REMARKS ON CERTAIN PATHOLOGICAL OPEN SUBSETS OF 3-SPACE AND THEIR FUNDAMENTAL GROUPS

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We shall consider several well known subsets of spherical 3-space S: the compact zero-dimensional set P described by Antoine [1][1] and the topological 3-cell C consisting of the "horned sphere" Σ of Alexander [2] together with its interior. That *the fundamental groups, $\pi_1(S-P)$ and $\pi_1(S-C)$, of their complementary domains are not finitely generated* (and hence in particular not trivial) was noted by Alexander [2, pp. 9 and 11]. It is the main purpose of this note to give a

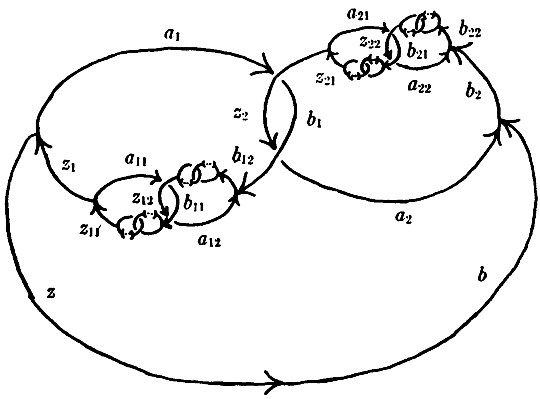

FIG. 1

complete proof of this remark, using explicit presentations of the groups.[2] We feel that this is justified by the historical significance of these examples, even though on the one hand Antoine's set occurs as a special case of a recent generalization [3] to n-dimensional space ($n \geqq 3$) and on the other Alexander's example has been supplanted by a much simpler one [4]. An unexpected consequence of the explicit presentation of the groups is the discovery, of which we make use, that $\pi_1(S-C)$ is a homomorph of $\pi_1(S-P)$; the geometrical meaning of this relationship between Antoine's and Alexander's examples is not evident.

Of great interest, although not as well known, is the simply con-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] A proof that $\pi_1(S-P)$ is non-trivial has been given by R. P. Coelho [6]. By a presentation $\{x/r=1\}$ of a group we mean a system of generators $\{x_j\}$ and defining relations $\{r_i=1\}$.

nected open subset M of S, which is not a topological open 3-cell, constructed by Whitehead and Newman [5]. In §3 we give a simplified proof that M is not homeomorphic to an open 3-cell. The simplifying ideas in this proof were used [4, example 1.3] in demonstrating the properties of a certain set Z which combines the essential features of the example of Alexander and that of Whitehead-Newman.

1. **Alexander's horned cell.** The complement, $S-C$, of Alexander's horned cell is homeomorphic to the complement of an infinite graph Γ, whose projection is shown in Fig. 1.

From this diagram a presentation of the group $\pi_1(S-C)=\pi_1(S-\Gamma)$ may be read, using a slight extension of a standard method.[3] The generators are z, z_α, a_α, b, b_α, where α ranges over all finite sequences of the form

$$\alpha = \alpha_1\alpha_2 \cdots \alpha_l, \qquad \alpha_i = 1, 2, \qquad l(\alpha) = l > 0.$$

The defining relations are of three types:

(1) At the crossings (cf. Fig. 2)

$$(1_\alpha) \qquad b_{\alpha 1} = z_{\alpha 2} a_{\alpha 1} z_{\alpha 2}^{-1}, \qquad z_{\alpha 2} = b_{\alpha 1}^{-1} a_{\alpha 2} b_{\alpha 1}.$$

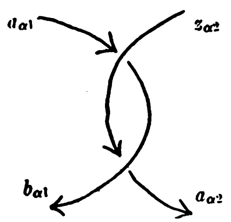

FIG. 2

(2) At the points of order three (cf. Fig. 3)

$$(2_\alpha) \qquad z_\alpha^{-1} z_{\alpha 1} a_{\alpha 1}^{-1} = 1, \qquad b_\alpha b_{\alpha 2}^{-1} a_{\alpha 2} = 1.$$

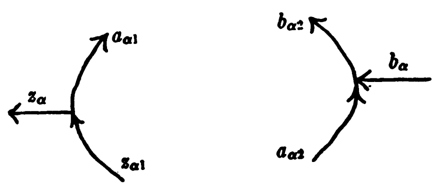

FIG. 3

(3) Around the singular points (cf. Fig. 4).

$$(3_\alpha) \qquad b_\alpha = z_\alpha,$$

[3] Cf. [4, p. 981].

where $\alpha 1$, $\alpha 2$ denote the sequences of length $l+1$ formed by adjoining 1 or 2, resp., to α.

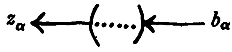

FIG. 4

Using (2_α) and (3_α) as definitions of the generators a and b we obtain from (1_α) the presentation:[4]

$$G = \pi_1(S - C) = \{z_\alpha, l(\alpha) \geqq 0 / z_\alpha = [z_{\alpha 1}, z_{\alpha 2}^{-1}]\}.$$

For each non-negative integer n we consider the free group G_n on generators z_α, $l(\alpha) = n$, and define a homomorphism ϕ_n of G_n into G_{n+1} by $\phi_n(z_\alpha) = [z_{\alpha 1}, z_{\alpha 2}^{-1}]$. Clearly G is the direct limit of the homomorphism sequence $G_0 \to G_1 \to G_2 \to \cdots$. Moreover, for each n, ϕ_n is an isomorphism, for if $w(z_\alpha)$ is a reduced word in G_n, then $\phi_n(w(z_\alpha)) = w([z_{\alpha 1}, z_{\alpha 2}^{-1}])$ is necessarily a reduced word in G_{n+1}. Hence the groups G_n may be identified with certain subgroups of G. Thus $G = \bigcup_{n=0}^{\infty} G_n$ is the union of an increasing sequence of finitely generated free groups. Since the rank of G_n is 2^n we obtain the result,[5]

$\pi_1(S-C)$ is a locally free group and does not have finite rank.

2. **Antoine's set.** Consider a solid torus P_0 in S and k solid tori $T_1, T_2, \cdots, T_k$ in P_0 arranged in cyclic order around P_0 in such a way that each torus is simply linked with each of its neighbors. Denote the union of $T_1, T_2, \cdots, T_k$ by P_1 and for each $i = 1, 2, \cdots, k$ select a homeomorphism f_i of P_0 onto T_i. For any finite sequence $\alpha = \alpha_1 \alpha_2 \cdots \alpha_l$, $1 \leqq \alpha_i \leqq k$, define $f_\alpha = f_{\alpha 1} f_{\alpha 2} \cdots f_{\alpha l}$, $T_\alpha = f_\alpha(P_0)$, and $P_l = \bigcup_{l(\alpha)=l} T_\alpha$. Assuming k sufficiently large and the tori $T_1, \cdots, T_k$ properly placed so that diam $T_\alpha \to 0$ as $l(\alpha) \to \infty$, the set $P = \bigcap_{l=0}^{\infty} P_l$ is recognizable as Antoine's set. We denote the closure of the set $P_l - P_n$ by P_n^l.

It is easily seen that P_1^0 is of the same homotopy type as the complement of the linkage whose projection is represented in Fig. 5, where we have taken k even.

Labelling and orienting the edges as indicated we find by a standard procedure[3] that $\pi_1(P_1^0)$ is generated by elements $x, u, v, w, a_1, b_1, \cdots, a_k, b_k$ subject to the defining relations[4]

[4] We use the notations $[u, v] = uvu^{-1}v^{-1}$, $u^v = vuv^{-1}$, $u^{-v} = vu^{-1}v^{-1}$.

[5] To show just that $\pi_1(S-C)$ is non-trivial, a simpler procedure is to exhibit a representation of G into the symmetric group of order 6. Cf. [3]. For definition of locally free group see [8].

$$
\begin{aligned}
x^{a_1} &= w = x^{b_1},\\
b_1^{x} &= v = u^{b_k^{-1}},\\
u^{x^{-1}} &= a_1 = b_1^{b_2},\\
a_2^{a_1} &= b_2 = a_2^{a_3},\\
b_3^{b_2} &= a_3 = b_3^{b_4},\\
a_4^{a_3} &= b_4 = a_4^{a_5},\\
&\cdots\cdots\cdots,\\
b_{k-1}^{b_{k-2}} &= a_{k-1} = b_{k-1}^{b_k},\\
a_k^{a_{k-1}} &= b_k = a_k^{u}.
\end{aligned}
$$

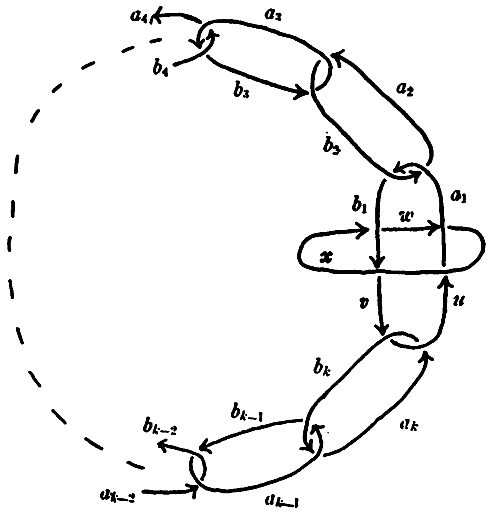

FIG. 5

From these relations we can deduce that

$$a_1^{-1}b_1 = a_2^{-1}b_2 = \cdots = a_k^{-1}b_k = u^{-1}v.$$

Denoting this element by y, we are now enabled to eliminate $u, v, w, b_1, \cdots, b_k$. As a result, we find that $\pi_1(P_1^0)$ is generated by $x, y, a_1, \cdots, a_k$, subject to the relations

$$
\text{I}\qquad
\begin{aligned}
&[x, y] = 1,\\
&y = [a_j^{-1}, a_{j-1}] = [a_j^{-1}, a_{j+1}], \qquad j = 2, 4, 6, \cdots, k-2,\\
&y = [a_k^{-1}, a_{k-1}] = [a_k^{-1}, a_1^{x}].
\end{aligned}
$$

Now since the closure of the set $T_i - U_j T_{ij}$ is just $f_i(P_1^0)$, we see that its fundamental group is given by generators x_i', y_i', a_{i1}', a_{i2}', $\cdots$, a_{ik}'

and defining relations I_i', obtained by replacing $x, y, a_1, \cdots, a_k$ in I by $x_i', y_i', a_{i1}', \cdots, a_{ik}'$, respectively. The intersection of P_1^0 and $f_i(P_1^0)$ is the torus $\dot{T}_i$ which is the boundary of T_i, and whose fundamental group is free abelian on the two generators x_i' and y_i'. Since x and y may be represented by loops bounding in the closure of $S-P_0$ and in P_0, respectively (see Fig. 5), it follows that $x_i'=f_i(x)$ and $y_i'=f_i(y)$ may be represented respectively by equatorial and meridional loops on $\dot{T}_i$. This observation enables us to compute from Fig. 5 the relations defining the injection $\pi_1(\dot{T}_i)\to\pi_1(P_1^0)$. They are

$$\mathrm{II}_i \qquad \begin{aligned} y_i &= a_i, \\ x_i &= a_{i+1}^{-1}a_{i-1}, && i = 2, 4, \cdots, k-2, \\ &= a_{i+1}a_{i-1}^{-1}, && i = 3, 5, \cdots, k-1, \\ &= a_2x^{-1}a_k^{-1}x, && i = 1, \\ &= xa_1^{-1}x^{-1}a_{k-1}, && i = k, \end{aligned}$$

where x_i, y_i (and later $a_{i1}, \cdots, a_{ik}$) denote the elements represented by loops obtained by conjugating a representative of x_i', y_i' $(a_{i1}', \cdots, a_{ik}')$ by a (properly chosen) fixed path joining the base point of $\pi_1(P_1^0)$ to that of $\pi_1(f_i(P_1^0))$.

Since the k sets $f_i(P_1^0)$ are mutually disjoint, we may obtain a presentation of the group $\pi_1(P_2^0)$ by k successive applications of the algorithm for computing the fundamental group of a union.[6] We thus obtain

$$\pi_1(P_2^0) = \{x_\alpha, y_\alpha, a_{\alpha i}/\mathrm{I}, \mathrm{I}_i, \mathrm{II}_i\}$$

where $0\leqq l(\alpha)\leqq 1$, $i=1, 2, \cdots, k$.

Proceeding similarly with the adjunction of the sets $P_3^2, P_4^3, \cdots$, we construct inductively presentations of the groups $\pi_1(P_n^0)$:

$$\pi_1(P_n^0) = \{x_\alpha, y_\alpha, a_{\alpha i}/\mathrm{I}, \mathrm{I}_{\alpha i}, \mathrm{II}_{\alpha i}\}$$

where $0\leqq l(\alpha)\leqq n-1$, $i=1, 2, \cdots, k$ and $\mathrm{I}_{\alpha i}$, $\mathrm{II}_{\alpha i}$ denote the relations obtained from I_i and II_i by replacing i by αi (see below).

Since $\pi_1(P_0-P)$ is the direct limit of the homomorphism sequence

$$\pi_1(P_1^0) \to \pi_1(P_2^0) \cdots ,$$

in which each homomorphism sends the generators of one group into the same-named generators of the other, we obtain $\pi_1(P_0-P)$ simply by removing the restrictions on $l(\alpha)$ in the presentations. A presentation of the group $\pi_1(S-P)$ is obtained by adjoining the relation $x=1$.

[6] Cf. [7, §52].

Thus $\pi_1(S-P)$ is given by generators x_α, y_α, $a_{\alpha i}$, $l(\alpha)\geqq 0$, with the defining relations:

$$
\text{I}_\alpha \qquad
\begin{aligned}
&x = 1,\\
&[x_\alpha, y_\alpha] = 1,\\
&y_\alpha = [a_{\alpha j}^{-1}, a_{\alpha j-1}] = [a_{\alpha j}^{-1}, a_{\alpha j+1}], \qquad j = 2, 4, \cdots, k-2,\\
&y_\alpha = [a_{\alpha k}^{-1}, a_{\alpha k-1}] = [a_{\alpha k}^{-1}, a_{\alpha 1}^{x_\alpha}],
\end{aligned}
$$

$$
\text{II}_{\alpha i} \qquad
\begin{aligned}
y_{\alpha i} &= a_{\alpha i},\\
x_{\alpha i} &= a_{\alpha i+1}^{-1} a_{\alpha i-1}, && i = 2, 4, \cdots, k-2,\\
&= a_{\alpha i+1} a_{\alpha i-1}^{-1}, && i = 3, 5, \cdots, k-1,\\
&= a_{\alpha 2} x_\alpha^{-1} a_{\alpha k}^{-1} x_\alpha, && i = 1,\\
&= x_\alpha a_{\alpha 1} x_\alpha^{-1} a_{\alpha k-1}^{-1}, && i = k.
\end{aligned}
$$

That $\pi_1(S-P)$ is not finitely generated (and therefore nontrivial) is demonstrated by the following homomorphism into $\pi_1(S-C)$:

$$
\begin{aligned}
a_\alpha = y_\alpha &\rightarrow z_{\sigma(\alpha)}, && \text{if } l(\alpha) \text{ is odd},\\
&\rightarrow z_{\sigma(\alpha)}^{-1}, && \text{if } l(\alpha) \text{ is even},
\end{aligned}
$$

where

$$\sigma(\alpha) = \sigma_1\sigma_2 \cdots \sigma_l, \qquad \sigma_i = 1, 2,$$

for $i = 1, 2, \cdots, l$ and

$$
\begin{aligned}
\sigma_i &\equiv \alpha_i \pmod 2 && \text{if } i \text{ is odd},\\
&\equiv 1 + \alpha_i \pmod 2 && \text{if } i \text{ is even}.
\end{aligned}
$$

3. **The Whitehead-Newman set.** To obtain the Whitehead-Newman set T_∞ one proceeds as in the construction of Antoine's set, using $k=1$ instead of $k\equiv 0 \pmod 2$. The solid torus corresponding to $S-P_0$ of the preceding section is denoted in [5] by $R=S-T_0$. The fundamental group G_∞ of $U_{0,\infty}=T_0-T_\infty=S-(R\cup T_\infty)$ is the limit of the direct homomorphism sequence

$$G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \cdots$$

where G_n denotes the fundamental group of $U_{0,n}=T_0-T_n=S-(R\cup T_n)$. From the presentation (3.1) of [5] it follows that G_∞ has the presentation

$$
\begin{aligned}
G_\infty = \{a_\lambda, b_\lambda, 0 \leqq \lambda < \infty / [a_\lambda, b_\lambda] = 1, b_\lambda &= [a_\lambda^{-1}, b_{\lambda+1}][a_\lambda^{-1}, b_{\lambda+1}^{-1}],\\
a_{\lambda+1} &= [b_{\lambda+1}^{-1}, a_\lambda][b_{\lambda+1}^{-1}, a_\lambda^{-1}]\}.
\end{aligned}
$$

The fundamental group of $S-T_\infty$, obtained by adjoining to G_∞ the relation $a_0=1$, is obviously trivial. Thus $S-T_\infty$ is a simply connected open subset of S.

On the other hand the group G_∞ has the representation

$$\begin{aligned} a_\lambda &\rightarrow (1\ 2\ 3\ 4\ 5) && \text{(for } \lambda \text{ even)} \\ &\rightarrow (1\ 5\ 3\ 2\ 4) && \text{(for } \lambda \text{ odd),} \\ b_\lambda &\rightarrow (5\ 4\ 3\ 2\ 1) && \text{(for } \lambda \text{ even)} \\ &\rightarrow (4\ 2\ 3\ 5\ 1) && \text{(for } \lambda \text{ odd),} \end{aligned}$$

in the symmetric group of order 5.

We assert that the complement $S-(T_\infty \cup J)$ of any compact subset J of $S-T_\infty$ containing R cannot be simply connected. For $S-(T_\infty \cup J) \subset S-(R\cup T_\infty) \subset U_{0,\infty}$, and we can always find λ so large that a_λ is represented by a loop in $U_{0,\infty}-J$, and this cannot be contracted to a point in $S-(T_\infty \cup J)$ because $a_\lambda \neq 1$ in $\pi_1(U_{0,\infty})$. This proves immediately that $S-T_\infty$ is not an open 3-cell, for in an open 3-cell any compact set R is contained in a closed 3-cell J whose complement is simply connected.

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