ON A SEMI-GROUP OF SUBSETS OF A LINEAR SPACE

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A family $\mathcal{J}$ of subsets of a space is defined to be a semi-group of subsets provided that the intersection of any two subsets of $\mathcal{J}$ is again a subset of $\mathcal{J}$. In giving a characterization of a Banach space of continuous functions in terms of the geometry of the space, Clarkson[1] considered a cone $\mathfrak{C}$ which had the property that the family of all of its translates formed a semi-group. This note is concerned with an investigation of the structure of a subset S of a linear space $\mathfrak{X}$ such that the family of all translates of S form a semi-group, that is, for any points x, y of $\mathfrak{X}$ there exists a $z \in \mathfrak{X}$ such that $(x+S) \cap (y+S) = z+S$. It will be shown that under certain rather weak restrictions, S must be a convex cone.

Let $\mathfrak{X}$ be an abstract linear space. The only topology which will be assumed for $\mathfrak{X}$ is that which arises from the topology of the straight lines of $\mathfrak{X}$, that is, if $u_n = \alpha_n x + \beta_n y$, $x \neq y$, $\alpha_n + \beta_n = 1$, $\alpha_n \to \alpha$, $\beta_n \to \beta$, then $u_n \to \alpha x + \beta y$. This will be called the linear topology of $\mathfrak{X}$. We shall assume that all straight lines of $\mathfrak{X}$ are complete in this topology. A point x is an extreme point of a set S if $x \in S$ and there exists no line segment with end points in S which contains x in its interior. The theorem to be proved is the following:

THEOREM. *Let S be a subset of $\mathfrak{X}$ which is closed in the linear topology and which has at least one extreme point. Let the family of all translates of S form a semi-group. Then S is a convex cone, that is, S is convex and there exists a point $v \in S$ such that*

$$S = \underset{v}{E}\,[y = v + \lambda x,\ x \in S,\ \lambda \geqq 0].$$

The theorem is proved by four lemmas.

LEMMA 1. *If $x+S=S$, then $x=\theta$.*

PROOF. It is evident that any translate of S can be used initially instead of S. Thus we may assume that S contains θ without loss of generality. If $x+S=S$, $x \in S$. Also $S=S-x$ and $-x \in S$. If $y \in S$, $x+y \in S$ and $-x+y \in S$. However, if $x \neq -x$ this shows that y is the midpoint of a line segment joining two points of S and con-

Presented to the Society, December 30, 1948; received by the editors January 4, 1949 and, in revised form, June 29, 1949.

[1] J. A. Clarkson, *A characterization of C spaces*, Ann. of Math. vol. 48 (1947) pp. 845–850.

tradicts the hypothesis that S has an extreme point.

LEMMA 2. *S contains only one extreme point.*

PROOF. Suppose that S contained at least two extreme points. It may be assumed that one is the point θ. Let the second be $v\neq\theta$. Consider the translation $S+v$. Since θ and v are both extreme points, $2v\notin S$ and $S+v$ does not contain S. Hence $(S+v)\cap S=y+S$ where $y\neq\theta$ and y is an extreme point of $y+S$. Also

$$\begin{aligned}\{v+[(S+v)\cap S]\}\cap S& \\ &= (v+y+S)\cap S \supset (v+y+S)\cap(y+S) \\ &= y+(y+S) = 2y+S.\end{aligned}$$

This shows that v, $y+v$, and $2y+v$ are in S and $y+S$ contrary to the conclusion that y is an extreme point of $y+S$.

LEMMA 3. *If v is the extreme point of S, S is a cone with vertex v.*

PROOF. It may be assumed that $v=\theta$. It must be shown that if $x\in S$, then $\lambda x\in S$ for all $\lambda\geqq 0$. Since λx is a complete ray, if there exist values of λ for which $\lambda x\notin S$, there must exist a smallest value λ_0 such that $\lambda_0 x\in S$ because of the linear closure of S. $(\lambda_0 x+S)\cap S=y+S$ where y is the extreme point of $y+S$. However, since $\lambda_0 x\in(\lambda_0 x+S)\cap S$, $\lambda_0 x$ is an extreme point of $y+S$ and since there is only one extreme point of the set, $y=\lambda_0 x$ by Lemma 1. Then $\lambda_0 x+S\subset S$ and $2\lambda_0 x\in S$. By induction it can be seen that if $\lambda_0 x\in S$, then $n\lambda_0 x\in S$ for any positive integer n. Also there exists a $z\in S$ such that $(2^{-1}\lambda_0 x+S)\cap S=z+S$. This implies the existence of a point $u\in S$ such that $z=u+2^{-1}\lambda_0 x$. Also, by the first part of the proof, $u+\lambda_0 x\in S$. Since z is the extreme point of $z+S$, $u+(n/2)\lambda_0 x\in z+S\subset S$ for every integer n by the same argument as was used in the first part of the proof. Then $S=S-z$ contains all points of the form $u+(n/2)\lambda_0 x-(u+2^{-1}\lambda_0 x)$. In particular, S contains the point $2^{-1}\lambda_0 x$, contrary to the assumption that λ_0 was the smallest nonzero value of λ for which $\lambda x\in S$. This shows that if $x\in S$, $\lambda x\in S$ for all $\lambda\geqq 0$ and S is a cone with vertex θ.

It is to be noted that if θ is the vertex, the above proof shows that if $x\in S$, then $\lambda x\in S$ for all non-negative λ and $\lambda x+S\subset S$. Hence, if $y\in S$, $y+\lambda x\in S$ for all $\lambda\geqq 0$. This shows that if S contains a ray through any point, it contains a parallel ray through every point of the set.

LEMMA 4. *S is convex.*

PROOF. It must be shown that if x and y are in S, then $\alpha x+\beta y\in S$ for α, $\beta\geqq 0$, $\alpha+\beta=1$. If S has vertex θ and if $y=\lambda x$, this is true by Lemma 3. If $y\neq\lambda x$, then the rays λx, νy, $\lambda\geqq 0$, $\nu\geqq 0$, are distinct and are in S. If for any point $\alpha x+\beta y$, $\alpha, \beta>0$, $\alpha+\beta=1$, there exists a $\mu_0\neq 0$ such that $\mu_0(\alpha x+\beta y)\in S$, then $\mu(\alpha x+\beta y)\in S$ for all $\mu\geqq 0$ by Lemma 3. S contains the ray $y+\lambda x$ where $\lambda\geqq 0$. If the existence of positive numbers μ and λ can be demonstrated such that $\mu(\alpha x+\beta y)=y+\lambda x$, the lemma is proved. If this is so, $(\mu\alpha-\lambda)x+(\mu\beta-1)y=\theta$ and if $\mu=1/\beta$, $\lambda=\alpha/\beta$, the rays $\mu(\alpha x+\beta y)$ and $y+\lambda x$ have a common point. Since all points of the form $y+\lambda x$ are in S, $(1/\beta)$ $(\alpha x+\beta y)\in S$ and hence $\mu(\alpha x+\beta y)\in S$ for all $\mu\geqq 0$, in particular for $\mu=1$. Hence S is convex.

By making use of this theorem it is possible to weaken the hypotheses of Clarkson's theorem slightly and to state it in the following form.

THEOREM. *A necessary and sufficient condition that a Banach space be equivalent to the space of all continuous functions over a bicompact Hausdorff space H is that there exist a closed set S with an extreme point, containing θ in its interior, such that all translates of S form a semi-group and such that the unit sphere in the space is the set $S\cap(-S)$.*

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