SUMMABILITY AND ANALYTIC CONTINUATION

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1. **Introduction.** In this paper we describe a new family of Toeplitz summability methods, and we study the regions in which these methods sum a Taylor series to the analytic continuation of the function which it represents.

Let $A=(a_{nm})$ and $x=\{s_m\}$ $(n, m=0, 1, \cdots)$ be a matrix and a sequence of complex numbers, respectively. We write formally

$$t_n \equiv A_n(x) \equiv \sum_{m=0}^{\infty} a_{nm}s_m, \tag{1}$$

and say that the sequence x (and the corresponding series $\sum_{m=0}^{\infty}(s_m-s_{m-1})$, with $s_{-1}=0$) is summable A to the sum t if each of the series in (1) converges and $\lim t_n$ exists and equals t. We say that the method A is regular provided it sums every convergent sequence to its limit. The method A is regular if and only if

$$\sum_{m=0}^{\infty} |a_{nm}| \leqq K \qquad (n = 0, 1, \cdots), \tag{2}$$

$$\lim_{n\to\infty} a_{nm} = 0 \qquad (m = 0, 1, \cdots), \tag{3}$$

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm} = 1, \tag{4}$$

where K is a constant independent of n (cf. Toeplitz[2] [2]).

2. **The methods F^r.** For each constant r $(r\neq 1)$ the element of the matrix $A=F^r$ shall be defined by the equations

$$\begin{aligned} a_{nm}(r) &= 0 & (m < n), \\ a_{nm}(r) &= (1-r)^{n+1}C_{m,n}r^{m-n} & (m \geqq n). \end{aligned} \tag{5}$$

We note first that, for $n=0, 1, \cdots$ and $|r|<1$,

$$\sum_{m=0}^{\infty} a_{nm}(r) = (1-r)^{n+1}\sum_{m=n}^{\infty} C_{m,n}r^{m-n} = 1$$

Presented to the Society, December 30, 1948; received by the editors June 2, 1949.

[1] The author wishes to acknowledge the constructive criticism of Professor G. Piranian on this paper.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

and

$$\sum_{m=0}^{\infty} |\, a_{nm}(r) \,| = \left(\frac{1 - r}{1 - |\, r\, |}\right)^{n+1}.$$

If on the other hand $|r| \geqq 1$ $(r \neq 1)$, then $\sum_{m=0}^{\infty} |a_{nm}(r)| = \infty$ $(n = 0, 1, \cdots)$. Therefore the method F^r is regular if and only if r is real and satisfies the condition $0 \leqq r < 1$.

THEOREM[3] 2.1. *If AB denotes the matrix product of A and B, $F^{r_1}F^{r_2} = F^{r_2}F^{r_1} = F^r$, where $r = r_1 + r_2 - r_1 r_2$.*

Let $F^{r_1} = (a_{nm})$, $F^{r_2} = (b_{nm})$, $F^{r_1}F^{r_2} = (c_{nm})$ and $F^{r_2}F^{r_1} = (\gamma_{nm})$. Then $c_{nm} = 0$ *when* $n > m$. *When* $n \leqq m$,

$$\begin{aligned}
c_{nm} &= \sum_{k=0}^{\infty} a_{nk} b_{km} \\
&= (1 - r_1)^{n+1} r_1^{-n} r_2^{m} \sum_{k=n}^{m} r_1^{k} r_2^{-k} (1 - r_2)^{k+1} C_{k,m} C_{n,k} \\
&= (1 - r_1)^{n+1} r_1^{-n} r_2^{m} \left(\frac{r_1}{r_2}\right)^{n} (1 - r_2)^{n+1} C_{m,n} \sum_{k=0}^{m-n} C_{m-n,k} \left(\frac{r_1}{r_2} - r_1\right)^{k} \\
&= (1 - r_1)^{n+1} (1 - r_2)^{n+1} C_{m,n} (r_1 + r_2 - r_1 r_2)^{m-n} \\
&= [1 - (r_1 + r_2 - r_1 r_2)]^{n+1} C_{m,n} (r_1 + r_2 - r_1 r_2)^{m-n}.
\end{aligned}$$

Since the last expression is symmetric in r_1 and r_2, $\gamma_{nm} = c_{nm}$, the proof is complete. We note the corollary that, for $r \neq 1$, the matrix F^r has the inverse F^ρ, where $\rho = -r/(1-r)$.

THEOREM 2.2. *If r_1 and r_2 are real constants $(0 \leqq r_1 < r_2 < 1)$, every bounded sequence summed by F^{r_1} is summed by F^{r_2}, and to the same sum.*

Let $r = (r_2 - r_1)/(1 - r_1)$; then the method F^r is regular, and $F^{r_2} = F^r F^{r_1}$, by Theorem 2.1. The present theorem now follows from a remark by Agnew [1] (cf. p. 328).

3. **Summability of Taylor series.** Henceforth, $\{u_m\}$ and $\{s_m\}$ shall denote sequences of complex numbers related by the equations

$$s_m = u_0 + u_1 + \cdots + u_m \qquad (m = 0, 1, \cdots).$$

LEMMA. *If*

[3] The author is indebted to A. Wilansky for helpful discussion regarding this result.

(i) *the series* $\sum u_m z^m$ *has a positive radius of convergence* R;
(ii) $|r| < R$;
(iii) $t_n = (1-r)^n \sum_{m=n}^{\infty} C_{m,n} U_m r^{m-n}$ $(n=0, 1, \cdots)$;
(iv) $T_n = t_0 + t_1 + \cdots + t_n$ $(n=0, 1, \cdots)$;
then

$$T_n = (1 - r)^{n+1} \sum_{m=n}^{\infty} C_{m,n} s_m r^{m-n} \qquad (n = 0, 1, \cdots). \tag{6}$$

Equation (6) asserts that a certain sum of n Taylor series is equal to another Taylor series. All the series in question converge because, for any fixed positive integer n, the respective radii of convergence of the four series

$$\sum_{m=0}^{\infty} u_m z^m, \qquad \sum_{m=0}^{\infty} s_m z^m, \qquad \sum_{m=0}^{\infty} C_{m,n} u_m z^m, \qquad \sum_{m=0}^{\infty} C_{m,n} s_m z^m$$

are equal. The validity of the equation will now be established by induction.

For $n=0$, the equation (6) reduces to the identity

$$(1 - r) \sum_{m=0}^{\infty} s_m r^m = \sum_{m=0}^{\infty} u_m r^m.$$

But if the equation holds for $n=k$, that is, if

$$T_k = \frac{(1 - r)^{k+1}}{r^k} \sum_{m=k}^{\infty} C_{m,k} s_m r^m,$$

then

$$\begin{aligned}
T_{k+1} &= T_k + t_{k+1} \\
&= \left(\frac{1 - r}{r}\right)^{k+1} \left\{ \sum_{m=k}^{\infty} C_{m,k} s_m r^{m+1} + \sum_{m=k+1}^{\infty} C_{m,k+1}(s_m - s_{m-1}) r^m \right\} \\
&= \left(\frac{1 - r}{r}\right)^{k+1} \left\{ \sum_{m=k+1}^{\infty} [C_{m,k} r + C_{m,k+1} - C_{m+1,k+1} r] s_m r^m \right. \\
&\qquad \left. + [C_{k,k} - C_{k+1,k+1}] s_k r^{k+1} \right\} \\
&= \left(\frac{1 - r}{r}\right)^{k+1} \sum_{m=k+1}^{\infty} \left\{ C_{m,k+1} - [C_{m+1,k+1} - C_{m,k}] r \right\} s_m r^m \\
&= \frac{(1 - r)^{k+2}}{r^{k+1}} \sum_{m=k+1}^{\infty} C_{m,k+1} s_m r^m.
\end{aligned}$$

In other words, equation (6) also holds for $n-k+1$. This proves the lemma. From the lemma it follows that if $0<|r|<R$, where R is the radius of convergence of the series $\sum u_m z^m$, the relation

$$\sum_{n=0}^{\infty}\left(\frac{1-r}{r}\right)^n \sum_{m=n}^{\infty} C_{m,n} u_m r^m = \lim_{n\to\infty} \frac{(1-r)^{n+1}}{r^n} \sum_{m=n}^{\infty} C_{m,n} s_m r^m$$

holds in the sense that the existence of either member implies the existence, with the same value, of the other member. The following result is now immediate:

THEOREM 3.1. *If the series $\sum_{m=0}^{\infty} u_m z^m$ has a positive radius of convergence R, the series $\sum u_m$ is summable F^r to the sum L for any constant r $(0<|r|<R)$ for which the series*

$$\sum_{m=0}^{\infty}\left(\frac{1-r}{r}\right)^n \sum_{m=n}^{\infty} C_{m,n} u_m r^m$$

converges to L.

4. **Analytic continuation by means of the methods F^r.** Let D be a simply connected region in the complex plane, C a simple closed Jordan curve lying in D and bounding the finite region D'. Corresponding to each complex number r $(r\neq 1)$ we define an open set $R(r, C)$ as follows: $R(r, C)$ is the set of all points in D' for which $|z-rz|<|t-rz|$ whenever t lies on C. A set B shall be said to be of type $R^*(r, D)$ if B is closed and if C can be chosen in such a way that B is a subset of $R(r, C)$. In the following theorem, $f^{(n)}(rz)$ denotes the nth derivative with respect to w of $f(w)$ at the point $w=rz$.

THEOREM 4.1. *If $f(z)$ is uniform and regular in the region D, r is a complex constant $(r\neq 1)$, and B is a set of type $R^*(r, D)$, then the series*

$$\sum_{n=0}^{\infty} f^{(n)}(rz)(1-r)^n z^n/n! \tag{7}$$

converges to $f(z)$ uniformly in B.

If z lies in B,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-rz}\,\frac{1}{1-(z-rz)/(t-rz)}\,dt$$
$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-rz}\sum_{n=0}^{\infty}\left(\frac{z-rz}{t-rz}\right)^n dt,$$

where C is a curve in D such that B is a subset of $R(r, C)$. The series in the last integrand converges uniformly with respect to t on C and z in B, because, for such values of t and z,

$$|(z - rz)/(t - rz)| < K < 1,$$

where the constant K depends only on B and C. Therefore

$$f(z) = \sum_{n=0}^{\infty} (z - rz)^n \frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t - rz)^{n+1}} .$$

But the point rz is inside of the curve C; for otherwise the line segment joining the points z and rz would meet the curve C at some point t', the inequality $|(z-rz)/(t'-rz)| <1$ would not be satisfied, and the point z would not lie in B. Therefore the last integral has the value $2\pi i f^{(n)}(rz)/n'$, and the theorem is proved.

The hypothesis that the region D is simply connected will now be removed. Let D be a connected open set, E its boundary, and r a complex constant ($r \neq 1$). By $Q(r, D)$ we shall denote the set of all points in D for which the inequality $|z-rz| < |t-rz|$ is satisfied whenever t lies in E.

THEOREM 4.2. *Let the function $f(z)$ be regular and uniform in the bounded region D, and let r be a complex constant* ($r \neq 1$). *Then the series* (7) *converges to $f(z)$ absolutely in $Q(r, D)$, and the absolute convergence is uniform in every closed subset of $Q(r, D)$.*

Let B be a closed subset of $Q(r, D)$, and z_0 a point in B. It is to be shown that the series

$$\sum_{n=0}^{\infty} f^{(n)}(rz_0)(1 - r)^n z_0^n/n! \tag{8}$$

converges to $f(z_0)$, and that the absolute convergence of the series (9) is uniform with respect to z_0 in B.

We observe that (8) is the Taylor series of $f(z_0)$ about the point $z = rz_0$.[4] Since B is a closed subset of $Q(r, D)$, there exists a positive number ϵ such that $|(z_0 - rz_0)/(t - rz_0)| < 1 - 2\epsilon$ when z_0 is in B and t is on E. The Cauchy estimate for the coefficients of our Taylor series gives the result

$$|f^{(n)}(rz_0)/n!| < M(rz_0)/[(1 - \epsilon)\delta_0]^n,$$

where δ_0 is the distance from rz_0 to the set E and $M(rz_0)$ denotes the maximum modulus of $f(z)$ on the circle $|z - rz_0| = (1 - \epsilon)\delta_0$. As the

[4] The author is indebted to A. M. Gleason who pointed out this fact in an oral communication.

point z_0 ranges over the set B, the points

$$z = rz_0 + e^{i\theta}(1 - \epsilon)\delta_0 \qquad (0 \leqq \theta < 2\pi)$$

range over a subset of D which is bounded away from the set E, and therefore the quantities $M(rz_0)$ have a common finite upper bound M. It follows that

$$\left| f^{(n)}(rz_0)(1 - r)^n \frac{z^n}{n!} \right| < \frac{[(1 - 2\epsilon)\delta_0]^n M}{[(1 - \epsilon)\delta_0]^n} < (1 - \epsilon)^n M,$$

and the theorem is proved.

THEOREM 4.3. *Let D be a connected open set containing the origin, $f(z)$ a uniform function regular in D; let the series $\sum a_m z^m$ converge to $f(z)$ in a circle of radius ρ; and let r be a complex constant ($r \neq 1$). Then, in every closed set which is contained in the intersection of the set $Q(r, D)$ and the circle $|z| < \rho/|r|$, the series $\sum a_m z^m$ is uniformly absolutely summable F^r to $f(z)$.*

If $r = 0$, the set $Q(r, D)$ is contained in the circle $|z| < \rho$, and the result is trivial. To prove the theorem for $r \neq 0$, we observe first that the series

$$\sum_{m=n}^{\infty} a_m m(m - 1) \cdots (m - n + 1)(rz)^{m-n}$$

converges to $f^{(n)}(rz)$ in the region $|z| < \rho/|r|$. It follows that the series (7) can be written in the form

$$\sum_{n=0}^{\infty} \sum_{m=n}^{\infty} a_m C_{m,n}(1 - r)^n z^m r^{m-n} = \sum_{n=0}^{\infty} \left(\frac{1 - r}{r}\right)^n \sum_{m=n}^{\infty} C_{m,n} r^m a_m z^m,$$

and upon application of Theorem 3.1 (with $u_m = a_m z^m$) the present theorem becomes a corollary of Theorem 4.2. We note that we have established analytic continuation of the function represented by the series $\sum a_m z^m$ by means of summability methods which need not even be regular.

5. **Summability of a special series.** We now subject the methods F^r to the customary test of applying them to the series $\sum z^n$. Here, the sole singularity of the function $f(z)$ is the point $z = 1$, and the boundary of the region D can therefore be taken to consist of the point $z = 1$ together with the circle $|z| = K$, where K is arbitrarily large. We shall restrict our considerations to the case where F^r is regular, that is, where r is real and $0 \leqq r < 1$. If z is any point in the plane, the

inequality $|z-rz|<|t-rz|$ is certainly satisfied if t is sufficiently large. It remains to examine the inequality for the case where $t=1$. Here we have, with $z=x+iy$,

$$(x^2 + y^2)(1 - r)^2 < (1 - rx)^2 + r^2y^2,$$
$$(1 - 2r)(x^2 + y^2) < 1 - 2rx.$$

Theorem 4.3 now gives the following result: If $0<r<1/2$, F^r sums the series $\sum z^m$ to the function $1/(1-z)$ in the intersection of the region $|z|<1/r$ with the interior of the circle having its center at $z=-r/(1-2r)$ and passing through the point $z=1$. If $1/2<r<1$, F^r sums the series in the intersection of the region $|z|<1/r$ with the exterior of the circle having its center at $z=r/(2r-1)$ and passing through the point $z=1$. The method $F^{1/2}$ sums the series in the intersection of the region $|z|<2$ with the half-plane $x<1$.

Finally, we recall that in §2 the relation $F^{r_2}\supset F^{r_1}$ $(0\leqq r_1<r_2<1)$ was shown to hold in the space of bounded sequences. The relation does not hold in the space of all sequences. For we have now established that the series $\sum(-5/3)^n$ is summable $F^{1/2}$; on the other hand, the transform of this series by F^r does not even exist when $r\geqq 3/5$.

Added in proof: After the present paper had been accepted for publication, an interesting paper by P. Vermes [Amer. J. Math. vol. 71 (1949) pp. 541–562] appeared which overlaps the results of the present paper. In addition R. P. Agnew [Math. Rev. vol. 11, p. 242] in a review of a paper on these methods by W. König-Meyer [Math. Zeit. vol. 52 (1949) pp. 257–304] points out that the first systematic study of these methods is in the thesis of R. Wais [*Das Taylorsche Summierungsverfahren*, Tubingen, 1935].

Bibliography

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