ON THE TWO-DIMENSIONAL DERIVATIVE OF A COMPLEX FUNCTION

VINCENT C. POOR

1. Introduction. We are considering here the two-dimensional derivative of the complex function $f(z)=u(x, y)+iv(x, y)$ over a domain D of the complex plane. This special case of higher-dimensional derivatives was treated by Cioranescu [1][1] and by making the limit independent of direction he arrived at a unique derivative, and in this sense monogenic. The class of functions arising was studied recently in the Bulletin by Haskell [2] and by Reade [3], in which some interesting results were obtained using certain mean value theorems.

The purpose of this paper is briefly to analyze and synthesize these results using a new and different viewpoint, and to characterize more explicitly this class of polygenic functions. In particular in extending these results a second class of monogenic functions will be introduced. The whole setup leads to a complete dual or analogue to the first class of functions. This dual situation will be developed; sufficient details to exhibit this completeness will be given.

AREOLAR MONOGENIC FUNCTIONS

2. Areolar monogeneity. Cioranescu arrives at this two-dimensional derivative by selecting a rectangle in D with vertices z, z_1, z', z_2 in positive order with ϕ as slope angle of one side. He then considers the

$$\lim_{z_1,z_2\to z} \frac{f(z') - f(z_1) - [f(z_2) - f(z)]}{(z_1 - z)(z_2 - z)}. \tag{2.1}$$

If h and k are the lengths of the sides of the rectangle, the numerator, a function of h and k, may be expanded into a double power series in h and k. The denominator will contain the area hk and an exponential factor function of ϕ. The limit is then taken and those terms involving the direction ϕ are set equal to zero, making the limit or two-dimensional derivative unique and independent of ϕ. Complex functions possessing this derivative at every point of the domain D have been called "areolar monogenic" in the domain. This type of monogeneity is characterized by the equations

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$2u_{xy} = -(v_{xx} - v_{yy}), \qquad 2v_{xy} = u_{xx} - u_{yy} \tag{2.2}$$

or according to Cioranescu these functions are solutions of the differential equation

$$\frac{\partial^2 f}{\partial \alpha^2} = 0 \tag{2.3}$$

which is the equivalent of (2.2). The derivative itself is $v_{xy} - iu_{xy}$ while u and v are biharmonic functions. The notation $\partial f/\partial \alpha$ is used for the Pompeiu [4] areal (or areolar) derivative.

3. **Basic theorems.** We turn now to some fundamental theorems, which will be useful in the sequel.

THEOREM 1. *If $f(z)$ possesses a differential of the form $df = f_x dx + f_y dy$ in a domain D of the complex plane, then for the circulation $\int_c f(z)dz$ around every rectifiable closed curve c in D to exist and be finite it is necessary and sufficient that the areal derivative $\partial f/\partial \alpha$ exist and be finite at every point of D.*

The proof given in another paper will be omitted here; however, the proof leads to the circulation theorem.

$$\frac{1}{2\pi i}\int_c f(z)dz = \frac{1}{\pi}\int_\sigma \frac{\partial f}{\partial \alpha}\, d\sigma \tag{3.1}$$

as a consequence.

COROLLARY. *If $f(z)$ is continuous, the necessary and sufficient condition that $f(z)$ be analytic is that $\partial f/\partial \alpha = 0$.*

The vanishing of the areal derivative is identical with the Cauchy-Riemann equations.

THEOREM 2. *If $f(z)$ is continuous in D, then the necessary and sufficient condition for $f(z)$ to be areolar monogenic in D is that the areal derivative $\partial f/\partial \alpha$ be analytic in D.*

The necessity. Here $f(z)$ is areolar monogenic by hypothesis; we wish to prove that $\partial f/\partial \alpha$ is analytic.

We construct a function $f(z, z')$ of two independent variables such that when z' is replaced by $\bar{z}$, $f(z, z')$ becomes $f(z)$. We first observe that $\partial f/\partial \alpha = \partial f(z, z')/\partial z' \big|_{z'=\bar{z}}$. We now solve the defining equation

$$\frac{\partial^2 f}{\partial \alpha^2} = \frac{\partial^2 f(z, z')}{\partial z'^2}\bigg|_{z'=\bar{z}} = 0$$

for $f(z)$; or

$$\frac{\partial^2 f(z, z')}{\partial z'^2} = 0$$

so that $\partial f/\partial z' = g(z)$ and

$$f(z, z') = z'g(z) + h(z)$$

where $g(z)$ and $h(z)$ are arbitrary functions of z subject to the defining equation $\partial^2 f(z, z')/\partial z'^2 = 0$. Hence

$$f(z) = \bar{z}g(z) + h(z). \tag{3.2}$$

Evidently then

$$\frac{\partial f(z)}{\partial \alpha} = g(z), \qquad \frac{\partial h}{\partial \alpha} = 0; \qquad \frac{\partial g}{\partial \alpha} = 0,$$

and since f is continuous $g(z)$ and $h(z)$ are analytic in D and this makes the areal derivative $\partial f/\partial \alpha$ analytic in D, which was to be proved.

The sufficiency. By hypothesis $\partial f/\partial \alpha = g(z)$, an analytic function of z in D, or $\partial f(z, z')/\partial z'|_{z'=\bar{z}} = g(z) = \partial f/\partial z'$. Hence

$$\frac{\partial^2 f(z, z')}{\partial z'^2} = 0 = \left.\frac{\partial^2 f(z, z')}{\partial z'^2}\right|_{z'=\bar{z}} = \frac{\partial^2 f}{\partial \alpha^2} = 0,$$

or $f(z)$ is areolar monogenic in D.

A simple alternate proof is to apply the corollary under Theorem 1; since $g(z)$ is analytic $\partial g(z)/\partial \alpha = 0 = \partial^2 f/\partial \alpha^2$.

THEOREM 3. *If $f(z)$ is continuous in D, then for $f(z)$ to be areolar monogenic in D it is necessary and sufficient that*

$$f(z) = \bar{z}g(z) + h(z) \tag{3.3}$$

where g and h are analytic functions of z.

The necessity. This part of Theorem 3 was established (3.2) in the proof of Theorem 2.

The sufficiency. Here (3.3) is given or

$$f(z, z') = z'g(z) + h(z)$$

while

$$\frac{\partial f}{\partial \alpha} = \left.\frac{\partial f(z, z')}{\partial z'}\right|_{z'=\bar{z}} = g(z),$$

then it is evident that $\partial^2 f(z)/\partial\alpha^2 = 0$, or $f(z)$ is areolar monogenic.

4. Related theorems. We wish to show here how some of the results obtained by Haskell [2] and by Reade [3] are related to the theorems obtained above. The principal theorem of the Haskell paper, though differing in appearance, is comprehended in Theorem 2 above. This is also true of Haskell's Theorem 2 for the domain $D-E$, where E is a countable set. This is then extended to D through the Menchoff [5] theorem.

The consideration of other theorems requires the application of the following

LEMMA. *If $\sigma(z, r)$ is a circular disk, center z, radius r, in D, then the average or mean value of the analytic function $f(z)$ integrated over the area of the disk is equal to the value of the function at the center of the disk and is given by the equation*

$$f(z) = \frac{1}{\pi r^2}\int_{\sigma(z,r)} f(\zeta)d\sigma. \tag{4.1}$$

PROOF. Beckenbach and Reade [8] have observed that if $f(z)$ is a harmonic polynomial, equation (4.1) or the lemma is valid. To extend this result to analytic functions, we call $f_n(z)$ the sum of the first n terms of the series representation of the analytic function; this gives

$$\frac{1}{\pi r^2}\int_{\sigma(z,r)} f_n(\zeta)d\sigma = f_n(z).$$

But this relation constitutes two identical sequences of harmonic polynomials; their limits $n\to\infty$ are therefore equal. This proves the lemma.

Briefly, we state that the lemma is applicable when $\sigma(z, r)$ is replaced by the area $P(z, r)$ of a regular n-gon which circumscribes the circle [8].

Continuing then: Theorem A and Theorem 1 of the Reade paper [2] require that there exist a function

$$g(z) = \frac{1}{2\pi i r^2}\int_{c(z,r)} f(\zeta)d\zeta \tag{4.2}$$

analytic in D as the necessary and sufficient condition for $f(z)$ to be areolar monogenic. The notation $c(r, z)$ is used for the circle radius r, center at z.

In the light of Theorem 2 we need only show that $g(z) = \partial f/\partial\alpha$. We first transform $g(z)$ by the circulation theorem (3.1) so that

$$g(z) = \frac{1}{2\pi i r^2} \int_{c(z,r)} f(\zeta)d\zeta = \frac{1}{\pi r^2} \int_{\sigma(z,r)} \frac{\partial f}{\partial \alpha} d\sigma, \tag{4.3}$$

σ here being the circular area. For $f(z)$ areolar monogenic, $\partial f/\partial\alpha$ is analytic by Theorem 2 above. The lemma just proved makes the right member of (4.3) equal to $\partial f/\partial\alpha$ at z, so that $g(z)$ is analytic. Conversely, if $g(z)$ is analytic, the limit as $r \to 0$ of the right member of (4.2) is by definition $\partial f/\partial\alpha$. This makes $\partial f/\partial\alpha$ analytic and $f(z)$ areolar monogenic.

The argument just concluded is also a proof of the Haskell analogue to the Morera theorem (Theorem 3). Similarly, Reade's Theorem 2 is obtainable by making some adjustments. In (4.3) we replace πr^2 by P, the area of a regular n-gon, and $c(r, z)$ by $p(r, z)$, the perimeter of the n-gon where r is the radius of the inscribed circle with z as center. The extension of the above lemma makes the right member of the adjusted equation (4.3) equal to $\partial f/\partial\alpha$ at z, while the limit as $r \to 0$ of the adjusted (4.2) is the areal derivative again by definition. This theorem is essentially a repetition of Reade's Thoerem 1.

Mean monogenic functions

5. **Mean monogeneity.** The class of mean monogenic functions may be arrived at in various ways. The following direct derivation seems the most satisfactory. Briefly then we modify definition (2.1) by replacing the denominator by its conjugate in which case the new limit becomes

$$\begin{aligned} &[f_{xy} \cos 2\phi - 2^{-1} (f_{xx} - f_{yy}) \sin 2\phi] e^{i(2\phi+\pi/2)} \\ &\quad = u_{xx} - u_{yy} - 2v_{xy} + i(2u_{xy} + v_{xx} - v_{yy}) \\ &\qquad + [2u_{xy} - v_{xx} + v_{yy} + i(2v_{xy} + u_{xx} - u_{yy})] e^{i(4\phi+\pi/2)}, \end{aligned} \tag{5.1}$$

omitting the many details.

To make this limit independent of direction we set the coefficient of the exponential function equal to zero making the limit independent of direction with the result that

$$2u_{xy} = v_{xx} - v_{yy}, \qquad 2v_{xy} = -(u_{xx} - u_{yy}) \tag{5.2}$$

or its equivalent

$$\partial^2 f(z)/\partial\beta^2 = 0 \tag{5.3}$$

where $\partial f/\partial\beta$ is the Kasner [7] mean derivative or

$$\frac{\partial f}{\partial \beta} = \lim_{\Delta\sigma_j \to 0} \frac{-1}{2i\Delta\sigma_j} \int_{c_j} f(z) d\bar{z}.$$

Solutions of (5.3) will be called *mean monogenic functions* because of this close relation with the Kasner mean derivative. The unique mean monogenic derivative itself is $-v_{xy}+iu_{xy}$ while, again, u and v are biharmonic functions.

6. **Theorems of mean monogenic functions.** We can now prove the analogue to Theorem 1. This theorem leads to the circulation theorem

$$\frac{-1}{2\pi i}\int_c f(z)d\bar{z} = \frac{1}{\pi}\int_\sigma \frac{\partial f}{\partial \beta}\,d\sigma \tag{6.1}$$

where c in D is the boundary of the area σ as before. We now state two theorems to emphasize the duality between the two classes of functions considered here.

THEOREM 4. *If $f(z)$ is continuous in D, then for $f(z)$ to be mean monogenic in D it is necessary and sufficient that the mean derivative $\partial f(z)/\partial\beta$ be an analytic function of $\bar{z}$ in D.*

THEOREM 5. *If $f(z)$ is continuous in D, then the necessary and sufficient condition for $f(z)$ to be mean monogenic in D is that it have the form given by the equation*

$$f(z) = zp(\bar{z}) + q(\bar{z}) \tag{6.2}$$

where p and q are analytic functions of $\bar{z}$.

The proofs of these two theorems will be omitted since the arguments follow those of Theorems 2 and 3, respectively. Analogues to the theorems of Haskell and of Reade may be stated and proved for the class of mean monogenic functions.

7. **Duality.** We shall here try to expose the completeness of the duality existing between the areolar and mean monogenic functions. In addition to the two theorems just stated together with the analogues just mentioned we supplement their weight by

THEOREM 6. *If $f(z)$ is areolar monogenic and at the same time mean monogenic in D, then*

$$f(z) = az\bar{z} + bz + c\bar{z} + d. \tag{7.1}$$

In our proof we have from our hypothesis the relation

$$f(z) = \bar{z}g(z) + h(z) \equiv zp(\bar{z}) + q(\bar{z}) \tag{7.2}$$

obtained from (3.3) and (6.2).

Since the highest degree term in (7.2) is $z\bar{z}$, it follows that $g(z)$

$=a_1z+b_1$ and $p(\bar{z})=a_2\bar{z}+b_2$; (7.2) then becomes

$$az\bar{z}+b_1\bar{z}+h(z)\equiv az\bar{z}+b_2z+q(\bar{z})$$

if we write a for a_1 and a_2. From this we have $h(z)=c_1z+d$ and $q(\bar{z})=c_2\bar{z}+d$; the adjustment of the constants gives (7.1).

To strengthen our conclusion we may note that if $f(z)$ is in the general form for an areolar monogenic function given by (3.3), then

$$\bar{f}(z)=z\bar{g}(z)+\bar{h}(z)=zp(\bar{z})+q(\bar{z}),$$

or if the coefficients of g and h in expanded form are real numbers, then $\bar{f}=zg(\bar{z})+h(\bar{z})$. In any case $\bar{f}(z)$ is the standard form for mean monogenic functions. On the other hand when $f(z)$ is given by (6.2)

$$\bar{f}(z)=\bar{z}\bar{p}(\bar{z})+\bar{q}(\bar{z})=\bar{z}g(z)+h(z)$$

or equals $\bar{z}p(z)+q(z)$ for real coefficients as before. Here $\bar{f}$ is areolar monogenic. In other words, every function of one class has an exact correspondent in the other. We are thus justified in saying that to every theorem pertinent to one class of functions there is an exact dual theorem valid for the other class.

With the above analysis we may conclude that the duality between areolar and mean monogenic functions is complete. The forms given by (3.3) and (6.2) also completely characterized these two classes of functions.

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University of Michigan