ON THE STRONG SUMMABILITY OF FOURIER SERIES

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This paper consists of three parts. In the first part we complete one of the deepest theorems due to Littlewood and Paley [3, 4].[1] Our theorem was already proved by Marcinkiewicz and Zygmund [4]. But the proof given here is very simple and direct. In the second part a strong summability theorem is proved. It is the completion of our former theorem [5]. We use the method due to Zygmund [8] for the proof. Finally we prove a strong summability theorem concerning lacunary sequences of partial sums (Zalcwasser [6]).

1. Let $f(\theta)$ be an integrable function with period 2π and its Fourier series be $f(\theta)\sim\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$, assuming $a_0=0$ for the sake of simplicity. If we put $z=\rho e^{i\theta}$ and

$$\phi(z)=\sum_{n=1}^{\infty}(a_n-ib_n)z^n,$$

then $f(\theta)$ becomes the boundary function of the harmonic function $\Re\phi(z)$. Let us denote by p, q, and r real numbers satisfying

$$r>1,\qquad 1<p\leqq 2\leqq q<\infty,$$

and by A, B, $\cdots$ absolute constants, not always the same from one occurrence to another.

THEOREM 1. *Let $f(\theta)$ be a function of the class L^r, then*

$$(1.1)\qquad \begin{aligned} A\int_0^{2\pi}|\phi(e^{i\theta})|^r d\theta &\leqq \int_0^{2\pi}\left\{\int_0^1(1-\rho)|\phi'(z)|^2 d\rho\right\}^{r/2}d\theta \\ &\leqq B\int_0^{2\pi}|\phi(e^{i\theta})|^r d\theta, \end{aligned}$$

$$(1.2)\qquad \int_0^{2\pi}\left\{\int_0^1(1-\rho)^{q-1}|\phi'(z)|^q d\rho\right\}^{r/q}d\theta\leqq C\int_0^{2\pi}|\phi(e^{i\theta})|^r d\theta,$$

$$(1.3)\qquad D\int_0^{2\pi}|\phi(e^{i\theta})|^r d\theta\leqq\int_0^{2\pi}\left\{\int_0^1(1-\rho)^{p-1}|\phi'(z)|^p d\rho\right\}^{r/p}d\theta.$$

(1.1) is due to Littlewood and Paley [4]. In order to prove (1.2) we put

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\Phi(\theta) = \max\left(|\phi(z)|; z \in S(\theta)\right)$$

where $S(\theta)$ is a kite-shaped region.[2] Then $|\phi'| \leqq A\Phi/(1-\rho)$. If we put

$$g(\theta) = \left(\int_0^1 (1-\rho)|\phi'(z)|^2 d\rho\right)^{1/2},$$

then we have

$$\left\{\int_0^{2\pi}\left(\int_0^1 (1-\rho)^{q-1}|\phi'|^q d\rho\right)^{r/q} d\theta\right\}^{1/r}$$

$$\leqq \left\{\int_0^{2\pi}\left(\int_0^1 |\phi'|^2(1-\rho)A\Phi^{q-2}d\rho\right)^{r/q} d\theta\right\}^{1/r}$$

$$\leqq \left\{\int_0^{2\pi}(A\Phi^{q-2}\cdot g^2)^{r/q}d\theta\right\}^{1/r}$$

$$\leqq \left(B\int_0^{2\pi}\Phi^{(1-2/q)r}g^{2r/q}d\theta\right)^{1/r}.$$

Since $0<2/q<1$, if we apply Hölder's inequality, the right-hand side becomes less than

$$C\left\{\left(\int_0^{2\pi}\Phi^r d\theta\right)^{1-2/q}\left(\int_0^{2\pi}|g|^r d\theta\right)^{2/q}\right\}^{1/r}$$

$$\leqq C\left\{\left(\int_0^{2\pi}\Phi^r d\theta\right)^{1/r}\right\}^{1-2/q}\cdot\left\{\left(\int_0^{2\pi}|g|^r d\theta\right)^{1/r}\right\}^{2/q}.$$

By the maximal theorem due to Hardy and Littlewood the last term is less than

$$D\left\{\left(\int_0^{2\pi}|\phi|^r d\theta\right)^{1/r}\right\}^{1-2/q}\cdot\left\{\left(\int_0^{2\pi}|g|^r d\theta\right)^{1/r}\right\}^{2/q}$$

$$\leqq E\left(\int_0^{2\pi}|\phi|^r d\theta\right)^{1/r}.$$

Thus we get (1.2) by (1.1).

For the proof of (1.3) we put

$$h(\theta) = \left(\int_0^1 (1-\rho)^{p-1}|\phi'(z)|^p d\rho\right)^{1/p}.$$

[2] See Littlewood and Paley [4].

Then we have

$$\left\{\int_0^{2\pi}\left(\int_0^1 (1-\rho)\,|\,\phi'(z)\,|^2 d\rho\right)^{r/2} d\theta\right\}^{1/r}$$

$$\leqq \left\{\int_0^{2\pi}\left(\int_0^1 |\,\phi'\,|^p(1-\rho)^{p-1}A\,\Phi^{2-p}d\rho\right)^{r/2} d\theta\right\}^{1/r}$$

$$\leqq \left\{\int_0^{2\pi} (A\,\Phi^{2-p}\,|\,h(\theta)\,|^p)^{r/2}d\theta\right\}^{1/r} \leqq B\left\{\int_0^{2\pi}\Phi^{(1-p/2)r}\cdot h^{pr/2}d\theta\right\}^{1/r}$$

$$\leqq C\left\{\left(\int_0^{2\pi}\Phi^r d\theta\right)^{1-p/2}\left(\int_0^{2\pi} h^r d\theta\right)^{p/2}\right\}^{1/r}$$

$$\leqq C\left\{\left(\int_0^{2\pi}\Phi^r d\theta\right)^{1/r}\right\}^{1-p/2}\left\{\left(\int_0^{2\pi} h^r d\theta\right)^{1/r}\right\}^{p/2}$$

$$\leqq D\left\{\left(\int_0^{2\pi}|\,\phi\,|^r d\theta\right)^{1/r}\right\}^{1-p/2}\left\{\left(\int_0^{2\pi} h^r d\theta\right)^{1/r}\right\}^{p/2}.$$

Thus we have

$$\left(\int_0^{2\pi} g^r d\theta\right)^{1/r} \leqq E\left(\int_0^{2\pi} h^r d\theta\right)^{1/r}.$$

Hence by (1.1) we get (1.3).

2. THEOREM 2. *Let $s_n(e^{i\theta})$ and $\tau_n(e^{i\theta})$ denote the partial sums and the arithmetic mean of Fourier power series of $\phi(e^{i\theta})$ belonging to the class H^r, respectively. Then we have*

$$(2.1)\qquad \begin{aligned} A\int_0^{2\pi}|\,\phi(e^{i\theta})\,|^r d\theta &\leqq \int_0^{2\pi}\left(\sum_{n=1}^{\infty}|\,s_n(e^{i\theta})-\tau_n(e^{i\theta})\,|^2/n\right)^{r/2} d\theta \\ &\leqq B\int_0^{2\pi}|\,\phi(e^{i\theta})\,|^r d\theta, \end{aligned}$$

$$(2.2)\qquad \int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}|\,s_n(e^{i\theta})-\tau_n(e^{i\theta})\,|^q\right)^{r/q} d\theta \leqq C\int_0^{2\pi}|\,\phi(e^{i\theta})\,|^r d\theta,$$

$$(2.3)\qquad D\int_0^{2\pi}|\,\phi(e^{i\theta})\,|^r d\theta \leqq \int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}|\,s_n(e^{i\theta})-\tau_n(e^{i\theta})\,|^p\right)^{r/p} d\theta.$$

(2.1) is due to Zygmund [**8, 9**], and (2.2) is given in my former paper [**5**]. For the proof of the theorem we require the following lemmas.

LEMMA 1. *Let $f_1, f_2, \cdots$ be a sequence of integrable functions and let $s_{n,\nu}$ denote the νth partial sum of the Fourier series of f_n. Then*

$$\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|s_{n,k_n}(\theta)|^m\right)^{r/m}d\theta \leqq A\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|f_n|^m\right)^{r/m}d\theta,$$

where $r>1$ and $m>1$.

The inequality is also valid for conjugate series.

This lemma is due to Boas and Bochner [1].

If we write

$$s_\nu(\rho,\theta) = \sum_{n=1}^{\nu}(a_n\cos n\theta + b_n\sin n\theta)\rho^n,$$

$$f(\rho,\theta) = \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\rho^n,$$

then we have the following lemma.

LEMMA 2. *Let $s_{n,\nu}(\rho,\theta)$ denote the sum derived from $f_n(\theta)$ similarly as $s_\nu(\rho,\theta)$ from $f(\theta)$. Then under the hypothesis of Lemma 1 we have*

$$\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|s_{n,k_n}(\rho_n,\theta)|^m\right)^{2/m}d\theta \leqq A\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|f_n(\theta)|^m\right)^{r/m}d\theta,$$

where $0\leqq\rho_n\leqq 1$. The sums $s_{n,\nu}$ may be replaced by the conjugate ones.

This lemma is easily derived from Lemma 1 by Zygmund's argument by a slight modification.

LEMMA 3. *Let $0\leqq\rho_n<1$ and Δ_n denote an arbitrary interval situated in $(\rho_n, 1)$. Then under the hypothesis of Lemma 1 we have*

$$\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|s_{n,k_n}(\rho_n,\theta)|^m\right)^{r/m}d\theta$$

$$\leqq A\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{1}{\Delta_n}\int_{\Delta_n}|f_n(\rho,\theta)|^m d\rho\right)^{r/m}d\theta.$$

The sums $s_{n,\nu}$ may be replaced by the conjugate ones.

This is immediate from Lemma 2.

PROOF OF (2.2). We have

$$s_n(e^{i\theta}) - \tau_n(e^{i\theta}) = -is_n'(e^{i\theta})/(n+1).$$

By Abel's transformation

$$s_n'(e^{i\theta}) = \rho^{-n}s_n'(\rho e^{i\theta}) - (1-\rho)\sum_{\nu=0}^{n-1}\rho^{-\nu-1}s_\nu'(\rho e^{i\theta}).$$

Therefore

$$\begin{aligned}|s_n'(e^{i\theta})|^q &\leqq A\left\{\rho^{-qn}|s_n'(\rho e^{i\theta})|^q + (1-\rho)^q\left(\sum_{\nu=0}^{n-1}\rho^{-\nu-1}|s_\nu'(\rho e^{i\theta})|\right)^q\right\}\\ &\leqq B\left\{\rho^{-qn}|s_n'(\rho e^{i\theta})|^q + \frac{1-\rho}{\rho^{(q-1)n}}\sum_{\nu=0}^{n-1}\rho^{-\nu-1}|s_\nu'(\rho e^{i\theta})|^q\right\}.\end{aligned}$$

Let $\rho_n = 1-1/(n+1)$ and $\Delta_n = (\rho_n, \rho_{n+1})$ and let I be the left-hand side integral of (2.2). Then we have

$$\begin{aligned}I &\leqq C\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{|s_n'(\rho_n e^{i\theta})|^q}{n^{q+1}\rho_n^{qn}}\right.\\ &\qquad\left.+\sum_{n=1}^{\infty}\frac{1-\rho_n}{n^{q+1}\rho_n^{(q-1)n}}\sum_{\nu=0}^{n-1}\rho_n^{-\nu-1}|s_\nu'(\rho_n e^{i\theta})|^q\right)^{r/q}d\theta\\ &\leqq D\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{|s_n'(\rho_n e^{i\theta})|^q}{n^{q+1}\Delta_n}\int_{\Delta_n}|\phi'(\rho e^{i\theta})|^q d\rho\right.\\ &\qquad\left.+\sum_{n=1}^{\infty}\frac{1}{n^{q+2}}\sum_{\nu=0}^{n-1}\frac{1}{\Delta_n}\int_{\Delta_n}|\phi'(\rho e^{i\theta})|^q d\rho\right)^{r/q}d\theta\\ &\leqq E\int_0^{2\pi}\left(\sum_{n=1}^{\infty}\frac{n^2\cdot n^{q-1}}{n^{q+1}}\int_{\Delta_n}(1-\rho)^{q-1}|\phi'(\rho e^{i\theta})|^q d\rho\right)^{r/q}d\theta\\ &\leqq F\int_0^{2\pi}\left(\int_0^1(1-\rho)^{q-1}|\phi'(\rho e^{i\theta})|^q d\rho\right)^{r/q}d\theta\\ &\leqq G\int_0^{2\pi}|\phi(e^{i\theta})|^r d\theta\end{aligned}$$

from (1.2). Thus we obtain (2.2).

PROOF OF (2.3). We may write

$$|\phi'(\rho e^{i\theta})| = \left|\sum_{\nu=1}^{\infty}\nu c_\nu\rho^{\nu-1}e^{i\nu\theta}\right| = (1-\rho)\left|\sum_{\nu=1}^{\infty}s_\nu'\rho^{\nu-1}\right|.$$

If we put $\rho_n = 1-1/n$, then we have

$$\int_0^1 (1-\rho)^{p-1}|\phi'|^p d\rho = \sum_{n=1}^{\infty}\int_{\rho_n}^{\rho_{n+1}} (1-\rho)^{p-1}|\phi'|^p d\rho$$

$$\leqq \sum_{n=1}^{\infty}\int_{\rho_n}^{\rho_{n+1}} (1-\rho)^{p-1}\left[(1-\rho)\sum_{\nu=1}^{\infty}|s_\nu'|\rho^{\nu-1}\right]^p d\rho$$

$$\leqq \sum_{n=1}^{\infty}\frac{1}{n^{p+1}}\left[(1-\rho_n)\sum_{\nu=1}^{\infty}|s_\nu'|\rho_{n+1}^{\nu-1}\right]^p$$

$$\leqq A\sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=1}^{\infty}|s_\nu'|\rho_{n+1}^{\nu-1}\right)^p$$

$$\leqq B\sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=1}^{n}|s_\nu'|\rho_{n+1}^{\nu-1}\right)^p + B\sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=n+1}^{\infty}|s_\nu'|\rho_{n+1}^{\nu-1}\right)^p$$

$$= P + Q$$

say. We have

$$P \leqq \sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=1}^{n}|s_\nu'|^p\right)\left(\sum_{\nu=1}^{n}\rho_{n+1}^{p'(\nu-1)}\right)^{p/p'}$$

$$\leqq C\sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=1}^{n}|s_\nu'|^p\right)\cdot n^{p/p'}$$

$$\leqq C\sum_{n=1}^{\infty}\frac{1}{n^{p+2}}\sum_{\nu=1}^{n}|s_\nu'|^p \leqq C\sum_{\nu=1}^{\infty}\frac{|s_\nu'|^p}{\nu^{p+1}}$$

where $1/p+1/p'=1$, and

$$Q \leqq \sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=n+1}^{\infty}|s_\nu'|\rho_{n+1}^{\nu-1}\right)^p$$

$$\leqq D\sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=n+1}^{\infty}\frac{|s_\nu'|^p}{\nu^{p+2}}\right)\left(\sum_{\nu=n+1}^{\infty}\nu^{(1+2/p)p'}\rho_{n+1}^{p'(\nu-1)}\right)^{p/p'}$$

$$\leqq D\sum_{n=1}^{\infty}\frac{1}{n^{2p+1}}\left(\sum_{\nu=n+1}^{\infty}\frac{|s_\nu'|^p}{\nu^{p+2}}\right)\left(\sum_{\nu=n+1}^{\infty}\cdot\nu^{3p'-2}\rho_{n+1}^{p'(\nu-1)}\right)^{p/p'}$$

$$\leqq D\sum_{n=1}^{\infty}\sum_{\nu=n+1}^{\infty}|s_\nu'|^p/\nu^{p+2} \leqq E\sum_{\nu=1}^{\infty}|s_\nu'|^p/\nu^{p+1}.$$

Since $|s_\nu'|/(\nu+1)=|s_\nu-\tau_\nu|$, we get (2.2) by (1.2).

3. THEOREM 3. *Let $f(\theta)$ be a function belonging to L^r $(r>1)$, and $\{p_n\}$ and $\{d_n\}$ be arbitrary increasing sequences of positive integers such that*

$$p_n/d_n = O(p_{n+1}-p_n).$$

Then we have

$$(3.1)\qquad \int_0^{2\pi}\left(\sum_{\nu=1}^{\infty}|s_{p_\nu}-\tau_{p_\nu}|^q/d_\nu\right)^{r/q}d\theta \leqq A\int_0^{2\pi}|f(\theta)|^r d\theta \quad (q \geqq 2).$$

Especially if

$$p_n/n = O(p_{n+1}-p_n),$$

then

$$(3.2)\qquad \sum_{\nu=1}^{\infty}|s_{p_\nu}-f|^m/\nu < \infty \qquad (m \geqq 1)$$

for almost all θ.

The proof of (3.1) runs similarly to that of (2.2). In this case we have to replace Δ_n and ρ_n by

$$\Delta_n = \begin{cases}(1-1/p_n,\ 1-1/p_{n+1}), & \text{if } p_{n+1} < 2p_n,\\ (1-1/p_n,\ 1-1/2p_n), & \text{if } p_{n+1} \geqq 2p_n,\end{cases}$$

and $\rho_n = 1-1/(p_n+1)$ respectively.

(3.2) is obvious from (3.1) by Kronecker's theorem.

REMARK. The formula

$$1/d(x) \sim cp'(x)/p(x)$$

gives the relation between the strong summability factor $d(x)$ and the lacunary sequence $p(x)$. If we put $d(x)=1$, then $p(x)=2^x$, which is nothing but the Kolmogoroff theorem [**2**]. If we put $d(x)=x$, $x^{1/2}$, and so on we get $p(x)=x^c$, $2^{cx^{1/2}}$, and so on, respectively.

We conclude with the following theorem which is proved by an argument due to Zygmund [**7, 8, 9**].

THEOREM 4. *If $f(\theta)$ belongs to L^r $(r>1)$, then for almost all θ, the sequence $\{1, 2, \cdots\}$ can be divided into two complementary subsequences $\{n_k\}$ and $\{m_k\}$, depending in general on θ, and such that*

$s_{p_{n_k}}(\theta)$ tends to $f(\theta)$,

the series $\sum 1/d_{m_k}$ converges,

where $d_n\uparrow$, $\sum 1/d_n=\infty$, and

$$p_n/d_n = O(p_{n+1}-p_n).$$

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ON THEOREMS OF M. RIESZ AND ZYGMUND

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Several proofs have been given of the results of M. Riesz and Zygmund.

(a) *The conjugate of the Fourier series of a function $f(x)$ of L^p, $p>1$, is Fourier series of a function $\bar{f}(x)$ of the same class, and*

$$\int_0^{2\pi} |\bar{f}(x)|^p dx \leqq A_p \int_0^{2\pi} |f(x)|^p dx$$

holds, A_p is a constant depending only on p.

(b) *If the function $|f(x)| \log^+ |f(x)|$ is integrable, the conjugate of the Fourier series of $f(x)$ is the Fourier series of a function $\bar{f}(x)$ of the class L. Moreover, there exist two constants A and B such that*

$$\int_0^{2\pi} |\bar{f}(x)| dx \leqq A \int_0^{2\pi} |f(x)| \log^+ |f(x)| dx + B.$$

In view of the importance of these theorems it may be of interest to give another proof of them based on a different idea. Actually it is

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