ON THE ALGEBRAIC INTEGRALS OF A SYSTEM OF DIFFERENTIAL EQUATIONS OF MECHANICS

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Introduction. If in the system of differential equations

$$
(1;\lambda,\rho)\qquad
\begin{aligned}
&\frac{dx_1}{dt} = -\frac{\lambda}{2}x_1x_3 + y_3, \qquad \frac{dx_2}{dt} = \frac{\lambda}{2}x_2x_3 - \rho y_3,\\
&\frac{dx_3}{dt} = \frac{1}{2-\lambda}(\rho y_1 - y_2),\\
&\frac{dy_1}{dt} = \rho x_1y_3 - x_3y_1, \qquad \frac{dy_2}{dt} = x_3y_2 - \rho x_2y_3,\\
&\frac{dy_3}{dt} = \frac{1}{2}(x_2y_1 - x_1y_2),
\end{aligned}
$$

the parameters λ, ρ are replaced by α, β, the resulting system will be denoted by $(1;\alpha,\beta)$. The present paper will present a study of the algebraic integrals of the system $(1;\lambda,1)$.

The equations $(1;\lambda,1)$ are, excepting for a change of variables, those of the motion of a solid about a fix point, in a uniform field of force, when the ellipsoid of inertia relative to the fix point is of revolution and the baricenter of the solid belongs to the equatorial plane of the same ellipsoid.

Theorems 3 and 4 are original. The demonstration of Theorem 3, presented by R. Liouville [1],[1] accepted by P. Burgatti [2], is incomplete. In fact, it is implicitly admitted that one of the polynomials belonging to the rational integral possesses terms independent of y_1, y_2, y_3. The method of demonstration of Theorem 4 was inspired from a paper of Husson [3].

For the sake of simplicity of nomenclature x will be used instead of (x_1, x_2, x_3) and y instead of (y_1, y_2, y_3) wherever possible.

Let us, "ab initio," make the following two observations about the system $(1;\lambda,1)$:

(a) The differential equations $(1;\lambda,1)$ partake of the following particular property of homogeneity: they are not altered when the x variables are multiplied by a constant k, the y variables by k^2; and t, by k^{-1}.

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[1] Numbers in brackets refer to references cited at the end of the paper.

We shall say that a function $f(x, y)$ is homogeneous (in the sense alluded to) if $f(kx, k^2y) = k^mf(x, y)$.

The scalar m will be called the degree of homogeneity of the function.

If the function $f(x, y)$ is homogeneous of the degree m, its derivative df/dt, as considered from (1; λ, 1) as a function of x, y, will be of the degree $m+1$.

In fact

$$\frac{df(kx, k^2y)}{d(k^{-1}t)} = k^{m+1}\frac{df(x, y)}{dt}.$$

(b) The system (1; λ, 1) is not altered when a permutation of x_1, x_2 is made as well as of y_1, y_2 so long as $-t$ is substituted for $+t$, or, when $-x_3$, $-y_3$ are substituted for $+x_3$, $+y_3$ as well as $-t$ for $+t$.

General case (1; λ, 1). For the sake of completeness the following theorem is included.

THEOREM 1. *Every algebraic integral of the system* (1; λ, 1),

$$\phi(x, y),$$

is an algebraic combination of rational integrals.

In fact, if a function ϕ is algebraic, it will be a root of an equation

$$\phi^p + A_1\phi^{p-1} + \cdots + A_{p-1}\phi + A_p = 0, \tag{2}$$

whose coefficients A_i are rational functions of x, y. Without any restriction in generality we may suppose that this equation is irreducible, as otherwise we would take one of its irreducible components instead of (2).

Since ϕ is an integral of (1; λ, 1), we have $d\phi/dt = 0$, and therefore, by deriving (2) we have

$$\frac{dA_1}{dt}\phi^{p-1} + \cdots + \frac{dA_{p-1}}{dt}\phi + \frac{dA_p}{dt} = 0, \tag{3}$$

the coefficients dA_i/dt being rational functions of x, y.

Equation (2) being, by hypothesis, irreducible, equation (3), of a lower degree, cannot subsist.

We must, therefore, have

$$\frac{dA_i}{dt} = 0, \qquad (i = 1, 2, \cdots, p).$$

These relations show that the functions A_i are (rational) integrals of system $(1;\lambda, 1)$. Owing to (2), ϕ is an algebraic combination of these integrals.

THEOREM 2. *Every rational integral of system* $(1;\lambda, 1)$,

$$\frac{R(x, y)}{S(x, y)},$$

is a combination of homogeneous and rational integrals.

Let us, initially, consider the developments

$$\begin{aligned}
&R(kx, k^2y) \\
&\quad = H(x, y, k)[L_0(x, y)k^p + L_1(x, y)k^{p-1} + \cdots + L_p(x, y)] \\
&\quad = H(x, y, k)L, \\
(4)\quad &S(kx, k^2y) \\
&\quad = H(x, y, k)[M_0(x, y)k^q + M_1(x, y)k^{q-1} + \cdots + M_q(x, y)] \\
&\quad = H(x, y, k)M,
\end{aligned}$$

where $L_0, L_1, \cdots, M_q$ are homogeneous polynomials and $H(x, y, k)$ is the H.C.F. of $R(kx, k^2y)$ and $S(kx, k^2y)$. According to the observation (a) of the introduction,

$$\frac{R(kx, k^2y)}{S(kx, k^2y)} = \frac{L}{M}$$

is still an integral of the system $(1; \lambda, 1)$, and therefore

$$\frac{d}{dt}\left(\frac{L}{M}\right) = 0,$$

that is,

$$M\frac{dL}{dt} - L\frac{dM}{dt} = 0,$$

whence,

$$\begin{aligned}
&k^{p+q}\left(M_0\frac{dL_0}{dt} - L_0\frac{dM_0}{dt}\right) \\
&\quad + k^{p+q-1}\left(M_0\frac{dL_1}{dt} + M_1\frac{dL_0}{dt} - L_1\frac{dM_0}{dt} - L_0\frac{dM_1}{dt}\right) + \cdots = 0.
\end{aligned}$$

The constant k being arbitrary, the following relations are obtained,

$$
\begin{aligned}
&M_0 \frac{dL_0}{dt} - L_0 \frac{dM_0}{dt} = 0, \\
(5) \qquad &M_0 \frac{dL_1}{dt} + M_1 \frac{dL_0}{dt} - L_1 \frac{dM_0}{dt} - L_0 \frac{dM_1}{dt} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots ,
\end{aligned}
$$

which constitute a system of $p+q+1$ homogeneous linear equations on the derivatives of the $p+q+2$ polynomials $L_0, L_1, \cdots, M_q$. The matrix of the coefficients of this system is

$$
\left\| \begin{matrix}
M_0 & 0 \cdots 0 & -L_0 & 0 \cdots & 0 \\
M_1 & M_0 \cdots 0 & -L_1 & -L_0 \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
M_q & \cdots & \cdots & \cdots & \cdots \\
0 & M_q \quad \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 \cdots M_q & 0 & 0 \cdots & -L_p
\end{matrix} \right\| .
$$

L and M being prime, their (Sylvester) resultant is not identically null. Thus the rank of this matrix must be $p+q+1$ and the system (5) will have one degree of indetermination. Under these conditions it necessarily follows from (5) that

$$
(6) \qquad \begin{aligned} \frac{dL_i}{dt} &= \alpha L_i \\ \frac{dM_j}{dt} &= \alpha M_j \end{aligned} \qquad (i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)
$$

α being a suitable factor of proportionality. In fact, system (5) is identically verified for the values (6).

From the above relations (6),

$$
\frac{L_i}{L_0}, \quad \frac{M_j}{L_0}
$$

are (homogeneous) integrals of system $(1; \lambda, 1)$. From (4) we conclude that R/S is a rational combination of these integrals.

THEOREM 3. *Every homogeneous and rational integral of system* $(1; \lambda, 1)$,

$$\frac{R_0(x, y)}{S_0(x, y)},$$

is the quotient of the division of homogeneous and whole integrals.

Without any loss of generality, let us suppose the polynomials R_0 and S_0 to be prime.

We have

$$\frac{d}{dt}\left(\frac{R_0}{S_0}\right) = 0,$$

that is,

$$\frac{dR_0}{dt} = \frac{1}{S_0} \times \frac{dS_0}{dt} \times R_0,$$

or, simply,

$$\frac{dR_0}{dt} = \mu R_0 \tag{7}$$

where μ is a polynomial. According to observation (a) of the introduction, μ must have degree 1 of homogeneity, and therefore

$$\mu = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3,$$

μ_1, μ_2, and μ_3 being constants. Thus (7) is transformed into

$$\frac{dR_0}{dt} = (\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3) R_0. \tag{8}$$

According to observation (b) of the introduction, the polynomial R_1 which is obtained from R_0 by permutation of x_1, y_1, and x_2, y_2 satisfies the relation

$$\frac{dR_1}{dt} = -\,(\mu_2 x_1 + \mu_1 x_2 + \mu_3 x_3) R_1. \tag{9}$$

From (8, 9) it follows that

$$\frac{dR_0R_1}{dt} = (\mu_1 - \mu_2)(x_1 - x_2) R_0 R_1. \tag{10}$$

Based on the same observation, the polynomial R_2 which is obtained from R_0R_1 by substituting $-x_3$, $-y_3$ for x_3, y_3 satisfies the relation

$$\frac{dR_2}{dt} = -(\mu_1 - \mu_2)(x_1 - x_2)R_2. \tag{11}$$

From (10, 11) it follows that

$$\frac{dR_0R_1R_2}{dt} = 0,$$

and, consequently, $R_0R_1R_2$ is an integral of system (1; λ, 1).

Finally, one can conclude that the rational integral R_0/S_0 may be considered as the quotient of the two homogeneous and whole integrals $R_0R_1R_2$ and $S_0R_1R_2$.

Kowalewski's case (1; 1, 1) [4]. It is known that the system (1; 1, ρ) admits the following four algebraic integrals

$$\begin{gathered}\rho x_1x_2 + \frac{1}{2}x_3^2 + \rho y_1 + y_2 = h_1, \quad x_1y_2 + x_2y_1 + x_3y_3 = h_2, \\ y_1y_2 + \rho y_3^2 = h_3, \quad (\rho x_1^2 - 2y_1)(x_2^2 - 2y_2) = h_4.\end{gathered} \tag{12; 1, ρ}$$

THEOREM 4. *The system* (1; 1, 1) *does not admit any other algebraic integral distinct from the integrals* (12; 1, 1).

Let

$$f(x_1, y_1, y_3, x_2, x_3, y_2) \tag{13}$$

be an algebraic integral of system (1; 1, 1). We shall prove that f is an algebraic combination of the integrals (12; 1, 1).

The system (1; 1, ρ) admits, whatever be the value of ρ, the algebraic integral

$$f(\rho x_1, \rho y_1, \rho y_3, x_2, x_3, y_2)$$

or, owing to (12; 1, ρ),

$$F(h_1, h_2, h_3, h_4, \rho, x_3, y_3),$$

F being algebraic with respect to every one of the arguments.

To prove that (13) is an algebraic combination of the integrals (12; 1, 1), it is sufficient to prove that F is independent of x_3 and y_3.

Let us, in the first place, notice that x_3 and y_3 are not integrals of the system (1; 1, 1), therefore F cannot depend only upon x_3 or upon y_3. Let us then suppose that it depends upon x_3 and y_3.

One can always multiply F by a convenient power of ρ, so chosen that F, considered as a function of ρ, will not admit the point $\rho=0$ neither as a zero nor as a pole. The algebraic function F of ρ is then developable, in the domains of the value $\rho=0$, according to the rising positive powers of ρ or of $\rho^{1/p}$. In such a development the coefficients of the various powers of ρ are algebraic functions of $h_1, h_2, h_3, h_4, x_3, y_3$.

We may always suppose that for $\rho=0$, F will not be reduced to a single function of h_1, h_2, h_3, h_4. In fact, let us carry on the development of F according to the powers of $\rho^{1/p}$, until the first term whose coefficient be not reduced to a single function of h_1, h_2, h_3, h_4. Let it be

$$F = \phi(h_1, h_2, h_3, h_4, \rho) + \rho^{k/p}F_0(h_1, h_2, h_3, h_4, \rho) + \rho^{(k+1)/p}F_1 + \cdots .$$

Substituting for F the expression

$$\frac{F - \phi(h_1, h_2, h_3, h_4, \rho)}{\rho^{k/p}},$$

which means subtracting from the integral f an algebraic function of the classic integrals, we see that one may always suppose

$$F = F_0(h_1, h_2, h_3, h_4, x_3, y_3) + \rho^{1/p}F_1 + \cdots \tag{14}$$

where F_0 is not independent of x_3 and y_3.

The system (1; 1, 0) admits the algebraic integrals (12; 1, 0) and must also admit the following algebraic integral

$$F_0(h_1, h_2, h_3, h_4, x_3, y_3)$$

which results from (14) for $\rho=0$. Let us seek for this integral.

From (12; 1, 0) one gets

$$\begin{aligned} 2y_2 &= 2h_1 - x_3^2 \\ x_1y_2 - x_2y_1 &= h_2 - x_3y_3 - h(2h_1 - x_3^2)^{-1/2} \end{aligned} \tag{15}$$

where $h=2[(4h_3-h_4)h_3]^{1/2}$.

From the third and the last equations of (1; 1, 0) it follows that

$$\frac{dx_3}{2y_2} = \frac{dy_3}{x_1y_2 - x_2y_1}.$$

Owing to (15), this equation becomes

$$\frac{dx_3}{2h_1 - x_3^2} = \frac{dy_3}{h_2 - x_3y_3 - h(2h_1 - x_3^2)^{-1/2}}.$$

The solution obtained is

$$y_3(2h_1 - x_3)^{-1/2} + \int [h(2h_1 - x_3^2)^{-1/2} - h_2](2h_1 - x_3^2)^{-3/2}dx_3 = c,$$

which is a nonalgebraic integral of the system (1; 1, 0).

This conclusion contradicts the fact of F_0 being an algebraic function, therefore the initial hypothesis that assumes F to be dependent upon x_3 and y_3 is absurd.

F being independent of x_3 and y_3, it follows, as initially said, that any algebraic integral of the system (1; 1, 1) is an algebraic combination of the four integrals (12; 1, 1), that is, such a system does not admit any other algebraic integral distinct from the integrals (12; 1, 1).

References

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NOTE ON THE FUNCTIONAL EQUATIONS $f(xy) = f(x) + f(y), f(x^n) = nf(x)$

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Introduction. M. Augustin-Louis Cauchy in his famous Cours d'Analyse de l'école Royale Polytechnique (Part I, *Analyse algêbrique*, chap. 5, p. 109), published in 1821, gives a beautiful proof that the only continuous solution of the functional equation $f(x)+f(y)=f(xy)$, where $f(x)$ is defined for all real numbers x, is the function $f(x)=a \ln x$. Cauchy's proof reduces the equation to the Cauchy equation $f(x)+f(y)=f(x+y)$. In 1905 G. Hamel in the Mathematische Annalen proved that the discontinuous solutions of Cauchy's equation are totally discontinuous. In 1919 B. Blumberg proved that the only measurable function satisfying Cauchy's equation is the function Ax, hence the only measurable solution of

$$f(xy) = f(x) + f(y), \qquad f(x^n) = nf(x) \tag{1}$$

is $a \ln x$. In the past thirty years there have been a number of papers on these functional equations where the domain of the variable has usually been the real number system. In 1946 Erdös proved that if

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