3) of Lemma 6. Consequently, by Lemma 6, G is a local Lie group. This completes our proof.

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NOTE ON A THEOREM OF KOKSMA

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In 1935 Koksma [2][2] showed, among other things, that the sequence $x, x^2, x^3, \cdots$ is uniformly distributed (mod 1) for almost all $x>1$; that is, that if $N(n, \alpha, \beta, x)$ denotes the number of elements x^i of the sequence $x, x^2, \cdots, x^n$ for which

$$0 \leqq \alpha \leqq x^i - [x^i] < \beta \leqq 1,$$

then

$$\lim_{n\to\infty} \frac{N(n, \alpha, \beta, x)}{n} = \beta - \alpha$$

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[1] The author is indebted to Professor Mark Kac for his help in connection with this paper.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

for almost all $x>1$. The purpose of this note is to provide another proof of this theorem, based on a lemma used in a recent paper [1] of Kac, Salem, and Zygmund on quasi-orthogonal functions. The assertion is contained in the more general theorem:

THEOREM 1. *Let $g(x, n)$ be any function of the real variable x and the positive integral variable n with the following properties in the interval $a<x<b$:*

(i) *dg/dx, d^2g/dx^2 exist,*

(ii) *$g'(x, j)-g'(x, k)$ is monotonic, and is different from zero for $j\neq k$,*

(iii) *for $x=a$ and $x=b$, the inequality*

$$| g'(x, j) - g'(x, k) | \geqq C | j - k |^{\epsilon}$$

is fulfilled for some $C>0$, $0<\epsilon\leqq 1$.

Then the sequence $g(x, 1)$, $g(x, 2)$, $\cdots$ is uniformly distributed (mod 1) *for almost all $x\in(a, b)$.*

This theorem is weaker than Theorem 3 of Koksma's paper, but it is easily verified that the conditions of the theorem are satisfied for

$$\begin{aligned} g(x, n) &= x^n, \quad 1 \leqq a < b, \\ g(x, n) &= n^t x, \quad \text{every } a, b \qquad (t \text{ a positive integer}), \\ g(x, n) &= n^x, \quad 1 \leqq a < b, \\ g(x, n) &= x^{M(n)}, \quad 1 \leqq a < b, \end{aligned}$$

where $M(n)$ is positive and such that $|M(j)-M(k)|\geqq N$ for some $N>0$ and all $j\neq k$. This last case is Theorem 2 of Koksma's paper.

We use the following specialization of Lemma 1 of [1]: Let $g(x, n)$ $(n=1, 2, \cdots)$ be any sequence of real continuous functions in (a, b), and let m be an integer different from zero. Suppose that for all positive integers j, k with $j\neq k$, we have

$$\left| \int_a^b e^{2\pi i m(g(x,j)-g(x,k))} dx \right| \leqq \frac{C_1}{|j-k|^{\epsilon}}$$

for some ϵ with $0<\epsilon\leqq 1$. Then the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i m g(x,n)}}{n^{1-\delta}}$$

converges almost everywhere in (a, b) for every $\delta<\epsilon/2$.

(Lemma 1 of [1] is stated for real-valued functions; in the case of complex-valued functions the integrand should be replaced by $f_j(x)\bar{f}_k(x)$, where $\bar{f}$ is the complex conjugate of f.)

We shall show that for a function g satisfying (i), (ii), and (iii) of the theorem, the conditions of the lemma are also satisfied for each m. The validity of the theorem then follows from Weyl's criterion [3, p. 91] for uniform distribution (mod 1) upon noting that the convergence of $\sum a_n/n$ implies that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = 0.$$

Assume (i), (ii), and (iii). Then

$$\left|\int_a^b \exp\{2\pi i m(g(x,j) - g(x,k))\}dx\right|$$

$$= \left|\int_a^b \exp\{2\pi i m(g(x,j) - g(x,k))\}\cdot\frac{g'(x,j) - g'(x,k)}{g'(x,j) - g'(x,k)}\,dx\right|$$

$$\leqq \frac{1}{2\pi|m|}\left(\left|\left[\frac{\exp\{2\pi i m(g(x,j) - g(x,k))\}}{g'(x,j) - g'(x,k)}\right]_a^b\right| + \left|\int_a^b \exp\{2\pi i m(g(x,j) - g(x,k))\}\frac{d}{dx}(g'(x,j) - g'(x,k))^{-1}dx\right|\right)$$

$$\leqq \frac{1}{2\pi|m|}\left(\frac{1}{|g'(a,j) - (g'a,k)|} + \frac{1}{|g'(b,j) - g'(b,k)|} + \int_a^b\left|\frac{d}{dx}(g'(x,j) - g'(x,k))^{-1}\right|dx\right).$$

By (ii), this is

$$\leqq \frac{1}{2\pi|m|}\left(\frac{1}{|g'(a,j) - (g'(a,k)|} + \frac{1}{|g'(b,j) - g'(b,k)|} + \left|\int_a^b \frac{d}{dx}(g'(x,j) - g'(x,k))^{-1}dx\right|\right)$$

$$\leqq \frac{1}{\pi|m|}\left(\frac{1}{|g'(a,j) - g'(a,k)|} + \frac{1}{|g'(b,j) - g'(b,k)|}\right),$$

and this, by (iii), is

$$\leqq \frac{2}{C\pi|m|\cdot|j-k|^{\epsilon}},$$

so that the assumptions of the lemma hold with $C_1 = 2/C\pi|m|$.

Actually, we have proved considerably more than is stated in Theorem 1, at least for the special functions $g(x, n)$ cited above. In each of these cases, the hypothesis of Theorem 1 holds with $\epsilon = 1$. It follows that for every integer $m \neq 0$, the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i m g(x,n)}}{n^{1/2+\theta}}, \qquad \theta > 0,$$

converges for almost all $x \in (a, b)$. Using Abel's partial summation formula we deduce the following theorem.

THEOREM 2. *Under the assumptions of Theorem* 1,

$$\sum_{n=1}^{N} e^{2\pi i m g(x,n)} = o(N^{1/2+\theta}), \qquad \theta > 0,$$

for every integer $m \neq 0$ *and for almost all* $x \in (a, b)$.

This is a much stronger statement than an assertion about uniform distribution. In view of its generality it is remarkably close to best possible, since it is known [4] that for $g(x, n) = xn^2$, this sum is not $o(N^{1/2})$ for any irrational x.

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